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THE PRICING OF EQUITY-LINKED LIFE INSURANCE POLICIES WITH AN ASSET VALUE GUARANTEE

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This paper considers the equilibrium pricing of equity-linked life insurance policies with an asset value guarantee; such policies provide for benefits which depend upon the performance of a reference portfolio subject to a minimum guaranteed benefit. The benefit is decomposed into a sure amount and an immediately exercisable call option on the reference portfolio. A numerical procedure for determining the value of the call option is presented and the risk minimizing investment strategy to be followed by the issuer of the policy is derived.

1. Introduction

The distinguishing feature of an equity-linked life insurance policy is that the benefit payable at expiration depends upon the market value of some reference portfolio. Thus, unlike traditional life insurance policies which provide a fixed benefit, or participating policies in which the benefit may be loosely related to the investment performance and mortality experience of the insurance company, the equity-linked policy imposes on the policyholder the full investment risk, a risk he may well be willing to assume under conditions of uncertain inflation if he regards equities as providing a hedge against inflation.

A pure equity-linked policy is in reality not an insurance policy at all, but an investment programme, in which the insurance company invests the premia less expenses in an investment portfolio, and at expiration pays the policyholder the market value of the investment portfolio. It is clear that this involves the insurance company in no risk, and that the company is performing no service which is not available from existing mutual fund investment plans. Of course, the pure equity-linked policy may be packaged with a traditional term or endowment policy, part of the premia being allocated to the equity element, and part to the term or endowment policy; but, while this may be regarded as a new insurance product, no new insurance principles are involved.

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In the United Kingdom and in Canada, sellers of equity-linked policies typically provide for a minimum benefit or asset value guarantee, payable on death or maturity, thus relieving the policyholder of a part of the investment risk; the insurance companies are then in fact selling an investment guarantee or insurance policy in addition to the straightforward investment plan: this involves them both in mortality risk since it is uncertain at what date the guarantee will be effective, and in investment risk since the cost of the guarantee will depend upon the investment performance of the portfolio. In the United States, where equity-based policies are referred to as 'variable life insurance contracts', controversy as to whether such contracts are securities which should be regulated by the Securities and Exchange Commission has till recently prevented their introduction. In addition, one U.S. insurance company offers an 'insured mutual fund redemption value program' which provides a minimum return to an investor who undertakes an investment plan in one of several specified mutual funds. The insured mutual fund redemption value program provides a guarantee only at the maturity of the plan, while the life insurance policies provide a guarantee at maturity or prior death: otherwise the two insurance products are similar.

This paper has two objectives: the first is to determine the equilibrium value of an equity-linked life insurance policy with an asset value guarantee (ELPAVG): this value is equilibrium in the sense that, given perfect frictionless securities markets, an insurance company which sold policies at this price would make no abnormal profits and losses, so that the equilibrium value corresponds to the perfectly competitive price. The second objective is to determine the appropriate investment strategy for the insurance company to follow in order to minimize its risk exposure.

The actuarial literature is replete with articles discussing the pricing of the asset value guarantee:[1] the simulation of investment performance and underwriting results appears to be the most popular technique; however, Kahn (1971) has used an approach identical to Sprenkle's (1961) warrant valuation model to value the guarantee on a single premium policy. The approach adopted here is to recognize that the benefit payable under an ELPAVG is equivalent to the guaranteed amount plus the value of an immediately exercisable call option on the reference portfolio with an exercise price equal to the guaranteed amount: the equilibrium value of this call option may then be determined by techniques derived from the option pricing model of Black–Scholes (1973) and Merton (1973), and hence the cost of providing the guarantee may be calculated.

The provision of asset value guarantees involves insurance companies in the acceptance of risks quite unlike those which they have traditionally borne, in that the investment risk is a non-diversifiable risk: a stock market collapse will render the company simultaneously liable under the guarantees of all its expiring

[1]E.g., Turner (1969, 1971), Leckie (1972), Di Paolo (1969).

policies. This has been a matter of some concern to actuaries and regulatory bodies concerned with the solvency of insurance companies.[2]

We shall show that an arbitrage or hedging strategy along Black–Scholes (1973), Merton (1973) lines enables the insurance company to eliminate these risks, much as immunization of its fixed value assets and liabilities allows an insurance company to hedge itself against unanticipated interest rate charges. It is to be hoped that once regulatory authorities become aware that the risks can be substantially eliminated in this manner, they will adopt a more positive attitude to the provision of these guarantees.

The remainder of the paper is organized as follows: in section 2, for a known date of policy expiration, the ELPAVG is shown to be equivalent either to a plan providing a fixed benefit plus a call option, or to a plan providing a benefit of the value of the reference portfolio plus a put option. In section 3, the equations determining the value of the call options are presented: an explicit formula is given in the case of the single premium policy, while for the periodic premium policy the differential equation and boundary conditions satisfied by the call option are given: a numerical procedure for solving this differential equation is presented in section 6. Section 4 allows for mortality risk and shows how the premia must be determined for the periodic premium contract, so that, allowing for mortality, the present value of the expected premia is equal to the present value of the expected benefits. Section 5 discusses the derivation of the insurance company's riskless investment strategy.

