Project Management Contracts with Delayed Payments

H. Dharma Kwon,[*] Steven A. Lippman, Kevin F. McCardle, and Christopher S. Tang[†]

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Abstract

When managing projects with considerable uncertainty such as those arising in construction, defense, and new product development, it is customary for a manufacturer (project manager) to offer contracts under which each supplier (contractor) receives a pre-specified payment when she completes her task. However, there are recent cases in which the manufacturer imposes "delayed payment" contracts under which each supplier is paid only when all suppliers have completed their tasks. By considering a model of one manufacturer and $n \geq 2$ identical and independent suppliers with exponential completion times, we analyze the impact of both a delayed payment regime and a no delayed payment regime on each supplier's effort level and on the manufacturer's net profit in equilibrium. When the suppliers work rates are unadjustable, we conjecture that the manufacturer is actually worse off under the delayed payment regime. However, when the suppliers work rates are adjustable, we obtain a different result: the delayed payment regime is more profitable for the manufacturer either when the project revenue is sufficiently small or when the number of suppliers is sufficiently large.

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[†]Kwon, Department of Business Administration, University of Illinois at Urbana-Champaign, 1206 South Sixth St., Champaign, IL 61820; Email: dhkwon@illinois.edu. Lippman, McCardle and Tang: UCLA Anderson School, UCLA, 110 Westwood Plaza, Los Angeles, CA 90095; Email: slippman@anderson.ucla.edu, kmccardl@anderson.ucla.edu, ctang@anderson.ucla.edu. The authors are grateful to the former editor Gerard Cachon, the senior editor, three anonymous reviewers, as well as seminar participants at the INFORMS meeting in Washington DC, Duke University, Hong Kong University of Science and Technology, London Business School, New York University, Purdue University, University of California at San Diego, University of Chicago, Singapore Management University, and Washington University in St. Louis and for their constructive comments.

1 Introduction

The growing importance of effective project management has led to the development of many project management tools since the 1950s such as Critical Path Method (CPM), Project Evaluation and Review Techniques (PERT), and cost-time tradeoff analysis (Klastorin (2004)). These tools are effective when there is little uncertainty in project completion times and/or operating costs. However, relatively little is known about ways to manage projects with considerable uncertainty such as those arising in construction, defense, and new product development. Although we have witnessed an increased research interest in examining supply contracts under uncertainty (Cachon (2003)), little research has been done in the area of project management contracts under uncertainty.

Consider a manufacturer who manages a project consisting of $n \geq 2$ separate and independent tasks that can be performed by different suppliers in parallel. The manufacturer's contract with each supplier specifies both the payment to the supplier and the payment terms. In practice, we observe two different payment regimes: no delayed payment and delayed payment. Under the conventional or no delayed payment regime, a supplier receives her payment immediately after she has completed her task. Under the delayed payment regime, however, each supplier receives her payment only when *all* suppliers have completed their tasks. We offer three examples to illustrate the existence of both payment regimes in practice. First, consider a translation agency that offers one-stop written translation services to customers who need to translate customer-specific materials such as employee handbooks, safety manuals, and web site content from a source language (e.g., English) to multiple target languages (e.g., Spanish and Italian). Typically, the agency receives full payment from the customer upon the completion of the entire translation project. Most agencies outsource the translation work associated with each target language to an external translator. According to our discussion with the managing director of Inline Translations Services (www.inlinela.com) based in Los Angeles, both payment regimes are common in practice. Second, consider a home warranty company that offers comprehensive home repair services to home owners. Upon receiving a repair service request from a customer, the company outsources the actual repair tasks to different independent contractors who specialize in different types of repair services (e.g., electrical, plumbing, flooring). For example, when one of the authors requested a home warranty company to repair his kitchen after an accidental flood, the home repair company managed his request by coordinating different repair tasks performed by a plumber, an electrician, a carpenter, and a carpet installer. According to a manager of First American Home Buyers

Protection Corporation (www.homewarranty.firstam.com), both payment regimes are common in practice. Third, when Boeing developed its 737 and 747 aircrafts, Boeing offered the no delayed payment regime to its suppliers. When developing the 787 aircraft, however, Boeing imposed the delayed payment regime (also known as the "risk-sharing" contract) upon its strategic suppliers. As reported in Greising and Johnsson (2007), the risk-sharing contracts stipulate that these strategic suppliers will not receive payments from Boeing to recoup their development costs until the *first* 787 plane is developed, certified, and delivered to Boeing's first customer (Japan's All Nippon Airways).

Even though both payment regimes exist in practice, we are unaware of any formal study regarding the rationale behind each payment regime. Based on our discussion with two translation agencies, various translators, and two major Boeing suppliers who request anonymity, we learned of the following issues. First, all suppliers believe that the no delayed payment regime is fair because the timing of each supplier's payment depends only upon her own performance (completion time). Because the timing of the payment to each supplier depends on the completion times of all suppliers under the delayed payment regime, there is a consistent perception among suppliers that the delayed payment regime penalizes those suppliers who finish early. Consequently, some suppliers are incentivized to work slower under the delayed payment regime. Second, because each supplier is paid when she completes her own task, the no delayed payment regime can create potential cash flow problems for the manufacturer, especially when the last supplier completes her task very late. As a way to reduce the manufacturer's financial risks, some manufacturers believe that the delayed payment regime provides an incentive for the suppliers to coordinate their tasks better so as to complete the entire project earlier. In summary, all suppliers prefer to receive their payments earlier, while the manufacturer prefers to issue his payments later. This sentiment suggests that both suppliers and manufacturers discount the value of future payments, either through mental calculations or actual financial discounting. Accordingly, we assume that there exists an "imputed" continuous time discount rate in our model. Also, we consider the case when the manufacturer and the suppliers are interested in maximizing their own expected discounted profit.

As an initial attempt to analyze these two payment regimes in the context of project contracts with uncertain completion times, we consider the case in which one manufacturer engages $n \geq 2$ identical suppliers in the project. By considering an abstraction of the aforementioned industry examples, we propose a stylized model to capture the salient features of the two regimes in order to gain intuition as to which regime yields a shorter project completion time and which regime imparts the larger manufacturer's profit. Although we compare the manufacturer's profits associated with two payment regimes that are simple and common in practice, there may be other payment regimes that dominate these two regimes. As such, our intent is to develop a basic model which can be used as a building block to examine more general settings. We discuss other payment regimes and future research in Section 5.

Our model consists of one risk-neutral manufacturer and $n \geq 2$ identical risk-neutral suppliers. The manufacturer will receive a total revenue of nq from his customer upon delivering the product or service which occurs when all suppliers complete their tasks. Henceforth, we refer to q (and not nq) as the manufacturer's revenue. The manufacturer acts as the leader in a Stackelberg game by selecting not only the payment p to be paid to each of the suppliers but also the payment regime (either the no delayed payment regime N or the delayed payment regime D). Given the payment p and the regime, each supplier acts as a follower by selecting her optimal work rate. The completion time of each task is uncertain. Because each supplier receives her payment only after all suppliers have completed their tasks under the delayed payment regime D, each supplier needs to take the other suppliers' work rates into consideration when selecting her own work rate.

Throughout this paper, we consider the case when each supplier is informed of the progress of the other suppliers. (For example, under the Boeing's 787 development program, the completion time as well as the progress of each task are commonly observed by all suppliers (Nolan and Kotha (2005)).) In Section 3, we present our base model wherein we assume that suppliers are unable to adjust their work rates once selected at time 0. This assumption is reasonable in many practical settings.[1] In Section 4, we relax this assumption so that each supplier can adjust her work rate dynamically over time. Our analysis answers the following questions:

1. Given the payment p, what is the supplier's optimal work rate under regimes N and D?

2. Given the revenue q, which regime will yield a higher expected profit for the manufacturer?

3. What conditions render one regime more profitable for the manufacturer?

[1]Based on our private communication with a law firm that specializes in construction laws, we learned that each supplier's work rate is usually specified in the contract, which cannot be adjusted dynamically because there are other involved parties including materials suppliers and subcontractors (Kromke (2009). For example, for freeway repair projects that require lane closures, the work rate cannot be adjusted dynamically due to a lengthy approval process.

4. How would a supplier's ability to adjust her work rate affect (a) her optimal work rate, (b) the manufacturer's optimal profit, (c) the dominance of one regime over the other?

The primary contributions of this paper are two-fold. First, our paper is the first to construct a model of a project management contract with and without delayed payments with uncertain completion times. Second, we derive conditions under which one payment regime dominates the other when the work rates are unadjustable and adjustable. Specifically, when the work rates are unadjustable, we conjecture that, contrary to the naive intuition shared among practitioners, the manufacturer is actually worse off under the delayed payment regime. This conjecture is supported by numerical and partially analytical results. However, when the work rates are adjustable, we obtain two additional interesting structural results: (1) under the delayed payment regime, it is optimal for each supplier to begin with a slow work rate and then switch to a faster rate when another supplier completes her task, and (2) the delayed payment regime is more profitable for the manufacturer either when the project revenue is sufficiently small or when the number of suppliers is sufficiently large.

This paper is organized as follows. Section 2 provides a brief review of related literature. Section 3 presents the base model for the case when suppliers' work rates are unadjustable. In Section 4, we consider a different setting in which each supplier is capable of adjusting her work rate over time. The analysis is more complex because it involves the analysis of an n-stage non-cooperative game. Despite certain technical challenges, we establish analytical conditions under which one payment regime dominates the other in equilibrium. We conclude in Section 5 with a brief summary of our results, a brief discussion of two payment regimes that are slightly more general than regimes N and D, and a discussion of the limitations of our model and potential future research topics. To streamline our presentation, all proofs are given in the Online Appendix.

2 Literature Review

To our knowledge, a time-based project contract with delayed payment has not been examined previously in the project management literature. In particular, there are three features of the time-based contract analyzed in this article which differ markedly from the existing supply contract literature (Cachon (2003) and Tang (2006)). First, under the delayed payment regime, each supplier receives payment at the time when all suppliers have completed their tasks. Consequently, each supplier needs to take into account the other supplier's behavior when selecting her own work rate. It is through this interaction among suppliers that the several underlying supply contracts are, in effect, transformed into a single joint supply contract between the manufacturer and his multiple suppliers. This linking of the several suppliers is a fundamental and crucial departure from the traditional supply contract. A related interaction among suppliers has been examined by Cachon and Zhang (2007). For an exogenously given price p, they consider the case when the manufacturer allocates randomly arriving jobs to different suppliers, and they develop a queueing game to evaluate the expected lead time for different allocation policies. In their model, each supplier selects her work rate so as to optimize her expected profit by taking other supplier's behavior into consideration. Their model differs from ours in that they focus on different allocation policies whereas we concentrate on pricing policies under different payment regimes. In addition, their model is based on substitutable tasks while ours focuses on complementary tasks.

The notion of substitutable tasks (or technologies) has been examined in the economics literature. For example, Reinganum (1982) analyzes a search game among competing firms who conduct new product R&D. The underlying technologies for the new product are complements: the profit of a given firm decreases as the costs of the other firms decrease. She establishes the existence of a Nash equilibrium in which each firm searches until it finds a cost below its reservation threshold. Naturally the R&D efforts of a given firm decreases as the other firms increase their efforts. In the same vein, the R&D model in Lippman and Mamer (1993) represents the extreme in substitutability. The firms engage in R&D, and the first firm to make the decision to bring its product to market wins the entire market. Bringing a low quality product to market results in a low firm profit, which spoils the market for the other firms. These R&D models are based on substitutable tasks (or technologies) while ours focuses on complementary tasks.

Wang and Gerchak (2003) present a model that deals with complementary tasks in the context of assembly operations: a manufacturer sells a product that requires different assembly components produced by different suppliers. To produce the components, the suppliers need to construct their individual component production capacities before observing the actual order quantities to be placed by the manufacturer. In this case, the effective production capacity of the product is dictated by the minimum of the component production capacities. As a way to induce proper component capacity installation, the manufacturer offers a per unit price to each supplier for its component; however, the manufacturer delays its order-quantity until demand uncertainty is resolved. By solving a Stackelberg game in which the manufacturer acts as the leader who specifies the per unit price of each component and the suppliers act as followers who install the component production capacities, Wang and Gerchak (2003) first determine each supplier's best response. By anticipating the supplier's best response, they determine the manufacturer's optimal per unit price. Their model differs from ours in that they focus on the suppliers' production capacities while we concentrate on the suppliers' work rates under time-based contracts with different payment regimes.

The economics literature on multi-agent incentive contract theory is vast: some seminal papers include Holmstrom (1982), Demski and Sappington (1984), Mookherjee (1984), McAfee and McMillian (1991), and Itoh (1991). While our model deals with multiple suppliers (agents), our setting and our focus are different from multi-agent incentive contract theory in the following sense. First, our model is intended to compare two common payment regimes in the context of project management contracts with uncertain completion times, while the multi-agent models focus on examining the existence of Nash equilibrium and general characteristics of optimal incentive contracts (e.g., Holmstrom (1982), Mookherjee (1984), and McAfee and McMillan (1991)). Second, in our model, the manufacturer receives his revenue at the instant when all suppliers have completed their tasks so the manufacturer's expected profit is a non-separable function of the suppliers' outputs (i.e., the completion times of different tasks). In most multi-agent models, the manufacturer's (principal's) expected profit is a separable function of the suppliers' outputs (e.g., Itoh (1991)). Third, in our model, the completion time of each task is a continuous random variable, while in most economic models, the outcome of each task takes on discrete values (e.g., Demski and Sappington (1984) and Itoh (1991)).

3 Base Model: Unadjustable Work Rates

The manufacturer will receive a total revenue nq from a customer when the project is complete. (To focus our analysis on the interaction between the manufacturer and n suppliers and to obtain tractable results, we assume that the revenue nq is given exogenously. Without this simplifying assumption, one needs to analyze a 3-level Stackelberg game with $n+2$ players, which is beyond the scope of this paper.) The project consists of $n \geq 2$ parallel tasks, each of which is to be performed by a distinct external supplier. Throughout this paper, we assume the tasks are of equal difficulty and the suppliers have equal capability so that the manufacturer will offer an identical payment p to all suppliers. (In many instances, the assumption of identical suppliers is reasonable and innocuous. For example, in translation services, the price for translating a document into Spanish or Italian is usually the same because the difficulty of translation is quite similar. While an approach similar to ours can be used to analyze the case of non-identical suppliers, the analysis is highly complex due to asymmetric equilibria and is beyond the scope of this paper.) In addition to the payment p, the manufacturer specifies the payment regime N or D. Under the no delayed payment regime N, each supplier is paid immediately after she completes her own task. Under D, each supplier is paid when all n suppliers have completed their tasks.

We assume that the completion time X_i of development task i is exponentially distributed with parameter r_i, where the work rate $r_i > 0$ is selected by supplier i at time 0, $i = 1, \cdots, n$. The exponential completion time assumption is commonly assumed in the project management literature (e.g., Adler et al. (1995), Maggott and Skudlarski (1993), and Pennings and Lint (1997)). Besides the empirical evidence for exponential completion times cited in the project management literature (Choen et al. (2004)), Dean et al. (1969) argue that an exponential completion time is more realistic in the context of project management than the Normally distributed completion times that are commonly assumed (e.g., Bayiz and Corbett (2005)).

In the base model, we assume that, due to practical reasons stated earlier, the supplier is unable to adjust her work rate r_i once selected at time 0. (We shall relax this assumption in Section 4.) Therefore, the project completion time T satisfies: $T = \max\{X_i : i = 1, \cdots, n\}$. To capture the sentiment that all suppliers prefer to receive their payments earlier and the manufacturer prefers to issue his payments later, let $\alpha > 0$ be the "imputed" continuous time discount rate. The expected discount factor associated with the project completion time $T = \max\{X_i : i = 1, \cdots, n\}$ (or the time for the suppliers to receive their payments under regime D) is denoted by $\beta_n(r_1, \cdots, r_n) = E(e^{-\alpha T})$. Because the distribution of X_i is $F_i(t) = 1 - e^{-r_i t}$, the distribution of T is $F(t) \equiv \prod_{i=1}^n F_i(t)$. Hence, the discount factor $\beta_n(r_1, \cdots, r_n) = E[e^{-\alpha T}] = \int_0^\infty e^{-\alpha t} d(F(t)) = \alpha \cdot \int_0^\infty e^{-\alpha t} F(t) dt$, where the last equality is obtained via integration by parts. Similarly, the expected discount factor associated with the completion time of task i (or the time for supplier i to receive her payment under regime N) is denoted by $\beta(r_i)$: $\beta(r_i) = E(e^{-\alpha X_i}) = \int_0^\infty r_i e^{-(r_i+\alpha)t} dt = \frac{r_i}{r_i+\alpha}$. Our analysis utilizes the following properties of $\beta_n(r_1, \cdots, r_n)$.

Lemma 1 *For any positive integer n, the expected discount factor* $\beta_n(r_1, \cdots, r_n)$ *satisfies:*

1. $\beta_n(r_1, \cdots, r_n) = E(e^{-\alpha T}) \le E(e^{-\alpha X_i}) = \beta(r_i)$ *for* $i = 1, \cdots, n$.

2. $\beta_n(r_1, \cdots, r_n)$ *is increasing and strictly concave in* r_i *for* $i = 1, \cdots, n$.

3. *$\beta_n(r_1,\cdots,r_n)$ is a submodular function of $(r_1,\cdots,r_n)$: $\frac{\partial^2\beta_n(r_1,\cdots,r_n)}{\partial r_j \partial r_i} > 0$ for $i \neq j$.*

4. *When $r_i = r\,\forall i$, $\beta_n(r_1,\cdots,r_n) = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{\alpha}{\alpha+jr}$. By letting $e^{-rt} = x$, we can express $\beta_n(r_1,\cdots,r_n) = \frac{\alpha}{r}\cdot\int_0^1 x^{(\alpha-r)/r}(1-x)^n dx = \frac{\alpha}{r}\cdot \mathrm{B}(\frac{\alpha}{r}, n+1) = \prod_{j=1}^{n}\frac{jr}{jr+\alpha}$, where $\mathrm{B}(.,.)$ is the Beta function (Chap. 6 of Abramowitz and Stegun, 1965).*

5. *When $r_i = r$ for all i, $\beta_n(r_1,\cdots,r_n)$ is decreasing in n and increasing in r.*

Because each supplier gets paid only when all suppliers have completed their tasks under regime *D*, statement 1 asserts that each supplier's payment is discounted more heavily under regime *D*. Statement 2 asserts that each supplier can reduce this "discounting penalty" under regime *D* by working faster, and statement 5 asserts that each supplier's payment is discounted more heavily under regime *D* as the number of suppliers *n* increases.

The supplier's operating cost $\kappa(r)$ per unit time associated with work rate r is a convex increasing function. To simplify our analysis, we assume that $\kappa(r) = kr^2$ with $k > 0$. Hence, supplier i's expected discounted total operating cost equals $E[\int_0^{X_i}\kappa(r_i)\cdot e^{-\alpha t}dt] = \int_0^{\infty}[\int_0^{x}\kappa(r_i)\cdot e^{-\alpha t}dt]\, r_i e^{-r_i x}dx = kr_i^2/(r_i+\alpha)$.

3.1 Profit functions and Participation Constraints

We now determine the supplier's and the manufacturer's expected discounted profit and their willingness to participate in the project under each regime. Under regime *N*, supplier *i* gets paid immediately when she completes her task. Given the manufacturer's payment *p*, supplier *i*'s expected discounted profit $\Pi_i^N(p;r_i)$ under regime *N* satisfies

$$\Pi_i^N(p;r_i) \quad = \quad p\cdot\beta(r_i) - \frac{kr_i^2}{r_i+\alpha}, \;\text{ for } i = 1,\cdots,n. \tag{3.1}$$

We assume that each supplier will participate in the project under regime *N* if the supplier's expected discounted profit exceeds a minimum target. We set this target to 0 so as to simplify our exposition. Hence, supplier *i*'s participation constraint is given by

$$\max_{r_i}\ \Pi_i^N(p;r_i) \geq 0. \tag{3.2}$$

Under regime *D*, supplier *i* gets paid when all suppliers have completed their tasks. Given *p*, the supplier's expected discounted profit $\Pi_i^D(p;r_1,\cdots,r_n)$ under regime *D* satisfies

$$\Pi_i^D(p;r_1,\cdots,r_n) \quad = \quad p\cdot\beta_n(r_1,\cdots,r_n) - \frac{kr_i^2}{r_i+\alpha}, \;\text{ for } i = 1,\cdots,n. \tag{3.3}$$

Hence, given the other suppliers' work rates, supplier i will participate in regime D if

$$\max_{r_i} \; \Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) \geq 0. \tag{3.4}$$

For any given payment p that ensures supplier participation, the manufacturer's expected discounted profit under regime N satisfies

$$\Pi_m^N(p;q) \quad = \quad nq \cdot \beta_n(r_1, \cdots, r_n) - p \sum_{i=1}^{n} \beta(r_i). \tag{3.5}$$

Because the manufacturer can elect not to participate in the project (by setting $p = 0$ and earning no profit), the manufacturer will participate in regime N if

$$\Pi_m^N(q) \geq 0, \;\; \text{where } \Pi_m^N(q) = \max_p \; \Pi_m^N(p;q), \; \text{subject to } (3.2) \;\; \forall \, i. \tag{3.6}$$

Similarly, the manufacturer's expected discounted profit under regimes D satisfies

$$\Pi_m^D(p;q) \quad = \quad n(q-p) \cdot \beta_n(r_1, \cdots, r_n), \tag{3.7}$$

and the manufacturer will participate in regime D if

$$\Pi_m^D(q) \geq 0, \;\; \text{where } \Pi_m^D(q) \equiv \max_p \; \Pi_m^D(p;q), \; \text{subject to } (3.4) \;\; \forall \, i. \tag{3.8}$$

Let us now compare the supplier's and the manufacturer's profit functions for the case when all parties participate under both regimes. For any given work rates r_i, we can use $\beta_n(r_1, \cdots, r_n) \leq \beta(r_i)$ given in Lemma 1 to show that $\Pi_i^N(p; r_i) > \Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n)$ for all i and that $\Pi_m^N(p;q) < \Pi_m^D(p;q)$. These observations confirm a basic intuition: when the price p and the work rates are the same under both regimes, supplier i prefers regime N while the manufacturer prefers regime D. However, when the manufacturer offers different prices and when the suppliers select different work rates under different regimes, it is unclear which regime will yield a higher expected profit for the manufacturer. To address this question, we analyze a Stackelberg game in which the manufacturer has the first move and the n suppliers simultaneously move second. The manufacturer starts by selecting the regime (either N or D) and the payment p. As in a backward recursion, each supplier determines her work rate r_i given the regime and p. Anticipating each supplier's work rate r_i, the manufacturer selects the payment p^* that maximizes his expected profit. The manufacturer selects the regime that yields the higher expected profit, and he informs the suppliers of the regime and the price p^*; in response, the suppliers select their optimal work rates.

Lastly, there is a property of the profit functions which is immensely useful for numerical analysis. It is straightforward to verify that there exist parameter-free functions $\tilde{\beta}_n(\cdot)$, $\tilde{\Pi}_i^N(\cdot)$, and $\tilde{\Pi}_m^N(\cdot)$ whose functional forms do not contain any model parameters such as α and k and which satisfy $\beta_n(r_1,...,r_n) = \tilde{\beta}_n(\frac{r_1}{\alpha},...,\frac{r_n}{\alpha})$, $\Pi_i^N(p;r_i) = k\alpha\tilde{\Pi}_i^N(\frac{p}{k\alpha};\frac{r_i}{\alpha})$, $\Pi_m^N(p;q) = k\alpha\tilde{\Pi}_m^N(\frac{p}{k\alpha};\frac{q}{k\alpha})$, and $\Pi_m^N(q) = k\alpha\tilde{\Pi}_m^N(\frac{q}{k\alpha})$. (The explicit functional forms of $\tilde{\beta}_n(\cdot)$, $\tilde{\Pi}_i^N(\cdot)$, and $\tilde{\Pi}_m^N(\cdot)$ appear in the Appendix.) Similarly, we also have $\Pi_i^D(p;r_1,...,r_n) = k\alpha\tilde{\Pi}_i^D(\frac{p}{k\alpha};\frac{r_1}{\alpha},...,\frac{r_n}{\alpha})$, $\Pi_m^D(p;q) = k\alpha\tilde{\Pi}_m^D(\frac{p}{k\alpha};\frac{q}{k\alpha})$, and $\Pi_m^D(q) = k\alpha\tilde{\Pi}_m^D(\frac{q}{k\alpha})$ for some parameter-free functions $\tilde{\Pi}_i^D(\cdot)$ and $\tilde{\Pi}_m^D(\cdot)$. This is a convenient property when we numerically compare the relative magnitudes of $\Pi_m^N(q)$ and $\Pi_m^D(q)$; For each n, the ratio of $\Pi_m^N(q)$ to $\Pi_m^D(q)$ is a function only of $q/\alpha k$, so we only need to vary a single variable $q/\alpha k$ to study $\Pi_m^N(q)/\Pi_m^D(q) = \tilde{\Pi}_m^N(\frac{q}{k\alpha})/\tilde{\Pi}_m^D(\frac{q}{k\alpha})$, and we do not need to vary q, α, and k independently.

3.2 N: The No Delayed Payment Regime

We begin by determining the supplier's optimal work rate and expected discounted profit.