2. The guarantee as an option

Deferring until section 4 the problem of mortality, we start by assuming that the policy matures at a known date, t, and that the benefit then payable consists of the greater of the value of some reference portfolio and some minimum guarantee. The reference portfolio is typically a portfolio formed by investing some pre-determined component of the policyholder's premium in common stocks. It is clearly not essential that the premiums actually be so invested; all that is required is that they are deemed to be invested in computing the policyholder's benefit. Indeed we shall find below in section 5 that it is in general not optimal for the insurance company to invest the whole of the investment component of the premium in the reference portfolio.

Then define:

[2]The Canadian Federal Department of Insurance, out of concern for insurance company solvency, has prohibited guarantees on policies with a term of less than 10 years, and limits the maximum guarantee to 100 percent of the gross premia paid. In the prospectus for the Harleysville Insurance Company's insured mutual fund redemption value program, the potential purchaser is warned that 'The greatest risk to participants in the program would appear to be a period of adverse general economic conditions of substantial duration', i.e., precisely when the guarantee is most likely to be required!

$x(t)$	the value of the reference portfolio at time t;
$g(t)$	the minimum guaranteed benefit;
$b(t)$	the benefit payable at time t;
$V_\tau(b(t))$	the market value at time τ of the uncertain benefit, $b(t)$, payable at time t;
$V_\tau(x(t))$	the market value at time τ of the right to receive the uncertain value of the reference portfolio, $x(t)$ at time t;
$W(x(\tau), t-\tau, g(t))$	the value at time τ of an option to purchase the reference portfolio at time t for the exercise price, $g(t)$;
$p(x(\tau), t-\tau, g(t))$	the value at time τ of an option to sell the reference portfolio for the exercise price, $g(t)$;
r	the known, constant, risk-free rate of interest.

The value of the benefit payable, $b(t)$, is given by

$$b(t) = \max [x(t), g(t)], \tag{1}$$

which may be written alternatively as

$$b(t) = g(t) + \max [x(t) - g(t), 0], \tag{2}$$

or

$$b(t) = x(t) + \max [g(t) - x(t), 0]. \tag{3}$$

Eq. (2) expresses the benefit as the sum of the sure amount $g(t)$ and the value of an immediately exercisable call option to purchase the reference portfolio for the price $g(t)$, $W(x(t), 0, g(t))$, while (3) expresses the benefit as the sum of the uncertain amount $x(t)$ and the value of an immediately exercisable put option on the reference portfolio with exercise price $g(t)$, $p(x(t), 0, g(t))$.

Then corresponding to (2) and (3) the present value of the benefit at time $\tau = 0$ may be written as

$$V_0(b(t)) = g(t)e^{-rt} + W(x(0), t, g(t)), \tag{4}$$

or

$$V_0(b(t)) = V_0(x(t)) + p(x(0), t, g(t)). \tag{5}$$

If $W(x(0), t, g(t))$ can be determined, then (4) may be used to calculate the equilibrium price the insurance company should charge for providing the uncertain benefit, $b(t)$. The next section will consider the determination of $W(x(0), t, g(t))$.

Note that $V_0(x(t))$ is equal to the present value of the investments to be made in the reference portfolio. This is because, while the amount $x(t)$ is uncertain, it can be purchased by following the known investment policy which defines the reference portfolio. Hence, from (5), $p(x(0), t, g(t))$ is the amount over and above the amount to be invested in the reference portfolio that the insurance company must charge for providing the guarantee. This guarantee premium may be found by equating (4) and (5),

$$p(x(0), t, g(t)) = g(t)e^{-rt} - V_0(x(t)) + W(x(0), t, g(t)). \tag{6}$$

The first two terms on the right-hand side are known, and the valuation of the call option $W(x(0), t, g(t))$ will be considered below. Given this, (6) is an expression for the premium to be charged for providing the guarantee: this premium is equal to the price of a put on the reference portfolio.

3. Equilibrium pricing of puts and calls on investment portfolios

An equity-linked life insurance policy typically calls either for a single investment to be made in the reference portfolio at the time the policy is purchased (the single premium contract) or for a regular series of periodic investments spread over the life of the contract (the periodic premium policy). We shall distinguish between these below.

At any instant at which no investment is being made in the reference portfolio, the value of the portfolio, x, is assumed to follow the stochastic differential equation

$$dx/x = \alpha d\tau + \sigma dz, \tag{7}$$

where dz is a Gauss–Wiener process and $E[dz] = 0$; $E[dz^2] = d\tau$; α is the instantaneous expected rate of return on the portfolio; and σ^2 is the instantaneous variance of the rate of return.

The market value at time τ of a call option on the reference portfolio exercisable at time t, $W(x(\tau), t-\tau, g(t))$ is assumed to depend only on the current value of the reference portfolio, $x(\tau)$, the remaining time to maturity, $t-\tau$, and the exercise price $g(t)$, as well as implicitly upon the remaining investments to be made in the reference portfolio.