Proposition 1 *Under regime N, supplier i's profit function $\Pi_i^N(p;r)$ given in (3.1) is concave in r, and $r_i^N(p)$, supplier i's optimal work rate, is given by*

$$r_i^N(p) \;=\; r^N(p) \;=\; \alpha\left(\sqrt{1+\frac{p}{\alpha k}}-1\right). \tag{3.9}$$

Supplier i's optimal expected profit $\Pi_i^N(p) \equiv \Pi_i^N(p;r^N(p))$ is given by

$$\Pi_i^N(p) \;=\; \frac{k}{\alpha}\cdot[r^N(p)]^2 = k\alpha\left(\sqrt{1+\frac{p}{\alpha k}}-1\right)^2. \tag{3.10}$$

Observe from (3.10) and (3.9) that each supplier i's participation constraint (3.2) is satisfied if and only if $p>0$.

Using the optimal work rate $r_i^N(p)$ given in (3.9), it is easy to compute the expected project completion time $E(T^N(p)) = E(\max\{X_i : i = 1,\cdots,n\})$, where X_i is exponentially distributed with parameter $r_i^N(p)$.

Corollary 1 *The expected project completion time $E(T^N(p))$ under regime N satisfies*

$$E(T^N(p)) = \frac{1}{r^N(p)}[\psi(n+1)-\psi(1)], \tag{3.11}$$

where $\psi(x)$ is the Digamma function.[2] *Also, $E(T^N(p))$ is increasing in n and decreasing in p.*

[2] The Digamma function $\psi(x)$ is the derivative of the logarithm of the Gamma function: $\psi(x) \equiv \frac{d}{dx}ln(\Gamma(x)) = \int_0^\infty(\frac{e^{-t}}{t} - \frac{e^{-xt}}{(1-e^{-t})})dt$. When n is a positive integer, $\psi(n+1)-\psi(1) = \sum_{k=1}^n k^{-1}$ (see Chap. 6 of Abramowitz and Stegun (1965)).

Corollary 1 confirms that, as the number of suppliers n increases, the expected completion time increases. Also, the expected completion time can be shortened if the manufacturer offers a larger payment p (because $r_i^N(p)$ given in (3.9) is increasing in p).

While Proposition 1 reveals that all suppliers will participate in the project under regime N when $p > 0$, the manufacturer will not participate if the revenue q is below a certain threshold. This is because, under regime N, the manufacturer has to pay each supplier when she completes her own task, but he must wait to receive his revenue until all suppliers have completed their tasks. This "time delay" can cause the manufacturer to suffer a loss when the revenue q is below a threshold q_n. By noting that the "time delay" becomes more severe as the number of suppliers n increases, we obtain the following result:

Lemma 2 *(Conditions for Participation under Regime N) Under regime N, each supplier will participate in the project if and only if $p > 0$. Also, there exists a unique threshold $q_n > 0$ such that the manufacturer will not participate if the revenue $q \leq q_n$, where q_n is increasing in n. Moreover, when n is sufficiently large, $q_n = \frac{k\alpha}{4}(\ln n)^2 + O(\ln n)$.*

Because q_n is increasing in n, Lemma 2 asserts that, for any fixed revenue q, the manufacturer will not participate when the number of suppliers $n > \tau_n^N$, where $\tau_n^N \equiv argmin_{n>0}\,\{q_n > q\}$.

3.3 D : The Delayed Payment Regime

Under regime D, each supplier receives her payment when all suppliers have completed their tasks: each supplier's expected discounted profit depends on all suppliers' work rates. We now show that there exists a symmetric Nash equilibrium.

Lemma 3 *Given $(r_1, ..r_{i-1}, r_{i+1}, ..., r_n)$, supplier i's expected discounted profit $\Pi_i^D(p; r_1, \cdots, r_n)$ given in (3.3) is concave in r_i. Also, supplier i's best response r_i^* (i.e., the value of r_i^* that maximizes $\Pi_i^D(p; r_1, \cdots, r_n)$) is increasing in r_j for $j \neq i$.*

Proposition 2 *There are no asymmetric Nash equilibria. There is a threshold $p_n > 0$ such that if $p > p_n$, then there are multiple symmetric Nash equilibria in which all suppliers work at the same rate r, where r satisfies*

$$\frac{p\alpha}{r^2}\mathrm{B}(\frac{\alpha+r}{r}, n)[\psi(\frac{\alpha+r}{r}+n) - \psi(\frac{\alpha+r}{r})] = \frac{(2k\alpha r + kr^2)}{(\alpha+r)^2}, \tag{3.12}$$

where $B(.,.)$ and $\psi(.)$ are the Beta function and the Digamma function, respectively. Among all possible equilibria, the Nash equilibrium with the largest work rate $r^D(n;p)$ has the following properties: both $r^D(n;p) > 0$ and its corresponding expected discounted profit for the supplier $\Pi_i^D(n;p) > 0$ are decreasing in n.

To ease our exposition, we defer our discussion of the threshold p_n and supplier participation till Lemma 4 below. Proposition 2 has three implications. First, observe from the last statement that the largest work rate in equilibrium $r^D(n;p)$ satisfies $r^D(n;p) < \cdots < r^D(2;p) < r^D(1;p) = r^N(p)$. This implies that, due to the delay in receipt of payment and the "gaming effect" among suppliers, the supplier's optimal work rate under regime D is lower than the optimal work rate under regime N: $r^D(n;p) < r^N(p)$. This result is intuitive because, under regime D, each supplier is effectively penalized for completing her task before other suppliers. Second, using the proof of Corollary 1, it is clear that

$$E(T^D(p)) = \frac{1}{r^D(n;p)}[\psi(n+1) - \psi(1)]. \tag{3.13}$$

Because $r^D(n;p) < r^N(p)$, (3.11) and (3.13) reveal that $E(T^D(p)) > E(T^N(p))$: the expected project completion time is longer under regime D. This result is expected because the supplier's optimal work rate under regime D is lower than the optimal work rate under regime N. Third, because $r^D(n;p) < r^N(p)$, the lower work rate $r^D(n;p)$ reduces supplier i's discounted operating cost $\frac{kr_i^2}{r_i+\alpha}$ as well as her discounted payment $p \cdot \beta_n(r_1, \cdots, r_n)$. It is not clear if supplier i's expected profit $\Pi_i^D(n,p) \equiv \Pi_i^D(p; r^D(n,p), \cdots, r^D(n,p))$ is lower under regime D. However, by combining the fact that $\Pi_i^D(n,p) = \Pi_i^N(p)$ when $n = 1$ (because $r^D(1;p) = r^N(p)$) with $\Pi_i^D(n;p)$ decreasing in n, we can conclude that $\Pi_i^D(n;p) < \cdots < \Pi_i^D(1;p) = \Pi_i^N(p)$. Therefore, given p, the supplier's profit under regime D is indeed lower than under regime N.

There is no closed form expression for the equilibrium work rate $r^D(n;p)$ that solves (3.12) for $n > 2$; however, we obtain a closed form expression when $n = 2$. When $n = 2$, (3.12) reduces to: $[\frac{p\alpha}{(\alpha+r)^2} - \frac{p\alpha}{(\alpha+2r)^2}] - \frac{(2k\alpha r + kr^2)}{(\alpha+r)^2} = -\frac{r \cdot k \cdot h(r)}{(r+\alpha)^2(2r+\alpha)^2} = 0$, where $h(r) = 4r^3 + 12\alpha r^2 + 9\alpha^2 r - 3\frac{p}{k}\alpha r + 2\alpha^3 - 2\frac{p}{k}\alpha^2$. By examining the cubic equation $h(r) = 0$, we get:

Corollary 2 *When $n = 2$, $r^D(2;p) = 0$ if $p \leq p_2$, where $p_2 = k\alpha$. If $p > p_2$, then $r^D(2;p) = 0$ is an equilibrium, and the only Nash equilibrium with $r^D(2;p) > 0$ satisfies*

$$r^D(2;p) = \alpha\left[\sqrt{1+\frac{p}{k\alpha}}\cos(\phi/3) - 1\right], \text{ where} \tag{3.14}$$

$$\phi \equiv \pi - \arctan\sqrt{\frac{p}{k\alpha}}. \tag{3.15}$$

Corollary 2 informs us that, when p exceeds p_2, there is a unique Nash equilibrium with positive work rate so that the suppliers earn positive profits. Consequently, it is Pareto optimal for the suppliers to select the equilibrium $r^D(2;p) > 0$ when $p > p_2$.

Observe from (3.7) that the manufacturer's profit $\Pi_m^D(p;q) = n(q-p)\cdot\beta_n(r_1,\cdots,r_n) = 0$ when the supplier's work rate in equilibrium $r^D(n;p)$ drops to zero. In particular, Corollary 2 reveals that, when $n = 2$, the supplier's equilibrium work rate $r^D(2;p)$ will drop to zero if $p \leq p_2$, where $p_2 = k\alpha$. Hence, in order for the manufacturer to obtain a positive profit, the revenue q needs to exceed a certain threshold so that his payment p satisfies $q > p > p_2$. Combine this observation with Proposition 2, we have:

Lemma 4 *(Conditions for Participation under Regime D) Under regime D, there exists a unique threshold $p_n > 0$ such that all suppliers will participate in the project if and only if $p > p_n$. Also, the manufacturer will participate if and only if the revenue $q > p_n$. Moreover, p_n is increasing in n, and $p_n = k\alpha n(\ln n + O(1))$.*

Unlike the case in which each supplier will participate in regime N if $p > 0$ (Lemma 2), Lemma 4 reveals that each supplier will participate in regime D if and only if $p > p_n$. Essentially, the supplier's "participation threshold" p_n under regime D captures the "imputed" penalty associated with the delayed payment under regime D. It follows from Proposition 2 that the supplier's expected profit $\Pi_i^D(n;p)$ is decreasing in n; hence, it is intuitive that the participation threshold p_n is increasing in n. In addition, there is another factor that contributes to the growth of p_n in n: as n increases, each supplier will work even slower due to the "gaming effect" among suppliers. Hence, as n increases, the delay in payment is exacerbated, which explains why the threshold p_n grows faster than n.

Given q, Lemma 4 implies that the manufacturer will not participate and will earn zero if the number of suppliers n exceeds τ_n^D, where $\tau_n^D \equiv argmin_{n>0}\,\{p_n > q\}$. By noting from Lemmas 2 and 4 that the threshold p_n grows faster than q_n when n is large, we have proved the following Corollary.

Corollary 3 *As the number of suppliers n increases, $\Pi_m^N(p;q) = 0$ when $n > \tau_n^N$ and $\Pi_m^D(p;q) = 0$ when $n > \tau_n^D$. Moreover, $\tau_n^D < \tau_n^N$.*

Intuitively speaking, the number of suppliers n has no effect on supplier participation under regime N because each supplier cares only about her own completion time. However, under regime N, the "time delay" between the manufacturer's payments and the receipt of his total revenue increases with the number of suppliers n so the manufacturer's expected profit decreases in n: he will not participate in the project under regime N if the revenue $q < q_n$. On the contrary, under regime D, the suppliers are concerned about their discounted payments, which depend on the completion times of other suppliers. As the number of suppliers n increases, the project completion time lengthens. To generate enough incentive for the suppliers to participate, each supplier demands a higher payment threshold p_n. By noting that the payment threshold p_n is exacerbated by the delay in payment and the "gaming effect" among suppliers revealed in Proposition 2 (i.e., each supplier will work slower under regime D), we make the following conjecture.

Conjecture 1: For $n > 2$, $p_n > q_n$, and p_n increases faster than q_n as n increases.

Conjecture 1 is supported by our numerical analysis in which we found that $p_n/k\alpha = q_n/k\alpha$ for $n = 2$ and and $p_n/k\alpha > q_n/k\alpha$ for $n = 3, 4, ..., 200$. Our numerical result is valid for all values of $k > 0$ and $\alpha > 0$; By the comment at the end of Section 3.1, for each n, the values of $p_n/k\alpha$ and $q_n/k\alpha$ are independent of the values of k and α. For example, when $\alpha = 1$ and $k = 1$, Figure 1 presents a plot of p_n and q_n as we vary n from 1 to 35. In the next section, we shall use Lemmas 2, 4, and Conjecture 1 to compare the manufacturer's optimal profits under regimes N and D.

Insert Figure 1 About Here.

3.4 Choosing the Payment Regime

We now compare the manufacturer's expected profits under regimes N and D. For any given p that satisfies $q > p > max\{p_n, q_n\}$ so that all parties will participate under both regimes, Proposition 2 asserts that under regime D each supplier works slower in equilibrium ($r^D(n; p) < r^N(p)$) and earns a lower profit ($\Pi_i^D(n; p) < \Pi_i^N(p)$). However, it is not clear if the manufacturer earns a higher profit under regime D for any given p. To elaborate, combine statements 5 and 1 of Lemma 1 and the fact $r^D(n; p) < r^N(p)$ to show that $\beta_n(r^D(n; p), \cdots, r^D(n; p)) < \beta_n(r^N(p), \cdots, r^N(p)) \leq \beta(r^N(p))$. Now we can compare the manufacturer's profit given in (3.7) and (3.5) to show that the profit comparison depends on two countervailing forces.

The first force is based on the fact that, for any given p, the suppliers optimal work rate in regime D is lower. Hence, $nq \cdot \beta_n(r^D(n; p), \cdots, r^D(n; p)) < nq \cdot \beta_n(r^N(p), \cdots, r^N(p))$: the manufacturer's discounted revenue is lower under regime D. The countering force stems from the fact that $np \cdot \beta_n(r^D(n; p), \cdots, r^D(n; p)) < np \cdot \beta(r^N(p)$: the manufacturer's discounted cost is also lower under regime D because the manufacturer benefits from not having to pay any of the suppliers until he receives his own revenue. Due to these two countervailing forces, it is inconclusive whether $\Pi_m^D(q;p) > \Pi_m^N(q;p)$ when the manufacturer offers the same payment p under both regimes. Because the manufacturer would offer different optimal payments under different regimes, this observation poses another technical challenge when we compare the manufacturer's optimal expected discounted profit under regimes N and D next.

There is no explicit analytical expression for the optimal payment p^N or the manufacturer's optimal expected profit $\Pi_m^N(q,p^N)$. Similarly, there is no explicit expression for the optimal payment p^D or for the manufacturer's optimal expected profit that maximizes $\Pi_m^D(q,p)$ given in (3.7). Therefore, it is challenging to compare $\Pi_m^N(q) \equiv \Pi_m^N(q;p^N)$ and $\Pi_m^D(q) \equiv \Pi_m^D(q;p^D)$. Despite this challenge, we are able to establish the following intuitive result.

Lemma 5 *For $n \geq 2$, $\Pi_m^N(q)$ and $\Pi_m^D(q)$ are convex and non-decreasing in q.*

We now compare the manufacturer's optimal profit functions $\Pi_m^N(q)$ and $\Pi_m^D(q)$ analytically for the case when q is sufficiently small and for the case when q is sufficiently large. (When q is in the intermediate range, we conduct our comparison numerically.) When the revenue q is small, we can use the participation conditions stated in Lemmas 2 and 4 to establish Proposition 3.

Proposition 3 *(Small Revenue q) If Conjecture 1 is true so that $p_n > q_n$, then regime N weakly dominates regime D: $\Pi_m^N(q) = \Pi_m^D(q) = 0$ for $q \in [0,q_n]$, and $\Pi_m^D(q) = 0 < \Pi_m^N(q)$ for $q \in (q_n, p_n]$. However, if Conjecture 1 is not true so that $p_n \leq q_n$, then regime D weakly dominates regime N: $\Pi_m^N(q) = \Pi_m^D(q) = 0$ for $q \in [0, p_n]$, and $\Pi_m^N(q) = 0 < \Pi_m^D(q)$ for $q \in (p_n, q_n]$.*

In light of Conjecture 1, we believe only statement (1) in Proposition 3 can occur when $n > 2$ in which case regime N dominates D when the revenue q is sufficiently small.

Proposition 4 *(Large Revenue q) Suppose q exceeds a unique threshold τ_l, where $\tau_l > \max\{p_n,q_n\}$. Then (1) regime N dominates regime D: $\Pi_m^N(q) > \Pi_m^D(q)$; (2) the manufacturer's optimal price is smaller under regime N: $p^N(q) < p^D(q)$; (3) the supplier's optimal work rate is larger under regime N: $r^N(p^N(q)) > r^D(p^D(q))$; and (4) the expected completion time of the project is shorter under regime N: $E(T^N(p^N(q))) < E(T^D(p^D(q)))$.*

Intuitively, the dominance of regime N can be explained as follows. When revenue q is large, the manufacturer is less concerned about his payments to the suppliers because the optimal payments p^N and p^D to each supplier under regimes N and D are small relative to q. However, despite a higher payment under regime D (statement (2)), each supplier will work at a slower rate in regime D (statement (3)) mainly due to the delay in payment and the "gaming effect" among suppliers under regime D. As the supplier's optimal work rate is larger under regime N (statement (3)), the earlier completion time (statement (4)) and the smaller payment under regime N (statement (2)) outweighed the disadvantage of having to make the payment earlier (that would have occurred under regime D). Consequently, regime N dominates regime D (statement (1)).

We now compare the manufacturer's optimal profits $\Pi_m^N(q)$ and $\Pi_m^D(q)$ as the number of suppliers n increases. By considering the supplier's and the manufacturer's non-participation along with the result stated in Corollary 3, we have:

Proposition 5 *(Large Number of Suppliers) For any given q, regime N dominates regime D for sufficiently large n. In other words, for sufficiently large n, there are thresholds τ_n^N and τ_n^D, where $\tau_n^D < \tau_n^N$ such that $\Pi_m^N(q) > \Pi_m^D(q) = 0$ for $n \in [\tau_n^D, \tau_n^N]$ and $\Pi_m^N(q) = \Pi_m^D(q) = 0$ for $n > \tau_n^N$.*

By observing from Propositions 3, 4, and 5 that regime N dominates regime D when q is sufficiently small (or sufficiently large), or when n is sufficiently large, we make the following conjecture.

Conjecture 2: When suppliers are unable to adjust their work rates dynamically, regime N always dominates regimes D.

While we are unable to prove Conjecture 2 analytically, Propositions 3, 4, and 5 together with our numerical analysis support conjecture 2: The manufacturer is actually worse off under the delayed payment regime D! In our numerical analysis, we computed the ratio $\Pi_m^N(q)/\Pi_m^D(q)$ for $2 \leq n \leq 20$ and for values of $q/k\alpha$ between 0 and 100 in the increment of 0.1, and we found that $\Pi_m^N(q)/\Pi_m^D(q) > 1$ for these values of $q/k\alpha$ and n and for every combination of these parameter values. (By the comment at the end of Section 3.1, the ratio $\Pi_m^N(q)/\Pi_m^D(q)$ is a function of $q/k\alpha$ and n only.) This result is surprising because it is contrary to the naive intuition shared among practitioners that we described in the Introduction.

4 Adjustable Work Rates

In the base model, we assumed that suppliers are unable to adjust their work rates over time. We now relax this assumption by considering a situation when each supplier is capable of adjusting her work rate at any time. Our goal is to examine whether the results and conjectures established in Section 3 continue to hold.

Specifically, based on our analysis of the delayed payment regime with adjustable work rates, we obtain two additional results: (1) it is optimal for each supplier to work at a slower rate initially and then increase her rate when another supplier completes her task, and (2) the delayed payment regime dominates the no delayed payment regime, reversing the conjecture we made under the unadjustable work rates assumption, when the revenue is sufficiently small or when the number of suppliers is sufficiently large.

4.1 NI: The No Delayed Payment Regime

Under regime NI, each supplier receives her payment p when she completes her own task. Because each supplier's expected profit is independent of the other suppliers' completion times and because of the memoryless property of her own completion time (which is exponentially distributed), it is optimal for supplier i to continue to work at her initial rate r_i selected at time 0 until she completes her task. Therefore, the capability to adjust her work rate has no effect on the supplier's behavior under the no delayed payment regime: all results reported in Section 3.2 continue to hold.

4.2 DI: The Delayed Payment Regime

At time 0, each of the n suppliers selects her work rate and begins working on her own task. Due to the memoryless property of the exponential distribution, there is no incentive for any supplier to adjust her work rate until one of the n suppliers completes her task. This observation suggests that each continuing supplier will consider adjusting her work rate only at the beginning of stage j, $j = n, (n-1), \cdots, 1, 0$, where stage n begins at time 0 with n continuing suppliers, stage $(n-1)$ begins at the instant when one of the n suppliers completes her task so that there are $(n-1)$ continuing suppliers, and so forth. Because the work rate decision is made only at the beginning of each of the n stages, we formulate the supplier's problem as an n-stage game. Specifically, at the beginning of stage j ($j = n, (n-1), \cdots, 1$), we analyze a non-cooperative game among j continuing suppliers. Due to the dynamic nature of the n-stage game and the delayed payment regime, each of the j continuing suppliers needs to take the other continuing suppliers' work rates at stage j and future stages (i.e., stages $(j-1), \cdots, 1$) into consideration when determining her work rate at stage j. Akin to the backward induction approach for solving a dynamic programming problem, we now solve this n-stage game backward in time: solve stage 1 first and solve stage n last.

4.2.1 Analysis of the supplier's n-stage game

At the beginning of stage 1, there is only 1 continuing supplier i who needs to determine her work rate λ, and there are $(n-1)$ "idle" suppliers who have completed their tasks earlier. For any work rate λ, supplier i's optimal expected profit discounted back to the beginning of stage 1 can be expressed as $R_i^{(1)}$, where $R_i^{(1)} \equiv \max_\lambda R_i^{(1)}(\lambda) = \max_\lambda [-\frac{k\lambda^2}{\lambda+\alpha} + p \cdot \frac{\lambda}{\lambda+\alpha}]$. (We use the superscript (j) to denote stage j, where $j = n, (n-1), \cdots, 1$.) It follows from the fact that the objective function is identical to (3.1), the optimal work rate for supplier i at stage 1 is $\lambda^{(1)} = \alpha(\sqrt{1+\frac{p}{\alpha k}} - 1)$, which equals $r^N(p)$ given in (3.9). By substituting $\lambda^{(1)}$ into $R_i^{(1)}(\lambda)$, it is easy to check that $R_i^{(1)} = \frac{k}{\alpha} \cdot [\lambda^{(1)}]^2 > 0$. Because the operating costs of those $(n-1)$ idle suppliers have already been incurred prior to the beginning of stage 1, the expected payment discounted back to the beginning of stage 1 for each of the $(n-1)$ idle suppliers, say, supplier i' , can be expressed as $S_{i'}^{(1)}$, where $S_{i'}^{(1)} = \frac{\lambda^{(1)}}{\lambda^{(1)}+\alpha} \cdot p$. This completes our analysis of stage 1.

At the beginning of stage 2, there are 2 continuing suppliers i and i' who need to decide on their work rates for stage 2, and there are . $(n-2)$ idle suppliers. Suppose supplier i works at rate λ and supplier i' works at rate μ throughout stage 2. Then the duration of stage 2 is $\tau^{(2)} = \min\{X_i, X_{i'}\}$, where X_i and $X_{i'}$ are exponentially distributed with parameters λ and μ, respectively. Hence, the probablity that supplier i finishes before supplier i' satisfies: $Prob\{X_i < X_{i'}\} = \frac{\lambda}{\lambda+\mu}$. Also, $Prob\{X_{i'} < X_i\} = \frac{\mu}{\lambda+\mu}$. By using these probabilities, the expected profit of supplier i discounted back to the beginning of stage 2 (for any given work rate μ of the other supplier i') satisfies:

$$\begin{aligned} R_i^{(2)}(\lambda,\mu) &= [-E[\int_0^{\tau^{(2)}} k\lambda^2 \cdot e^{-\alpha t} dt] + E(e^{-\alpha\cdot\tau^{(2)}}) \cdot \frac{\lambda}{\lambda+\mu} \cdot S_i^{(1)} + E(e^{-\alpha\cdot\tau^{(2)}}) \cdot \frac{\mu}{\lambda+\mu} \cdot R_i^{(1)}] \\ &= [-\frac{k\lambda^2}{\lambda+\mu+\alpha} + \frac{\lambda}{\lambda+\mu+\alpha} S_i^{(1)} + \frac{\mu}{\lambda+\mu+\alpha} R_i^{(1)}]. \end{aligned} \tag{4.1}$$

By examining the expected profit functions $R_i^{(2)}(\lambda,\mu)$ and $R_{i'}^{(2)}(\lambda,\mu)$ (omitted) associated with both continuing suppliers i and i', we have:

Proposition 6 *At stage 2, there exists a unique equilibrium in which both continuing suppliers work at the same rate $\lambda^{(2)}$, where*

$$\lambda^{(2)} = \frac{[(S_i^{(1)} - R_i^{(1)}) - 2k\alpha] + \sqrt{[(S_i^{(1)} - R_i^{(1)}) - 2k\alpha]^2 + 12k\alpha S_i^{(1)}}}{6k}. \tag{4.2}$$

Also, $0 < \lambda^{(2)} < \lambda^{(1)} = r^N(p)$ *and* $0 < R_i^{(2)} < R_i^{(1)}$.