Then consider the hedging strategy of forming a portfolio consisting of investments in the reference portfolio, the call option and the risk-free asset, in such proportions that the net investment is zero and the return on the portfolio is non-stochastic.[3] Then in equilibrium, the return on this portfolio must be zero, and as Black–Scholes (1973), Merton (1973) and Black (1974) have shown, this implies that between the dates of deemed contribution to the reference portfolio, the value of the call option must satisfy the partial differential equation

$$\tfrac{1}{2}\sigma^2 x^2 W_{xx} + rxW_x - Wr + W_\tau = 0, \tag{8}$$

[3]See Black–Scholes (1973) and Merton (1973) for an assurance that this is possible.

where the subscripts denote partial derivatives. In addition, the value of the call option must satisfy certain boundary conditions:

(i) At expiration, $\tau = t$,

$$W(x(t), 0, g(t)) = \max\,[x(t)-g(t), 0]. \tag{9}$$

This is the standard condition for a call option, that its value at maturity is equal to the greater of its value when exercised and zero.

(ii) At any time at which a contribution is deemed to be made to the reference portfolio,

$$W(x(\tau^-), t-\tau^-, g(t)) = W(x(\tau^+), t-\tau^+, g(t)), \tag{10}$$

where $[x(\tau^+)-x(\tau^-)] = D(\tau)$ is the amount of the contribution to the reference portfolio, and τ^- and τ^+ denote the instants immediately before and after the deemed contribution respectively. (10) is a statement of the obvious condition that the value of the option is unaffected by the fully anticipated contribution to the reference portfolio, $D(\tau)$.

(iii) At any time prior to maturity,

$$\lim_{x(\tau)\to\infty} W_x(x(\tau), t-\tau, g(t)) = 1. \tag{11}$$

The proof of this parallels Merton's (1973) demonstration of the corresponding proposition for a stock option. First, W is a convex function of the value of the reference portfolio; secondly, since the reference portfolio pays no dividends, it would never pay to exercise the option prior to maturity, were that possible, which implies that the value of the option is bounded from below by its exercise value $x(\tau)-g(t)$. Finally, the option is bounded from above by the sum of the present value of the remaining contributions to be made to the reference portfolio and the current value of the reference portfolio. These three conditions jointly imply (11).

(iv) When there are no further contributions to be made to the reference portfolio,

$$W(0, t-\tau, g(t)) = 0. \tag{12}$$

The economic justification for this boundary condition is that if the value of the reference portfolio is zero and there are no further contributions to be made, its value at maturity will be zero with certainty, which implies that the value of the call option must be zero. Condition (12) holds everywhere for the single premium contract, while for the periodic premium contract it holds only after the last contribution has been made to the reference portfolio. The corresponding boundary condition for the periodic premium contract prior to the date of the final contribution to the reference portfolio is given by:

(v) At any time prior to the final contribution to the reference portfolio,

$$W_\tau(0, t-\tau, g(t)) = rW(0, t-\tau, g(t)). \tag{13}$$

This condition may be obtained by setting $x = 0$ in (8). Its economic justification is that if the value of the reference portfolio falls to zero, its value at the next contribution date will be equal to the value of the known contribution. This implies that the value of the option on that date is also known, so that over the interval up to the next contribution date the option is a riskless security: it must therefore earn the riskless rate of interest as implied by (13).

The value of the call option for the single premium contract for which there are no contributions to the reference portfolio after the initiation of the contract is given by the solution to the partial differential equation (8) subject to the boundary conditions (9), (11) and (12). This is identical to the problem of valuing a call option on a non-dividend paying stock whose solution, derived by Black–Scholes (1973) and Merton (1973), is that the equilibrium value of the call option is given by

$$W(x(\tau), t-\tau, g(t)) = x(\tau)\, N(d_1) - g(t)\mathrm{e}^{-r(t-\tau)}N(d_2), \tag{14}$$

where

$$d_1 = [\ln x(\tau) - \ln g(t) + (r+\tfrac{1}{2}\sigma^2)(t-\tau)]/\sigma\sqrt{(t-\tau)},$$

$$d_2 = d_1 - \sigma\sqrt{(t-\tau)},$$

$$N(d) = (\sqrt{2\pi})^{-1}\int_{-\infty}^{d} \mathrm{e}^{-\frac{1}{2}\lambda^2}\,\mathrm{d}\lambda.$$

The value of the call option at the time the contract is issued, $\tau = 0$, is obtained by setting $\tau = 0$ in (14) and noting that $x(0) = D$, the value of the deemed contribution to the investment portfolio.

For the periodic premium case (8) must be solved subject to the boundary conditions (9), (10), (11), (12) and (13). Since there exists no known analytic solution in this case, resort must be made to numerical methods, to be described in section 6 below. The value of the call option at time of issue, $\tau = 0$, may then be determined.