Proposition 6 states that both suppliers will work at rate $\lambda^{(2)} > 0$ in equilibrium. Then, as soon as one of the suppliers completes her task, it is optimal for the remaining supplier to expedite her task by increasing her work rate from $\lambda^{(2)}$ to $\lambda^{(1)} = r^N(p)$. Also, by substituting $\lambda = \mu = \lambda^{(2)}$ into (4.1), we obtain $R_i^{(2)} \equiv R_i^{(2)}(\lambda^{(2)}, \lambda^{(2)})$. Similarly, it is easy to check that $S_i^{(2)} = E(e^{-\alpha\cdot\tau^{(2)}}) \cdot S_i^{(1)} = \frac{2\lambda^{(2)}}{2\lambda^{(2)}+\alpha} S_i^{(1)}$. This completes the analysis of the game associated with stage 2.

Using the same approach, we can solve the supplier's game at any stage j, getting:

Proposition 7 *Under regime DI, there exists a unique Nash equilibrium at stage j, where $j = 1, \cdots, n$. It entails each continuing supplier working at rate $\lambda^{(j)} > 0$, where*

$$\lambda^{(j)} = \frac{[(j-1)(S_i^{(j-1)} - R_i^{(j-1)}) - 2k\alpha] + \sqrt{[(j-1)(S_i^{(j-1)} - R_i^{(j-1)}) - 2k\alpha]^2 + 4(2j-1)k\alpha S_i^{(j-1)}}}{2(2j-1)k} \quad (4.3)$$

By observing from (4.3) and (4.2) that $S_i^{(j-1)}$ and $R_i^{(j-1)}$ are functions of $\lambda^{(j-1)}$, Proposition 7 exhibits that we can compute $\lambda^{(j)}$, $R_i^{(j)}$ and $S_i^{(j)}$ in a recursive manner. This completes our analysis of the n-stage game.

4.2.2 Profit functions under regime *DI*

For any given payment p, we computed $\Pi_i^{DI}(n;p)$, supplier i's expected discounted profit at time 0, and we found that

$$\Pi_i^{DI}(n;p) \quad = \quad R_i^{(n)}, \text{ for } i = 1, \cdots, n. \quad (4.4)$$

We now determine the expected project completion time. To do so, let $\tau^{(j)}$ denote the duration of stage j. Note that all j continuing suppliers work at rate $\lambda^{(j)}$ at stage j, $\tau^{(j)} = \min\{X_1, X_2, \cdots, X_j\}$. Because X_i, $i = 1, \cdots, j$, are independent and exponentially distributed random variables with parameter $\lambda^{(j)}$, $E(\tau^{(j)}) = \frac{1}{j\lambda^{(j)}}$. Noting that the project completion time is equal to the sum of the duration of all n stages, we have

$$E(T^{DI}(p)) \quad = \quad \sum_{j=1}^{n} E(\tau^{(j)}) = \sum_{j=1}^{n} \frac{1}{j\lambda^{(j)}}. \quad (4.5)$$

Also, it is easy to check that

$$E(e^{-\alpha\cdot T^{DI}(p)}) \quad = \quad \prod_{j=1}^{n} \frac{j\cdot\lambda^{(j)}}{j\cdot\lambda^{(j)} + \alpha}. \quad (4.6)$$

Finally, the manufacturer's expected discounted profit in equilibrium under regime *DI* satisfies

$$\Pi_m^{DI}(q,p) = n(q-p)\cdot E(e^{-\alpha\cdot T^{DI}(p)}) = n(q-p)\cdot \prod_{j=1}^{n} \frac{j\cdot\lambda^{(j)}}{j\cdot\lambda^{(j)} + \alpha}. \quad (4.7)$$

Letting p^{DI} denote the manufacturer's optimal payment, his expected discounted profit under regime *DI* is given by $\Pi_m^{DI}(q) \equiv \max_{p>0} \Pi_m^{DI}(q,p) = \Pi_m^{DI}(q,p^{DI})$.

Using mathematical induction, one can check from (4.3) that the supplier's work rate $\lambda^{(j)}$ is strictly positive for $j = 1,\cdots,n$ if $p > 0$. This is because, under regime *DI*, each supplier can adjust her rates so she can start slow and work faster later so as to lower her discounted total cost and to generate a positive profit (i.e., $\Pi_i^{DI}(n;p)$ given in (4.4) is strictly positive) provided $p > 0$. Combining this observation with the fact that the manufacturer's expected profit $\Pi_m^{DI}(q,p)$ given in (4.7) is strictly positive if and only if $q > p > 0$, we have:

Lemma 6 *(Conditions for Participation under Regime DI) Under regime DI, all suppliers and the manufacturer will participate if and only if $q > p > 0$.*

Observe from Lemma 6 that the conditions for participation are less stringent than the conditions for participation stated in Lemmas 2 and 4. This suggests that the manufacturer can obtain a higher expected profit under regime *DI* than under regimes *N* and *D*. We investigate this matter next.

4.3 Choosing the Payment Regime

Due to the recursive formula for the supplier's optimal work rate $\lambda^{(j)}$ given in (4.3), there is no explicit expression for the optimal payment p^{DI} that maximizes the manufacturer's profit function $\Pi_m^{DI}(q,p)$ given in (4.7) or for the manufacturer's optimal discounted profit $\Pi_m^{DI}(q)$ under regime *DI*. Nevertheless, we are able to use the approach presented in Section 3.4 and the participation conditions stated in Lemmas 6 and 2 to compare $\Pi_m^{DI}(q)$ and $\Pi_m^{N}(q)$ analytically when q is either small or large.

Proposition 8 *(Small and Large Revenue q) (1) When q is sufficiently small, regime DI dominates regime N: $\Pi_m^{DI}(q) > \Pi_m^{N}(q)$. (2) When q is sufficiently large, regime N dominates regime DI: $\Pi_m^{N}(q) > \Pi_m^{DI}(q)$.*

Proposition 8 is similar to Proposition 4: regime *N* dominates when q is sufficiently large. However, when q is sufficiently small, Proposition 8 exhibits the opposite result reported in Proposition 3. Specifically, if Conjecture 1 is true, then we can combine the results stated in Propositions 8 and 3 to show that $\Pi_m^{DI}(q) > \Pi_m^{N}(q) = \Pi_m^{D}(q) = 0$ when $q \in (0, q_n]$. Hence, the delayed payment regime is beneficial (not beneficial) to the manufacturer when the suppliers work rates are adjustable (unadjustable). This result is due to the fact that, when the revenue q is sufficiently small, the manufacturer will participate under regime *DI* and will not participate under regimes N and D. Therefore, when choosing a payment scheme, the suppliers' capability to adjust their work rates plays an important role.

We now compare the manufacturer's optimal profits under regimes DI, D, and N as the number of suppliers n increases. First, let us examine the manufacturer's expected profit under regime D. Observe from Lemma 4 that the manufacturer's optimal expected profit $\Pi_m^D(q)$ will drop to 0 (non-participation) under regime D when the number of suppliers $n > \tau_n^D$, where $\tau_n^D \equiv argmin_{n>0}\ \{p_n > q\}$. Lemma 6 states that the manufacturer's optimal profit is positive for any n under regime DI. Hence, regime DI dominates D when the number of suppliers n is sufficiently large: $\Pi_m^{DI}(q) > \Pi_m^D(q)$ when $n > \tau_n^D$. Next, under regime N, one can check from Lemma 2 that the manufacturer's optimal expected profit $\Pi_m^N(q) = 0$ (non-participation) under regime N when the number of suppliers $n > \tau_n^N$, where $\tau_n^N \equiv argmin_{n>0}\ \{q_n > q\}$. Combining this observation with Lemma 6, we can conclude that regime DI dominates N when the number of suppliers n is sufficiently large: $\Pi_m^{DI}(q) > \Pi_m^N(q)$ when $n > \tau_n^N$. Coupling these observations with Corollary 3 ($\tau_n^N > \tau_n^D$) produces

Proposition 9 *(Large Number of Suppliers) For any given q, regime DI dominates both regimes N and D when the number of suppliers* $n > \tau_n^N$.

Coupling Proposition 9 and Proposition 5 reveals that $\Pi_m^{DI}(q) \geq \Pi_m^N(q) \geq \Pi_m^D(q)$ when the number of suppliers n is sufficiently large. Thus, when the number of suppliers n is large, the delayed payment regime is beneficial to the manufacturers when the work rates are adjustable but not beneficial when the work rates are unadjustable. Therefore, when choosing a payment regime for a project contract, the number of suppliers and the suppliers' capability to adjust their work rates are determining factors.

Our analytical results stated in Proposition 8 and numerical examples motivate us to develop the following conjecture:

Conjecture 3: For any given n, regime DI dominates regime N if and only if q is below a certain threshold.

Conjecture 3 is based on the intuition that, when q is small, the manufacturer can only afford to offer a small payment under regimes DI and N. Although the payment is delayed under regime DI, each supplier can adjust her work rate over time optimally so as increase her profit by operating more efficiently. This explains why the supplier's participation condition under regime DI (as stated in Lemma 6) is less stringent than that of under regime N (as stated in Lemma 2). Because suppliers are more eager to participate under regime DI than under regime N even when the payment p is small, we develop Conjecture 3. While we are unable to prove Conjecture 3 analytically, our numerical analysis supports the above conjecture. In our numerical analysis, we computed the ratio $\Pi_m^N(q)/\Pi_m^{DI}(q)$ for $2 \leq n \leq 20$ and for values of $q/k\alpha$ between 0 and 100 in the increment of 0.1, and we found that, for each n, $\Pi_m^N(q)/\Pi_m^{DI}(q) < 1$ if and only if $q/\alpha k$ is below a certain critical value. (The comment at the end of Section 3.1 also applies to the ratio $\Pi_m^N(q)/\Pi_m^{DI}(q)$, i.e., $\Pi_m^N(q)/\Pi_m^{DI}(q)$ is a function of $q/k\alpha$ and n only.) For example, by considering the case when $\alpha = 1$ and $k = 1$, and by varying q from 0 to 35 and n from 2 to 15, Figure 2 depicts the region within which one payment regime dominates the others. As shown in Figure 2, regime D is always dominated by regime N or regime DI. Moreover, regime DI dominates regime N if and only if q is small for any given n. Figure 2 is representative of our numerical results for $2 \leq n \leq 20$ and $0 \leq q/\alpha k \leq 100$. Overall, Figure 2 as well as our numerical results supports Conjectures 2 and 3.

Insert Figure 2 About Here.

5 Discussion and Concluding Remarks

Our model enabled us to examine how a delayed payment affects the supplier's optimal work rate, the manufacturer's optimal payment, the supplier's and the manufacturer's expected discounted profits, and the expected project completion time. When work rates are unadjustable, we obtained numerical and partially analytical results that support a conjecture that, relative to the no delayed payment regime N, each supplier operates at a slower rate and obtains a lower expected profit under regime D for any given p. Consequently, for any given p, use of regime D lengthens the project completion time. To induce suppliers to increase their work rates under regime D, the manufacturer will offer a higher payment. Partly because of the need to offer a higher price under regime D, our analytical and numerical results suggested that, contrary to the naive intuition shared among practitioners that we described in the Introduction, the manufacturer is actually worse off under regime D.

We have investigated the effect of the suppliers' ability to adjust their work rates. Whereas the capability to adjust work rates has no value to the suppliers under regime N, this ability is definitely beneficial to the suppliers under regime DI. By modeling the case of adjustable work rates as an n-stage game, we have shown that under the delayed payment regime, there exists an equilibrium at each stage in which all continuing suppliers work at the same rate. Also, we have shown that regime *DI* dominates regime *N* when the revenue q is small or when the number of suppliers n is large.

Our results generate the following insights. Based on our discussion with manufacturers and the suppliers, there is a naive intuition shared among practitioners that, under the delayed payment regime *D*, (1) the manufacturer is better off and (2) the suppliers are worse off. While the latter notion is true, the former notion is not always true. Specifically, we have exhibited conditions under which the manufacturer is worse off under the delayed payment regime. These conditions depend upon the revenue q, the number of suppliers n, and the suppliers' capability to adjust their work rates dynamically. These results are interesting because they run counter to the naive intuition shared among practitioners that the manufacturer is always better off under regime *D*. In summary, when designing a project contract, it is important for the manufacturer to understand the interactions among different factors (revenue, number of suppliers, supplier's ability, and supplier's behavior) and their impact on the manufacturer's profit. Our model is an initial attempt to examine some of the underlying dynamics; it has several limitations. Removing these limitations can serve as the starting point for future research.

5.1 Other Payment Schemes

The model presented in this paper is motivated by two simple payment regimes commonly observed in practice. Essentially, both regimes *N* and *D* are based on a single decision variable p and the timing of the payment. However, if the manufacturer (and the suppliers) are willing to entertain other contracts with more decision variables, then many other forms of contracts deserve attention. We now briefly discuss two other payment schemes and refer the reader to Kwon et al. (2008) for details. First, let us consider a regime $N+D$ that combines regimes *N* and *D*. Under regime $N+D$, each supplier receives a portion of her payment δp when she completes her own task and then receives the remaining portion of her payment $(1-\delta)p$ after all suppliers have completed their task. In this case, the manufacturer has to make two decisions: δ and p, where $\delta \in [0,1]$. By using an approach similar to the one used in Sections 3 and 4, one can determine the supplier's work rate in equilibrium, the supplier's expected profit and the manufacturer's expected profit. While the analysis is complex, it is clear that regime $N+D$ dominates both regimes *N* and *D* because regimes *N* and *D* are special cases of regime $N+D$ when $\delta = 1$ and $\delta = 0$, respectively. As such, the manufacturer's optimal profit is higher under regime $N+D$. The same result holds when the suppliers are able to adjust their work rates dynamically.