Given the value of the call option at $\tau = 0$, the corresponding value of the put option may be derived from eq. (6). This represents the amount the investor must pay in excess of the deemed investment in the reference portfolio in return for the guarantee. In the case of a single premium contract this would ordinarily be paid in a lump sum as part of the single premium, whereas for a multiple premium contract it would be converted into an equivalent annuity and paid over the life of the contract.

Thus far we have assumed that the contract expires on a known date. While this is true of the insured mutual fund redemption value program referred to above, the expiration of life insurance contract depends upon mortality considerations to which we now turn.

4. Mortality risk

The benefit payable under an ELPAVG if the policy expires in year t either through maturity or premature death is given from section 2 by

$$b(t) = x(t)+p(x(t), 0, g(t)), \tag{15}$$

where $p(x(t), 0, g(t))$ denotes the value at time t of an immediately exercisable put option on the reference portfolio.

Consider first the value of the reference portfolio, $x(t)$. The value of $x(t)$ is uncertain, as also is the date at which it becomes payable because of the mortality risk. However, the insurance company is assured of being able to pay this amount on the uncertain date of policy expiration if it follows the policy of investing in the reference portfolio the amounts deemed to be invested in the portfolio under the policy contract. Hence the contract premium must be equal to the sum of the amount to be invested in the reference portfolio and an amount sufficient to cover the cost of providing the put option. Our problem then is that of determining how much the insurance company should charge in excess of the investment in the reference portfolio in return for providing the put option, given that the date at which the put option is exercised is uncertain.

Let $\alpha(\tau, t)$, for $t = 1, \ldots, T-1$, denote the probability that a given policyholder, alive in year τ, will die in year t, and let $\alpha(\tau, T)$ be given by $(1-\sum_{t=1}^{T-1} \alpha(\tau, t))$. Thus if T is the term of the policy, $\alpha(0, t)$, $t = 1, \ldots, T$, represents the probability that the policy will expire in year t, given that the policyholder is alive at the time the policy is written, $\tau = 0$. $\alpha(0, t)$ will of course depend upon the age of the policyholder at the time the policy is written, as well as on sex, race and the other factors considered by actuaries in constructing mortality tables.

In this paper we follow standard actuarial practice by assuming that sufficient contracts are written to eliminate mortality risk. Then the average purchaser of a policy of a given age may be thought of as purchasing a contract whose benefits in excess of the value of the reference portfolio are $[\alpha(0, t)\ p(x(t), t, g(t))]$, $t = 1, \ldots, T$. Therefore the present value of the premia which must be paid for provision of this guarantee or overall put option is given by

$$P = \sum_{t=1}^{T} \alpha(0,t) p(x(0), t, g(t)), \tag{16}$$

and (16) represents the present value of the payments which must be made by the policyholder in excess of the amounts deemed to be invested in the reference portfolio.

We have now determined how much the insurance company should charge for a single premium policy. This is equal to the sum of the amount deemed to be invested in the reference portfolio, D, and the amount P, given by (16), which is the cost of the overall put option. P itself depends on the mortality factors $\alpha(0, t)$ and the present value of individual put options of known maturity on a reference portfolio to which no further contributions will be made after the commencement of the policy. This in turn is given by (6) and (14).

For the periodic premium policy the cost of the overall put option is still given by (16), and the value of an individual put option of known maturity is given by (6). However $W(x(0), t, g(t))$ must be determined by solving the differential equation (8) subject to the relevant boundary conditions by numerical methods to be discussed in section 6. Finally, since a periodic premium contract typically calls for a series of level premia, we must determine by how much the periodic premium must exceed the deemed periodic contribution to the reference portfolio, D, to cover the cost of the overall put option, P.

Let Y denote the annual premium charged for the overall put option. Then, allowing for mortality factors, the present value of the payments made for the put contract is

$$Y \sum_{t=1}^{T} \alpha(0,t) \sum_{k=0}^{t-1} \mathrm{e}^{-rk},$$

where it is implicitly assumed that, if the policyholder dies in period t, no payment is made. Since this present value must equal P, we have

$$Y = P \Big/ \sum_{t=1}^{T} \alpha(0,t) \sum_{k=0}^{t-1} \mathrm{e}^{-rk}, \tag{17}$$

and the total periodic premium for the contract is $(D+Y)$.

In this section we have determined the equilibrium premia that an insurance company must charge for an ELPAVG. These premia are equilibrium in the sense that if they are charged and the insurance company follows the hedging strategy referred to in section 3 it will incur neither profits nor losses. In the following section we consider the hedging strategy in more detail.