Second, consider the payment regime *PB* under which each supplier is compensated according to her own performance. For example, under regime *PB*, the manufacturer can pay the suppliers according to the order of their completion times: pay $p_{(1)}$ to the first supplier the moment she finishes, pay $p_{(2)}$ to the second supplier the moment she finishes, and pay $p_{(n)}$ to the last supplier the moment she finishes. In this case, the manufacturer has to make n decisions: $p_{(1)}, \cdots, p_{(n)}$). By using a similar approach presented in this paper, one can determine the supplier's work rate in equilibrium, the supplier's expected profit, and the manufacturer's expected profit. While the analysis is complex, we are able to show that the manufacturer's optimal profit under regime *PB* is strictly larger than under regime N for the case when $n = 2$ and when the suppliers are able to adjust their work rates dynamically. The results associated with regimes $N + D$ and *PB* reveal that the manufacturer can benefit from payment regimes that involve more decision variables.

5.2 Other Future Research Topics

There are many research opportunities for addressing the limitations of the model presented in this paper. First, our model is based on the assumption that the completion time of each task is exponentially distributed. It would be of interest to examine other probability distributions, develop near-optimal heuristics for the suppliers' time-varying work rates, and conduct simulation experiments to examine the robustness of the results presented in this paper. Second, we have assumed that the operating costs of all n suppliers are identical. This assumption is critical to establish the existence of symmetric equilibria and to establish the analytical results presented in this paper. One potential future research direction is to examine the case of non-identical suppliers and to numerically investigate the robustness of the results presented in this paper. Third, our model assumes that all parties are risk-neutral. It would be of interest to examine the behavior and the performance metrics when the suppliers are risk-averse. Fourth, our model is based on the assumption that the manufacturer has perfect information about the supplier's cost structure including the value of k. In reality, the manufacturer will not possess perfect information. Because imperfect information can create another technical challenge for the manufacturer to design an effective project contract, it would be of interest to explore the use of mechanism design theory to develop effective project contracts. Fifth, when the information regarding each supplier's cost structure is private, it would be of interest for the manufacturer to consider using auction mechanisms instead of incentive contracts. Sixth, even though supply contracts have been well studied, the issue of channel coordination in the context of project management contracts is not well-understood. This is another potential future research topic.

Appendix

The parameter-free functions introduced at the end of Section 3.1 are given below.

$$
\begin{aligned}
\tilde{\beta}_n(r_1,\ldots,r_n) &= \int_0^{\infty} e^{-t}\prod_{i=1}^{n}(1-e^{-r_i t})\,dt \\
\tilde{\Pi}_i^N(p;r_i) &= p\tilde{\beta}(r_i) - \frac{r_i^2}{r_i+1} \\
\tilde{\Pi}_i^D(p;r_1,\ldots,r_n) &= p\tilde{\beta}_n(r_1,\ldots,r_n) - \frac{r_i^2}{r_i+1} \\
\tilde{\Pi}_m^N(p;q) &= nq\tilde{\beta}_n(r_1,\ldots,r_n) - p\sum_{i=1}^{n}\tilde{\beta}(r_i) \\
\tilde{\Pi}_m^D(p;q) &= n(q-p)\tilde{\beta}_n(r_1,\ldots,r_n)\,.
\end{aligned}
$$

References

[1] Abramowitz, M., and Stegun, I.A., *Handbook of Mathematical Functions*, Dover Publications, 1965.

[2] Adler, P., Mandelbaum, A., Nguyen, V., and Schwerer, E., “From Project to Process Management: An Empirical-based Framework for Analyzing Product Development Time,” *Management Science*, Vol. 41, 3, pp. 458-482, 1995.

[3] Bayiz, M., and Corbett, C.J., “Coordination and Incentive Contracts in Project Management under Asymmetric Information,” working paper, UCLA Anderson School, 2005.

[4] Cachon, G., “Supply Chain Coordination with Contracts,” in *Handbooks in Operations Research and Management Science*, (Eds. De Kok, A.G., and Graves, S.,), Elsevier Publisher, 2003.

[5] Cachon, G., and Zhang, F., “Obtaining Fast Service in a Queueing System via Performance-Based Allocation of Demand,” *Management Science*, Vol. 53, pp. 408-420, 2007.

[6] Cohen, I., Mandelbaum, A., and Shtub, A., “Multi-Project Scheduling and Control: A Project-Based Comparative Study of the Critical Chain Methodology and Some Alternatives,” *Project Management Journal*, Vol. 35, 2, pp. 39-50, 2004.

[7] Dean, B., Merterl Jr. S., and Roepke, L., “Research Project Cost Distribution and Budget Forecasting,” *IEEE Transactions on Engineering Management*, Vol. 16, 4, pp. 176-191, 1969.

[8] Dempski, J., and Sappington, D., “Optimal Incentive Contracts with Multiple Agents,” *Journal of Economic Theory*, Vol. 17, pp. 152-171, 1984.

[9] Gradshteyn, I.S., and Ryzhik, I.M., *Table of Integrals, Series, and Products*, 7th ed., Eds Alan Jeffrey and Daniel Zwillinger, Academic Press, 2007.

[10] Greising, D., and Johnsson, J., “Behind Boeing’s 787 Delays,” *Chicago Tribune*, December 8, 2007.

[11] Holmstrom, B., “Moral Hazard in Teams,” *The Bell Journal of Economics*, Vol. 13, 2, pp. 324-340, 1982.

[12] Itoh, H., “Incentives to Help in Multi-agent Situations,” *Econometrica*, Vol. 59, 3, pp. 611-636, 1991.

[13] Klastorin, T., *Project Management: Tools and Trade-offs*, John Wiley and Sons, Inc., Somerset, New Jersey, 2004.

[14] Kromke, E., *Private Communication*, Feldman & Associates, Inc., Los Angeles, 2009.

[15] Kwon, D., Lippman, S.A., McCardle, K.F., and Tang, C.S., “Managing Time-Based New Product Development Contracts with Delayed Payments,” unpublished manuscript, 2008.

[16] Lippman, S.A., and Mamer, J.W., “Pre-emptive Innovation,” *Journal of Economic Theory*, Vol. 61, pp. 104-119, 1993.

[17] Magott, J., and Skudlarski, K., “Estimating the mean completion time of PERT networks with exponentially distributed durations of activities,” European Journal of Operational Research, Vol. 71, 8, pp. 7079, 1993.

[18] McAfee, R.P., and McMillan, J., “Optimal Contracts for Teams,” *International Economic Review*, Vol. 32, 3, pp. 561-577, 1991.

[19] Mookherjee, D., “Optimal Incentive Schemes with Many Agents,” *Review of Economic Studies,* Vol. 51, pp. 433-446, 1984.

[20] Nolan, R., and Kotha, S., “Boeing 787: The Dreamliner,” Harvard Business School case 9-305-101, 2005.

[21] Pennings, E., and Lint, O., “The Option Value of advance R & D,” *European Journal of Operational Research*, Vol. 103, 16, pp. 83-94, 1997.

[22] Reinganum, J.F., “Strategic Search Theory,” *International Economic Review*, Vol. 23, 1, pp. 1 - 17, 1982.

[23] Ross, S., *Introduction to Stochastic Dynamic Programming*, Academic Press, 1983.

[24] Tang, C.S., “Perspectives in Supply Chain Risk Management,” *International Journal of Production Economics*, Vol. 103, pp. 451-488, 2006.

[25] Tavares, L.V., “A Review of the Contribution of Operations Research to Project Management,” *European Journal of Operational Research*, Vol. 136, pp. 1-18, 2002.

[26] Wang, Y., and Gerchak, Y., “Capacity Games in Assembly Systems with Uncertain Demand,” *Manufacturing and Service Operations Management*, Vol. 5, 3, pp. 252-267, 2003.

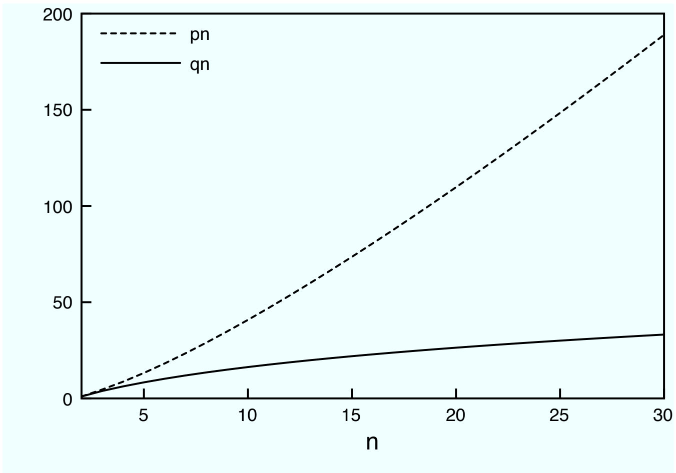

Figure 1: The participation thresholds p_n and q_n when $\alpha = k = 1$.

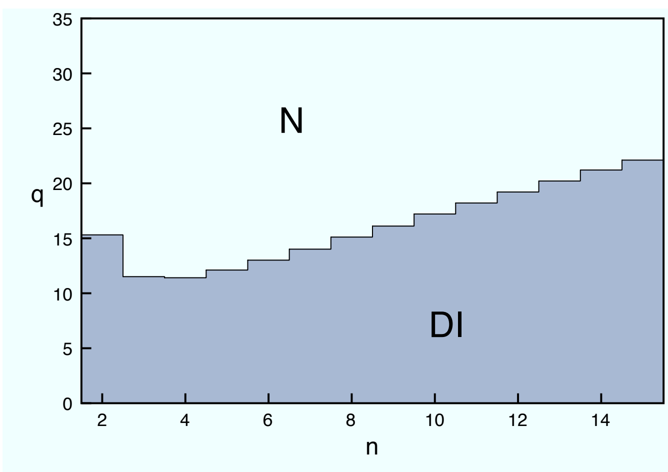

Figure 2: Regimes N, D, and DI for $2 \leq n \leq 15$ and $0 \leq q \leq 35$ when $\alpha = k = 1$. The shaded area is where DI regime dominates while the unshaded area is where N regime dominates. Regime D is always dominated by regime N.

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Proof of Lemma 1: The first statement follows immediately from the fact that $T = \max\{X_1, \cdots, X_n\}$ is stochastically larger than X_i, while the other statements follow from basic calculus. Statement 4 results from the transformation of variable of having $e^{-rt} = x$ so that $\beta_n(r_1, \cdots, r_n) = \frac{\alpha}{r} \cdot \int_0^1 x^{(\alpha-r)/r}(1-x)^n dx = \frac{\alpha}{r} \cdot \mathrm{B}(\frac{\alpha}{r}, n+1) = \prod_{j=1}^n \frac{jr}{jr+\alpha}$, where $\mathrm{B}(.,.)$ is the Beta function (Chap. 6 of Abramowitz and Stegun, 1965). Notice that the last equality follows from 3.312 of Gradshteyn and Ryzhik (2007). ∎

Proof of Proposition 1: Combining statement 2 of Lemma 1 along with the fact that $-\frac{kr_i^2}{r_i+\alpha}$ is concave in r_i, it is easy to check from (3.1) that $\Pi_i^N(p; r_i)$ is concave in r_i. By considering the first-order condition, we obtain $r^N(p)$. Substitute $r^N(p)$ into $\Pi_i^N(p; r_i)$, we obtain (3.10) after some algebra. ∎

Proof of Lemma 2: First, observe from (3.10) and (3.9) that supplier i's expected profit $\Pi_i^N(p) > 0$ if and only if $p > 0$. This observation suggests that the supplier participation constraint (3.2) holds if and only if $p > 0$.

Next, to establish the conditions under which the manufacturer participation constraint (3.6) does not hold, let us first rewrite (3.9) as: $p = (\alpha k)[(r/\alpha + 1)^2 - 1]$. By applying statement 4 of Lemma 1, it is easy to check that the manufacturer's expected profit $\Pi_m^N(q, p)$ given in (3.5) can be re-expressed as a function of r as follows:

$$\Pi_m^N(q, p) = nq \cdot \frac{\alpha}{r}\mathrm{B}(\frac{\alpha}{r}, n+1) - n(\alpha k)\frac{r}{r+\alpha}[(\frac{r}{\alpha}+1)^2 - 1]\,. \tag{5.1}$$

The condition $\Pi_m^N(q, p) \leq 0$ can be equivalently expressed as

$$q \leq f_N(r) \equiv \frac{(\alpha k)\frac{r}{r+\alpha}[(\frac{r}{\alpha}+1)^2 - 1]}{\frac{\alpha}{r}\mathrm{B}(\frac{\alpha}{r}, n+1)}\,. \tag{5.2}$$

By using the identity $\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and the asymptotic properties $\Gamma(x) = x^{x-1/2}e^{-x}\sqrt{2\pi}(1 + O(x^{-1}))$ (Stirling's formula) from 8.327 of Gradshteyn and Ryzhik (2007), we obtain

$$f_N(r) = 2(k\alpha)(\frac{\alpha}{r})^{n-2}\frac{1}{\Gamma(n+1)}[1 + O(r)]$$

in the small-r limit and

$$f_N(r) = k\alpha(\frac{r}{\alpha})^2[1 + O(r^{-1})]$$

in the large-r limit. If $n = 2$, $\lim_{r\to 0} f_N(r) = k\alpha$ and $\lim_{r\to\infty} f_N(r) = \infty$; if $n > 2$, $f_N(r) \to \infty$ in both limits $r \to 0$ and $r \to \infty$. Thus, $f_N(\cdot)$ achieves a minimum somewhere in $(0, \infty)$, so there exists a threshold $q_n = \min_{r>0} f_N(r) > 0$ such that $\Pi_m^N(q) = 0$ unless $q \geq q_n$. Hence, we can conclude that the manufacturer participation constraint (3.6) does not hold if $q \le q_n$. By observing (5.2) that $f_N(r)$ is increasing in n, it is easy to check that q_n is also increasing in n.

Next, the large-n asymptotic properties of $f_N(\cdot)$ can be obtained from the asymptotic properties of $\Gamma(\cdot)$. The asymptotic form of $f_N(\cdot)$ is

$$f_N(r) = k\alpha \frac{r}{r+\alpha}[(\frac{r}{\alpha}+1)^2 - 1]\frac{n^{\alpha/r}}{\Gamma(\frac{\alpha}{r}+1)}(1+O(n^{-1}))\,.$$

The minimum value of $f_N(\cdot)$ can be obtained from the first-order condition $d f_N(r)/dr = 0$, which yields $r = \frac{1}{2}\alpha \ln n + O(1)$. After substituting r into (5.2), we obtain

$$\min_{r>0} f_N(r) = \frac{k\alpha}{4}(\ln n)^2 + O(\ln n). \ \blacksquare$$

Proof of Corollary 1: By noting that $T = \max\{X_1, \cdots, X_n\}$ is stochastically increasing in n for any non-negative random variables $X_1, X_2, \cdots$, we can conclude that $E(T^N(p))$ under regime N is increasing in n. Next, when X_i is exponentially distributed with rate $r^N(p)$ for $i = 1, \cdots, n$, the distribution of $T^N(p)$ is equal to $(1-e^{-r^N(p)t})^n$. Hence, $E(T^N(p)) = \int_0^\infty t d((1-e^{-r^N(p)t})^n) = nr\int_0^\infty t \cdot e^{-r^N(p)t} \cdot (1-e^{-r^N(p)t})^{n-1}dt$. With some algebra and from the integral formula 4.253 of Gradshteyn and Ryzhik (2007), we obtain (3.11). By noting that $r^N(p)$ is increasing in p, (3.11) implies that $E(T^N(p))$ is decreasing in p. $\blacksquare$

Proof of Lemma 3: First, by applying statement 2 in Lemma 1 along with the fact that $-\frac{kr_i^2}{r_i+\alpha}$ is concave in r_i, it is easy to check that $\Pi_i^D(p; r_1, \cdots, r_n)$ given in (3.3) is concave in r_i. Second, by using the submodularity of the discount factor $\beta_n(.)$ as established in statement 3 of Lemma 1, we can conclude that the optimal r_i^* is increasing in r_j for $j \ne i$. $\blacksquare$

Proof of Proposition 2: We use contradiction to establish the existence of only symmetric Nash equilibria. Suppose an asymmetric equilibrium $r = (r_1, \cdots, r_n)$ exists that has $r_i = x$ and $r_j = y$ for some $i \ne j$, where $y > x$. By symmetry among all suppliers, there is another asymmetric equilibrium that has $r' = (r_1^w, \cdots, r_n^w)$ with $r_i^w = y$ and $r_j^w = x$, and all other work rates remain the same as in r. However, as we increase r_i from x to y, Lemma 3 proves that the best response for supplier j is to increase her rate r_j from y to a higher value $z > y$. This contradicts the assumption that r' is a Nash equilibrium with $r_i^w = y$ and $r_j^w = x$ because it implies that supplier j's best response is to reduce her rate from y to x when i increases her rate from x to y. Therefore, asymmetric equilibria do not exist.

We now establish the existence of a symmetric equilibrium and determine its value. To do so, differentiate the supplier's expected discounted profit $\Pi_i^D(p; r_1, \cdots, r_n)$ given in (3.3) with respect to r_i. By considering the first-order condition and by considering the case when $r_1 = \cdots = r_n = r$, it is easy to check the first-order condition can be simplified as:

$$p\alpha \int_0^\infty t e^{-(\alpha+r)t}(1-e^{-rt})^{n-1} dt - \frac{2k\alpha r + kr^2}{(\alpha+r)^2} = 0. \tag{5.3}$$

By change of variables, $\int_0^\infty t e^{-(\alpha+r)t}(1-e^{-rt})^{n-1} dt = -\frac{1}{(\alpha+r)^2}\int_0^1 \ln x\,(1-x^{\frac{r}{\alpha+r}})^{n-1} dx$ and using the formula 4.253 of Gradshteyn and Ryzhik (2007), we obtain (3.12).

Now we determine the optimal rate $r^D(n;p)$ for the suppliers. To begin, let us rewrite (3.12) as:

$$p = f_D(r) \equiv \frac{\frac{(2k\alpha r + kr^2)}{(\alpha+r)^2}}{\frac{\alpha}{r^2}\mathrm{B}(\frac{\alpha+r}{r}, n)[\psi(\frac{\alpha+r}{r}+n) - \psi(\frac{\alpha+r}{r})]}. \tag{5.4}$$

Using the identity $\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and the asymptotic properties $\Gamma(x) = x^{x-1/2}e^{-x}\sqrt{2\pi}(1+O(x^{-1}))$ (Stirling's formula) from 8.327 of Gradshteyn and Ryzhik (2007) and $\psi(x) = \ln x - (2x)^{-1} + O(x^{-2})$ from 6.3.18 of Abramowitz and Stegun (1965), we obtain

$$f_D(r) = 2(k\alpha)(\frac{\alpha}{r})^{n-2}\frac{1}{\Gamma(n+1)}[1+O(r)]$$

in the small-r limit and

$$f_D(r) = k\alpha(\frac{r}{\alpha})^2\frac{n}{\psi(n+1)-\psi(1)}[1+O(r^{-1})] \tag{5.5}$$

in the large-r limit. If $n=2$, $f_D(0) = k\alpha$ and $\lim_{r\to\infty} f_D(r) = \infty$; if $n>2$, $f_D(r) \to \infty$ in both limits $r \to 0$ and $r \to \infty$. Moreover, from (5.4), $f_D(\cdot)$ is not identically infinite, so $f_D(r) < \infty$ for some values of r. Thus, $f_D(\cdot)$ achieves a minimum positive value at some $r < \infty$. Hence, a positive solution to $p - f_D(r) = 0$ exists if p is sufficiently large. In particular, for $n>2$, $p = f_D(r)$ for sufficiently large p has at least two solutions (at least two Nash equilibria). We simply choose the Nash equilibrium with the largest work rate and call it $r^D(n;p)$.

The function $p - f_D(r)$ has three properties. First, it is locally decreasing in r at $r = r^D(n;p)$ because we chose $r^D(n;p)$ to be the largest root of $p - f_D(r) = 0$ where $f_D(\cdot)$ achieves a global minimum somewhere in $(0,\infty)$ and $f_D(r) \to \infty$ in the limit $r \to \infty$. Second, $p - f_D(r)$ obviously increases in p. Finally, it can be shown from (5.4) that $f_D(r)$ increases in n. It follows that $r^D(n;p)$ decreases in n and increases in p.

Finally, by substituting $r_i = r^D(p)$ for $i = 1,\cdots,n$ into (3.3), by differentiating the resulting profit function with respect to n, and then by applying the fact that $r^D(n;p)$ is decreasing in n, we can conclude that the supplier's expected discounted profit $\Pi_i^D(n;p)$ is also decreasing in n. ■

Proof of Corollary 2: We prove that there is a unique positive root of $h(x) = 4x^3 + 12\alpha x^2 + 9\alpha^2 x - 3\frac{p}{k}\alpha x + 2\alpha^3 - 2\frac{p}{k}\alpha^2 = 0$ when $p > k\alpha\}$, and it is given in (3.14). First, we can verify that r^D defined in (3.14) and (3.15) satisfies $h(r^D) = 0$ by substituting r^D in the function $h(\cdot)$. Second, at $p = k\alpha$, we have $\phi = 3\pi/4$ and $\cos(\phi/3) = 1/\sqrt{2}$ so that $r^D(k\alpha) = 0$; it is straightforward to show that $r^D(p)$ is increasing in p by directly calculating $dr^D(p)/dp$, and hence, $r^D > 0$ if and only if $p > k\alpha$. It remains to show that r^D is the only positive root of $h(x) = 0$ when $p > k\alpha$ and that there are no positive roots when $p \leq k\alpha$.

The larger root of the quadratic equation $h'(x) = 12x^2 + 24\alpha x + 9\alpha^2 - 3(p)\alpha/k = 0$ is $\widetilde{x} = \alpha[\sqrt{1+(p)/(\alpha k)} - 2]/2$ which is greater than $-\alpha$. Also from $h''(x) = 24(x+\alpha) > 0$ for $x > -\alpha$, we find that $h(\cdot)$ is strictly decreasing in the interval $(-\alpha, \widetilde{x})$ and strictly increasing in $(\widetilde{x}, \infty)$. We also note that

$$h(-\alpha) = \alpha^3 + p\alpha^2/k > 0\,, \text{ and}$$

$$h(\widetilde{x}) = -(p + k\alpha)\alpha^2(\sqrt{1+p/k\alpha} - 1)/k < 0\,.$$

Hence, from the continuity of $h(\cdot)$ and the limits $h(-\infty) = -\infty$ and $h(\infty) = \infty$, there is exactly one root of $h(x) = 0$ in each of the intervals $(-\infty, -\alpha)$, $(-\alpha, \widetilde{x})$, and $(\widetilde{x}, \infty)$.

If $p \leq k\alpha$, then $\widetilde{x} \leq \alpha(\sqrt{2} - 2)/2 < 0$ and $h(0) = 2\alpha^3 - 2\frac{p}{k}\alpha^2 \geq 0$ so there is no positive root of $h(x) = 0$. If $p > k\alpha$, then $h(0) < 0$ and $h(-\alpha) > 0$ so there is exactly one root each in the intervals $(-\infty, -\alpha)$, $(-\alpha, 0)$ and $(0, \infty)$. Thus, there is exactly one positive root if and only if $p > k\alpha$.

The optimal work rate can written as $r^D(p) = \alpha f(\frac{p}{\alpha k})$ where

$$f(x) \equiv \sqrt{1+x}\cos[(\pi - \arctan\sqrt{x})/3] - 1\,.$$

The function $f(x)$ is strictly increasing and unbounded in x. Thus, $r^D(p)$ is strictly increasing and unbounded in p. Moreover, $xf(x^{-1})$ is unimodal in x, so $r^D(p)$ is unimodal in α. ■

Proof of Lemma 4: Observe from the proof of Proposition 2 and the definition of $f_D(\cdot)$ in (5.4) that there exists a threshold $p_n > 0$ such that $\min_{r>0} f_D(r) = p_n$. Thus, a positive solution r to (3.12) exists if and only if $p \geq p_n$. (If $p < p_n$, then the only Nash equilibrium for the contractors is $r = 0$.) This implies that the work rate in equilibrium $r^D(n;p) > 0$ if and only if $p \geq p_n$. We can prove that the supplier participation constraint (3.4) holds when $p > p_n$ by showing that $r^D(n;p) > 0$ if and only if $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) > 0$ when $r_j = r^D(n;p)$ for $j = 1, \cdots, n$. First, if $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) > 0$ when $r_j = r^D(n;p)$ for $j = 1, \cdots, n$, then $r^D(n;p)$ must be positive because $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) = 0$, otherwise. Next, if $r^D(n;p) > 0$, we now prove $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) > 0$ by contradiction. Suppose not. Then $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) \leq 0$.