5. The riskless investment strategy

The basis for the differential equation (8), which describes the behavior of the call option price, is that it is possible to form a portfolio whose return is non-stochastic, by combining investments in the call option and in the reference portfolio. As Black–Scholes (1973) and Merton (1973) demonstrate, such a hedge position can be created by selling short one call option and investing an amount to $x(\tau)W_x(x(\tau), t-\tau, g(t))$ in the reference portfolio. Now, since, as eq. (2) shows, the benefit payable under an ELPAVG can be decomposed into the guaranteed amount $g(t)$, and an immediately exercisable call option on the reference portfolio with exercise price $g(t)$, an insurance company which has sold such a contract with a known date of expiration is implicitly short one call option: it can therefore eliminate the risk associated with this short position by taking an appropriate offsetting long position in the reference portfolio. Then, taking into account mortality factors, the insurance company which has sold an ELPAVG is implicitly short $\alpha(\tau, t)$ call options on the reference portfolio of maturity t ($t = \tau+1, \ldots, T$), so that the amount to be invested in the reference portfolio under the riskless investment strategy at time τ, $H(x(\tau), \tau)$, is

$$H(x(\tau), \tau) = x(\tau) \sum_{t=\tau+1}^{T} \alpha(\tau, t) W_x(x(\tau), t-\tau, g(t)). \tag{18}$$

Note that the riskless investment strategy is a function both of time and of the current value of the reference portfolio. If the insurance company continuously maintains its investment in the reference portfolio at the level indicated by (18), and charges the equilibrium price for its insurance contract as determined in sections 3 and 4, then it will make neither profits nor losses, and, ignoring the possible risks of mortality experience, it will have completely eliminated the risks associated with the provision of the asset value guarantee. This is an important result, since, as mentioned in section 1, the risk to company solvency of granting such guarantees has been a cause of major concern, and these guarantees do pose a substantial risk to solvency since they are non-diversifiable risks, unless the investment strategy outlined above is followed.

For the single premium policy the results of Black–Scholes (1973) and Merton (1973) may again be adduced to show that

$$W_x(x(\tau), t-\tau, g(t)) = N(d_1), \tag{19}$$

where $N(d_1)$ is as defined in eq. (14). Substitution from (19) in (14) yields an analytical expression for the riskless investment strategy.

For the periodic premium policy, $W_x(x(\tau), t-\tau, g(t))$ must be evaluated by numerical methods. As explained below, the algorithm calculates the values of $W(x(\tau), t-\tau, g(t))$ for discrete values of $x(\tau)$ and τ: the partial derivative may then be approximated by a finite difference.

The feasibility of the riskless investment policy for an insurance company which is prohibited from borrowing depends upon whether the policy requires the company to borrow in order to make the necessary investment in the reference portfolio. We therefore consider briefly the borrowing requirement under the riskless investment strategy.

If the company charges the equilibrium price for the ELPAVG and follows the riskless investment strategy then by definition it will make no profits or losses, so that the present value of its assets and liabilities will always be equal. The present value of its liabilities is the present value of the mortality weighted benefits as given in eq. (2),

$$\sum_{t=\tau+1}^{T} \alpha(\tau, t) g(t) \mathrm{e}^{-r(t-\tau)} + \sum_{t=\tau+1}^{T} \alpha(\tau, t) W(x(\tau), t-\tau, g(t)),$$

while the present value of its assets is

$$C(x(\tau), \tau) + (D+Y) \sum_{t=\tau+1}^{T} \alpha(\tau, t) \sum_{t=0}^{t-1} \mathrm{e}^{-rk},$$

where the first term is the total investible funds at the company's disposal relating to this contract, and the second term is the present value of the mortality weighted future premia payable: this second term is of course zero for the single premium contract. Equating these two expressions, the amount of available investible funds is given by

$$\begin{aligned} C(x(\tau), \tau) = & \sum_{t=\tau+1}^{T} \alpha(\tau, t)[g(t) \mathrm{e}^{-r(t-\tau)} + W(x(\tau), t-\tau, g(t))] \\ & -(D+Y) \sum_{t=\tau+1}^{T} \alpha(\tau, t) \sum_{k=0}^{t-1} \mathrm{e}^{-rk}, \end{aligned} \tag{20}$$

and the company's net lending, $L(x(\tau), \tau)$, is equal to the amount of investible funds less the required investment in the reference portfolio,

$$L(x(\tau), \tau) = C(x(\tau), \tau) - x(\tau) \sum_{t=\tau+1}^{T} \alpha(\tau, t) W_x(x(\tau), t-\tau, g(t)). \tag{21}$$

For the single premium contract, substituting from (14), (19) and (20) in (21) and using the fact that the present value of the future premia payable is zero, yields

$$L(x(\tau), \tau) = \sum_{t=\tau+1}^{T} \alpha(\tau, t) g(t) \mathrm{e}^{-r(t-\tau)}[1 - N(d_2(\tau, t))] \geqq 0, \tag{22}$$

where

$$d_2(\tau, t) = [\ln x(\tau) - \ln g(t) + (r - \tfrac{1}{2}\sigma^2)(t-\tau)]/\sigma \sqrt{(t-\tau)}.$$

It is shown in the appendix that for the periodic premium contract the borrowing requirement is bounded from above by the present value of the future premia payable in excess of the amounts deemed to be invested in the reference portfolio, i.e.,

$$L(x(\tau),\tau) \geqq -Y \sum_{t=\tau+1}^{T} \alpha(\tau,t) \sum_{k=0}^{t-1} \mathrm{e}^{-rk}. \tag{23}$$

Thus for the single premium contract the riskless investment strategy can always be affected without borrowing, while for the periodic premium contract the maximum borrowing requirement is likely to be modest.

6. The solution algorithm for the periodic premium contract

While explicit expressions have been derived for the single premium contract, valuation of the multiple premium contract requires a solution to the differential equation (8) subject to the boundary conditions (9), (10), (11), (12) and (13). Since no analytical solution exists, numerical methods must be employed to determine $W(x(\tau), t-\tau, g(t))$. Letting u ($\equiv t-\tau$) represent the time to expiration of the option, (8) may be re-written as[4]

$$\tfrac{1}{2}\sigma^2 x^2 W_{xx} + rxW_x - rW - W_u = 0. \tag{8'}$$

Then by writing finite differences in place of partial derivatives in (8′), the differential equation can be approximated by

$$a_i W_{i-1,j} + b_i W_{i,j} + c_i W_{i+1,j} = W_{i,j-1}, \qquad i = 1,\ldots,n-1$$
$$j = 1,\ldots,m, \tag{24}$$

where

$$a_i = \tfrac{1}{2}ki(r-\sigma^2 i),$$
$$b_i = (1+rk)+\sigma^2 k i^2,$$
$$c_i = -\tfrac{1}{2}ki(r+\sigma^2 i),$$

$$W(x,u) = W(x_i,u_j) = W(ih,jk) = W_{i,j},$$

and $W(x,u)$ is the value of the option when the value of the reference portfolio is x, and the time to expiration is u; h and k are the discrete increments in the value of the reference portfolio and in time to expiration, respectively. By reducing these step sizes any desired degree of accuracy in the solution may be achieved, but at the expense of increased computational cost; n and m represent

[4]See McCracken and Dorn (1964) for a detailed explanation of the solution procedure.

the number of steps in the time dimension and the reference portfolio value dimension, respectively. The former is chosen to correspond to the maturity of the option under consideration, while the latter must be sufficiently large for the boundary condition (11) to be well approximated at the maximum reference portfolio value considered.

Boundary condition (11) may be written in finite difference form as

$$-W_{n-1,j}+W_{n,j}=h, \qquad j=1,\ldots,m. \tag{25}$$

Boundary conditions (12) and (13), relating respectively to the time after and prior to the date of the final deemed contribution to the reference portfolio, may be written as

$$W_{0,j}=0, \qquad j<j^*, \tag{26}$$

$$W_{0,j}=(1-rk)W_{0,j-1}, \qquad j>j^*, \tag{27}$$

where j^* corresponds to the date of the final deemed contribution to the reference portfolio.

Then the $(n-1)$ eq. (24), together with eq. (25) and eq. (26) or (27) depending on the value of j, constitute a system of $(n+1)$ linear equations in the $(n+1)$ unknowns $W_{i,j}$, $i=0,1,\ldots,n$. The values of $W_{i,0}$, $i=0,\ldots,n$, are given from the boundary condition at expiration, (9),

$$\begin{aligned} W_{i,0} &= ih-g, && i \geqq g/h, \\ &= 0, && i < g/h. \end{aligned} \tag{28}$$

Hence by repetitive solution of the set of linear equations values of $W_{i,j}$, $i=0,\ldots,n$ and $j=1,2,\ldots,m$, may be determined.

When the date of a deemed contribution to the reference portfolio is reached, boundary condition (10) is introduced by the substitution

$$W_{i,j}=W_{i+D/h,j}, \qquad i=0,1,\ldots,(n-D/h). \tag{29}$$

The values of $W_{i,j}$ on the right-hand side of (29) represent the post-contribution values which have been generated by our algorithm which works backwards from time of expiration, while the values on the left-hand side represent the pre-contribution values. The remaining pre-contribution values $(i=n-D/h,\ldots,n)$ are then generated using a modified version of the boundary condition (25) which reflects the fact that $W_{i,j}$ is linear in i for sufficiently high values of i.

7. Numerical examples

The examples which follow relate to the periodic premium policy which is the most common type of policy issued. In addition certain other standard assumptions are made throughout:

(i) The investment component of the periodic premium is $100.
(ii) The premiums are paid annually at the beginning of the year.
(iii) If death takes place during a year, it is assumed to occur at the end of the year.
(iv) The mortality table used is the Canadian Assured Lifes 1958–64 Select for males.
(v) The guaranteed amount is 100 percent of the investment component of the premia paid up to the expiration of the policy: the policyholder is thus guaranteed that the return on his deemed investment in the reference portfolio will not be negative.

The additional parameters of the basic example are given in table 1. The variance rate corresponds to the historical variance rate on the Toronto Stock Exchange Industrials Index for the period September 1968–August 1973.