(a) $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n)$ cannot be strictly negative when $r_j = r^D(n; p)$ for $j = 1, \cdots, n$. This is because if $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) < 0$ when $r_j = r^D(n; p)$ for $j = 1, \cdots, n$, supplier i can set her work rate to 0 and earns a higher profit 0, which contradicts that $r^D(n; p)$ is the work rate in equilibrium. (b) $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n)$ cannot be zero when $r_j = r^D(n; p)$ for $j = 1, \cdots, n$. This is because if $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n) = 0$ when $r_j = r^D(n; p)$ for $j = 1, \cdots, n$, the function $\Pi_i^D(p; r_1, \cdots, r_i, \cdots, r_n)$ when $r_j = r^D(n; p)$ for $j \neq i$ cannot be a strictly concave function in r_i because this function also equals zero when $r_i = 0$. Knowing the fact that $-\frac{kr_i^2}{r_i+\alpha}$ is concave, this contradicts statement 2 of Lemma 1. Hence, we have shown that the supplier participation constraint (3.4) holds when $p > p_n$.

Similarly, observe that a positive solution r to (3.12) exists if and only if $p \geq p_n$. Hence, in view of condition for the supplier participation, the manufacturer's optimal expected profit $\Pi_m^D(q) > 0$ if and only if $q > p_n$. Observe from (5.4) that $f_D(\cdot)$ is increasing in n, and hence, p_n is also increasing in n.

Next, the large-n asymptotic properties of $f_D(\cdot)$ and $f_N(\cdot)$ can be obtained from the asymptotic properties of $\Gamma(\cdot)$ and $\psi(\cdot)$. The asymptotic form of $f_D(\cdot)$ is

$$f_D(r) = \frac{(2k\alpha r + kr^2)r^2 n^{1+\alpha/r}}{\alpha(\alpha+r)^2\Gamma(\frac{\alpha}{r}+1)\ln n}(1+O(n^{-1}))\,.$$

The minimum value of $f_D(\cdot)$ can be obtained from the first-order condition $d f_D(r)/dr = 0$. For large n, keeping the leading-order terms of $\ln n$, we find that $r = \alpha \ln n + O(1)$ minimizes $f_D(\cdot)$ and that $\min_{r>0} f_D(r) = k\alpha n(\ln n + O(1))$. ■

Proof of Lemma 5: Consider the case when $n = 2$. For any pair $q_1 < q_2$, let $p_1 = p^N(q_1)$ and $p_2 = p^N(q_2)$ so that

$$\begin{aligned}
\Pi_m^N(p_1, q_1) &= 2q_1\beta(r^N(p_1), r^N(p_1)) - 2p_1\beta(r^N(p_1)) \\
&\leq 2q_2\beta(r^N(p_1), r^N(p_1)) - 2p_1\beta(r^N(p_1)) \leq \Pi_m^N(p_2, q_2)\,.
\end{aligned}$$

Hence, $\Pi_m^N(q)$ is non-decreasing in q. Moreover, if $r^N(p_1) > 0$, then $\beta(r^N(p_1) > 0$, so $\Pi_m^N(p_1, q_1) < \Pi_m^N(p_2, q_2)$. Therefore, $\Pi_m^N(q)$ is strictly increasing in q for $q > z$. Next, let $q_x = xq_1 + (1-x)q_2$ where $x \in (0,1)$ and define $p_x = p^N(q_x)$. Then

$$\begin{aligned}
\Pi_m^N(q_x) &= 2q_x\beta(r^N(p_x), r^N(p_x)) - 2p_x\beta(r^N(p_x)) \\
&= x[2q_1\beta(r^N(p_x), r^N(p_x)) - 2p_x\beta(r^N(p_x))] \\
&\quad +(1-x)[2q_2\beta(r^N(p_x), r^N(p_x)) - 2p_x\beta(r^N(p_x))] \\
&\leq x\Pi_m^N(q_1) + (1-x)\Pi_m^N(q_2) \quad .
\end{aligned}$$

These arguments apply to prove that $\Pi_m^D(q)$ is convex and non-decreasing in q and that it is strictly increasing q for $q > \alpha k$. For general $n \geq 2$, we can use the same approach to obtain the results. ■

Proof of Proposition 3: The first two statements follow immediately from the participation conditions under regime N and regime D as stated in Lemmas 2 and 4. ■

Proof of Proposition 4: First, we show that in the regime D there is a Nash equilibrium in which $r^D(p)$ grows in p without bound: from (5.5), the first-order condition (3.12) yields

$$r^D(n;p) = \alpha\sqrt{(\frac{p}{n\alpha k})[\psi(n+1)-\psi(1)]}(1+O(p^{-1}))$$

for large p. Moreover, it can be shown that this is the Nash equilibrium which yields the highest profits for the manufacturers for large p. From $\mathrm{B}(\frac{\alpha}{r}, n+1)(\frac{\alpha}{r}) = 1 - \frac{\alpha}{r}[\psi(n+1)-\psi(1)] + O(r^{-2})$ for large r,

$$\Pi_m^D(q,p) = nq - nq\cdot(\frac{\alpha}{r^D(n;p)})[\psi(n+1)-\psi(1)] - np + O(r^D)\,.$$

Differentiating the above with respect to p and imposing the first-order condition, we obtain

$$\begin{aligned} p^D(n;q) &= (\frac{q}{2})^{2/3}\{\alpha k n[\psi(n+1)-\psi(1)]\}^{1/3}(1+O(\frac{1}{r^D})) \\ r^D &= \alpha(\frac{q[\psi(n+1)-\psi(1)]^2}{2n\alpha k})^{1/3}(1+O(1/r^D))\,. \end{aligned}$$

Finally,

$$\Pi_m^D(q) = nq - 3n(\frac{q}{2})^{2/3}(\alpha k)^{1/3}\{n[\psi(n+1)-\psi(1)]\}^{1/3} + O(q^{1/3})\,.$$

Similarly, we study Π_m^N in (5.1) in the large-r limit:

$$\Pi_m^N(q,p) = nq - nq\cdot(\frac{\alpha}{r})[\psi(n+1)-\psi(1)] - n\alpha k(\frac{r}{\alpha})^2 + O(r)\,.$$

The first-order condition with respect to r yields

$$\begin{aligned} r^N &= \{\frac{q\alpha^2[\psi(n+1)-\psi(1)]}{2k}\}^{1/3}(1+O(1/r^N))\,, \\ p^N(q) &= \frac{k}{\alpha}\{\frac{q\alpha^2[\psi(n+1)-\psi(1)]}{2k}\}^{2/3}(1+O(1/r^N))\,, \\ \Pi_m^N(q) &= nq - 3n(\frac{q}{2})^{2/3}(\alpha k)^{1/3}[\psi(n+1)-\psi(1)]^{2/3} + O(q^{1/3})\,. \end{aligned}$$

Noticing that $\psi(n+1)-\psi(1) = \sum_{k=1}^{n} k^{-1} < n$ for all positive integers n, we can prove the statements of the Proposition through direct comparisons. ■

Proof of Proposition 7: The proof follows exactly the same approach as in the proof of Proposition 6. We omit the details. ■

Proof of Lemma 6: The proof follows immediately from the supplier's and the manufacturer's participation constraints. ■

Proof of Proposition 8: When q is sufficiently small, we obtain the first statement immediately from the manufacturer participation conditions as stated in Lemmas 2 and 6.

When q is sufficiently large, the manufacturer and the suppliers participate in the project under regimes DI and N. Let us consider regime DI. We first claim that $\lambda^{(j)} = \alpha\ell(j)\sqrt{\frac{p}{\alpha k}} + O(1)$, $S^{(j)} = p + O(\sqrt{p})$, $R^{(j)} = p + O(\sqrt{p})$, and $S^{(j)} - R^{(j)} = m(j)\sqrt{p\alpha k} + O(1)$ for sufficiently large p where $\ell(j)$ and $m(j)$ are positive functions of the stage index j such that $\ell(j) < \frac{j+1}{2j-1} \leq 1$ and $m(j) < 1$ for $j \geq 2$ and $\ell(1) = m(1) = 1$. We prove this claim via the mathematical induction.

From $\lambda^{(1)} = \alpha\sqrt{\frac{p}{\alpha k}} + O(1)$, we obtain $R^{(1)} = p - \frac{p}{\lambda^{(1)}/\alpha} - k\lambda^{(1)} + O(1) = p - 2\sqrt{p\alpha k} + O(1)$ and $S^{(1)} = p - \frac{p}{\lambda^{(1)}/\alpha} + O(1) = p - \sqrt{p\alpha k} + O(1)$ so that $S^{(1)} - R^{(1)} = \sqrt{p\alpha k} + O(1)$.

Suppose that the claim is true for all stage indices up to $j-1$. From (4.3),

$$\begin{aligned}
\lambda^{(j)} &= \frac{(j-1)m(j-1)\sqrt{p\alpha k} + \sqrt{[(j-1)m(j-1)\sqrt{p\alpha k}]^2 + 4(2j-1)k\alpha p}}{2(2j-1)k} + O(1) \\
&\leq \sqrt{p\alpha k}\frac{(j-1) + \sqrt{j^2+6j-3}}{2(2j-1)k} + O(1) < \sqrt{p\alpha k}\frac{(j-1) + \sqrt{j^2+6j+9}}{2(2j-1)k} + O(1) \\
&= \alpha\sqrt{\frac{p}{\alpha k}}\frac{j+1}{2j-1} + O(1)\,,
\end{aligned}$$

$$\begin{aligned}
R^{(j)} &= R^{(j-1)} - k\lambda^{(j)}/j + (S^{(j-1)} - R^{(j-1)})/j - R^{(j-1)} \cdot (\frac{\alpha}{j\lambda^{(j)}}) + O(1) \\
&= p + \sqrt{p\alpha k}[-\frac{\ell(j)}{j} + \frac{m(j-1)}{j} - \frac{1}{j\ell(j)}] + O(1)\,,
\end{aligned}$$

and

$$S^{(j)} = \frac{j\lambda^{(j)}}{j\lambda^{(j)} + \alpha}S^{(j-1)} = S^{(j-1)}(1 - \frac{\alpha}{j\lambda^{(j)}} + O(p^{-1}))\,.$$

Hence,

$$\begin{aligned}
S^{(j)} - R^{(j)} &= (S^{(j-1)} - R^{(j-1)})(1 - \frac{1}{j}) + \frac{k\lambda^{(j)}}{j} + O(1) \\
&= \sqrt{p\alpha k}(1 - \frac{1}{j} + \frac{\ell(j)}{j}) + O(1) \; < \sqrt{p\alpha k} + O(1)\,.
\end{aligned}$$

Thus, the claim is proved for all j.

Next, we study $\Pi_m^{DI}(q,p)$ for large q. Assume that p is also large. Then

$$\Pi_m^{DI}(q,p) = n(q-p)\prod_{j=1}^{n}\frac{j\lambda^{(j)}}{j\lambda^{(j)} + \alpha} = n(q-p)[1 - \sum_{j=1}^{n}\frac{\alpha}{j\lambda^{(j)}} + O(p^{-1})]$$

$$= \quad nq - nq[\sum_{j=1}^{n} \frac{\alpha}{j\lambda^{(j)}} + O(p^{-1})] - np + O(\sqrt{p})\,. \tag{5.6}$$

Then the first-order condition for the optimal p is given by

$$-q\frac{d}{dp}\sum_{j=1}^{n} \frac{\alpha}{j\lambda^{(j)}} = 1 + O(p^{-1/2})\,.$$

This is consistent with the assumption that the optimal p grows unboundedly as q grows. Now we compare (5.6) to $\Pi_m^N(q,p)$ in the large-p limit:

$$\Pi_m^N(q,p) \quad = \quad nq - nq\cdot(\frac{\alpha}{r^N(p)})[\psi(n+1)-\psi(1)](1+O(\frac{1}{r^N(p)})) - np + O(\sqrt{p})$$

where $\psi(n+1)-\psi(1) = \sum_{j=1}^{n} j^{-1}$. Because $r^N(p) = \alpha\sqrt{\frac{p}{\alpha k}} + O(1) \geq \lambda^{(j)}$ where the inequality is strict for $j \geq 2$, we have $\Pi_m^N(q,p) > \Pi_m^{DI}(q,p)$ for large p and for any $n \geq 2$. Thus, for large q in which case p is also large, $\Pi_m^N(q) > \Pi_m^{DI}(q)$. ■

Proof of Proposition 9: Observe from Lemma 2 that q_n increases in n without bound. Hence, for any fixed value of q, $\Pi_m^{DI}(q) > \Pi_m^N(q) = 0$ when n is sufficiently large. Similarly, from Lemma 4, p_n increases in n without bound. Because $\Pi_m^D(q) = 0$ for $q \leq p_n$, for any fixed value of q, $\Pi_m^{DI}(q) > \Pi_m^D(q) = 0$ when n is sufficiently large. ■