Table 2 displays the results of an intermediate stage of the calculations for the basic example in which mortality is ignored; thus each line of the table corresponds to a policy with a known, predetermined time to expiration. The second column gives the present value of the call option on the reference portfolio with exercise price equal to the guarantee; for example the exercise price for the five-year call would be $500 since this is the amount deemed to have been invested in the reference portfolio by the end of year 5. The third column is simply the present value of the amount guaranteed. The fourth column, the present value of the contract is simply the sum of the present value of the guarantee and the present value of the call option [see eq. (2)]. The fifth column, the present value of the reference portfolio is what we have designated $V_0(x(t))$ and is simply equal to the present value of the investments deemed to be made in the reference portfolio. The present value of the put contract, determined from eq. (5), is equal to column (4) less column (5). The three remaining columns are simply the annuitized values of columns (4), (6) and (2), respectively.

The data in table 2 are fundamental in the sense that, given these data for a particular reference portfolio, interest rate and guarantee, the equilibrium premium on a policy of any given term on a life of any given age may be determined by combining these data with the mortality factors as described in section 4. Our main focus of interest is the annual premium on the put, column (8), since this is the amount which must be paid in excess of the deemed investment in the reference portfolio. Notice that the annual put premium is never more than 3.5 percent and declines towards 1 percent as the maturity of the contract is increased.

Table 1
Parameters of basic example.

Variance rate on the reference portfolio:	0.01846 per annum
Risk-free rate:	0.04 per annum compounded continuously
Guarantee:	100% of the investment component of the premiums paid at the time of expiration

Table 2
Basic example: Intermediate results.[a]

No. of years to expiration (1)	Value call (2)	PV guarantee (3)	Total PV contract (4)	PV reference portfolio (5)	Value put (6)	Total annual premium (7)	Annual premium put (8)	Annual premium call (9)
1	7.4	96.1	103.5	100.0	3.5	103.5	3.5	7.4
5	67.8	409.4	477.1	462.3	14.8	103.2	3.2	14.7
10	193.5	670.3	863.8	840.8	23.0	102.7	2.7	23.0
15	353.2	823.2	1176.4	1150.7	25.8	102.2	2.2	30.7
20	530.8	898.7	1429.5	1404.4	25.1	101.8	1.8	37.8

[a]In this table the date of expiration of the contract is assumed to be known. The value of the contract [col. (4)] is expressed as the sum of col. (2) and (3) or col. (5) and (6). The level annual premium [col. (7)] is equal to the investment component of $100 plus the annual put premium [col. (8)].

Table 3a shows the results of combining the data of table 2 with the mortality factors to obtain the overall annual put premium. Not surprisingly we find that it declines with the term of the contract. The put premium increases with the age of the purchaser at entry essentially because the older is the purchaser the less is likely to be the effective term of the policy, and of course this effect is more pronounced for longer-term policies which take the policyholder into the years of high mortality. To obtain the annual premium on a policy with a $100 annual investment component one simply adds $100 to the annual put premiums given in table 3a.

Table 3b gives the annual put premium on the basic policy when the interest rate is assumed to be 8 percent per annum rather than 4 percent. The sensitivity of the model to the interest rate assumption is striking and it should be borne in mind that a limitation of this model is its assumption of a known, constant

Table 3a
Level of annual put premium (dollar): Basic example.[a]

Term of contract (in years)	Age of purchaser at entry (years)			
	20	30	40	50
10	2.74	2.74	2.75	2.75
15	2.24	2.25	2.26	2.29
20	1.80	1.80	1.83	1.90

[a]The put premium is the amount in excess of the $100 annual investment which must be paid for the guarantee.

Table 3b
Level of annual put premium (dollar): Risk-free rate = 0.08 per annum.

Term of contract (in years)	Age of purchaser at entry (years)			
	20	30	40	50
10	0.39	0.40	0.40	0.41
15	0.16	0.17	0.17	0.20
20	0.10	0.11	0.11	0.14

Table 3c
Level of annual put premium (dollar): Variance rate = 0.04 per annum.

Term of contract (in years)	Age of purchaser at entry (years)			
	20	30	40	50
10	2.04	2.04	2.05	2.07
15	1.32	1.32	1.34	1.39
20	0.88	0.89	0.92	1.02

risk-free rate of interest. However, insurance companies are accustomed to making interest rate assumptions in pricing their more traditional products.

In table 3c the effect of varying the basic example by increasing the assumed variance rate from 0.01864 to 0.04 is shown. It is well known that an increase in the variance rate increases the value of a call option and must therefore decrease the value of a put option[5] and this effect is apparent in the table.

Table 4
Ratio of actual to deemed investment in the reference portfolio (%) under riskless investment strategy: $H(x(\tau), \tau)/x(\tau)$.

	Annual increase in value of reference portfolio (%)								
Year	−20	−15	−10	−5	0	5	10	15	20
0	85	85	85	85	85	85	85	85	85
1	83	83	84	84	85	85	86	86	87
2	77	79	81	82	84	85	87	88	89
3	68	72	76	79	83	85	88	91	92
4	55	62	69	75	81	86	90	93	97
5	37	42	59	69	80	87	93	96	98
6	18	29	44	61	76	88	95	99	100
7	4	11	26	49	73	90	98	100	100
8	0	1	9	32	70	94	100	100	100
9	0	0	0	11	64	89	100	100	100

Table 5
Net lending under riskless investment strategy: $L(x(\tau), \tau)$ (dollar).

	Annual increase in value of reference portfolio (%)								
Year	−20	−15	−10	−5	0	5	10	15	20
0	17	17	17	17	17	17	17	17	17
1	41	39	38	38	37	35	34	34	32
2	76	72	68	63	58	54	50	46	41
3	132	120	108	97	84	72	60	49	39
4	221	197	169	144	114	86	64	43	27
5	347	308	258	205	150	100	59	29	9
6	511	461	392	294	194	104	43	11	−3
7	678	646	567	423	246	98	22	−1	−4
8	823	817	774	615	311	71	4	−4	−3
9	961	961	958	880	395	25	0	0	0

The riskless investment strategy is evaluated using the data of the basic example and assuming a policy with a 10-year term sold to a 40-year old male. Table 4 shows the ratio between the amount actually invested in the reference portfolio under the riskless strategy, $H(x(\tau), \tau)$, and the amount deemed to be invested in the reference portfolio, $x(\tau)$. From eq. (18), this ratio is given by

$$H(x(\tau), \tau)/x(\tau) = \sum_{t=\tau+1}^{T} \alpha(\tau, t) W_x(x(\tau), t-\tau, g(t)), \tag{30}$$

[5]For a European option which is the relevant one here, the difference between the call price and the put price is the interest earned on the stock price over the period to expiration. See Stoll (1969).

and the table shows the values of this ratio at different stages in the contract life assuming different rates of return on funds deemed to be invested in the reference portfolio. For negative rates of return on the reference portfolio, the ratio monotonically approaches zero, reflecting the increasing probability that the insurance company's liability will actually be the guaranteed amount, while for rates of return above zero the ratio monotonically approaches unity as it becomes increasingly probable that the insurance company's liability will actually be the value of the reference portfolio rather than the guaranteed amount.

Table 5 shows the net lending called for by the riskless investment strategy, $L(x(\tau), \tau)$, given by eq. (21). This is also shown for different stages in the contract's life and for different assumed rates of return on the investment portfolio. As shown in the appendix the net lending is a decreasing function of the value of the reference portfolio and hence of the assumed rate of return on the reference portfolio. As suggested earlier, the maximum borrowing requirement is modest, being limited in this example to 4 percent of the annual deemed investment in the reference portfolio. It seems unlikely therefore that borrowing constraints could significantly impede the ability of an insurance company to pursue this riskless investment strategy.

Appendix

In this appendix we offer a proof of eq. (23) which places an upper bound on the maximum amount of borrowing required under the periodic premium contract. Substituting from (20) in (21),

$$\begin{aligned} L(x(\tau), \tau) = \sum_{t=\tau+1}^{T} & \alpha(\tau, t)[g(t)\mathrm{e}^{-r(t-\tau)} + W(x(\tau), t-\tau, g(t)) \\ & - x(\tau)W_x(x(\tau), t-\tau, g(t))] \\ & -(D+Y)\sum_{t=\tau+1}^{T} \alpha(\tau, t)\sum_{k=0}^{t-1} \mathrm{e}^{-rk}. \end{aligned} \tag{31}$$

Differentiating with respect to $x(\tau)$,

$$\delta L/\delta x = -\sum \alpha(\tau, t)[x(\tau)W_{xx}(x(\tau), t-\tau, g(t))] < 0, \tag{32}$$

since $W_{xx} > 0$ [see Merton (1973)].

Hence net lending decreases as $x(\tau) \to \infty$. But in the limit as $x(\tau) \to \infty$, $W_x \to 1$ from eq. (11), so (31) becomes

$$\begin{aligned} \lim_{x(\tau)\to\infty} L(x(\tau), \tau) = \sum_{t=\tau+1}^{T} & \alpha(\tau, t)[W(x(\tau), t-\tau, g(t)) - x(\tau) \\ & + g(t)\,\mathrm{e}^{-r(t-\tau)} - (D+Y)\sum_{k=0}^{t-1} \mathrm{e}^{-rk}]. \end{aligned} \tag{33}$$

But

$$W \geqq x - g\,\mathrm{e}^{-r(t-\tau)} + D \sum_{k=0}^{t-1} \mathrm{e}^{-rk}, \tag{34}$$

since the expression on the right-hand side of (34) is the value of a contract to buy the reference portfolio at time t for the exercise price g at time t, while W is the value of an option to buy the portfolio.

Hence

$$\lim_{x(\tau)\to\infty} L(x(\tau),\tau) \geqq -Y \sum_{t=\tau+1}^{T} \alpha(\tau,t) \sum_{k=0}^{t-1} \mathrm{e}^{-rk}. \tag{35}$$

Eqs. (34) and (35) jointly imply (23).

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