# THE TERM STRUCTURE OF DISCOUNT RATES

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ABSTRACT

Practical implementations of asset pricing models to estimate discount rates for real investment projects typically assume that risk premia are constant over time, and treat the time variation in riskless interest rates in an ad hoc fashion. This paper considers the implications of recent evidence of time-varying risk premia, and shows how a model of time variation in risk premia and riskless interest rates may be used to derive discount rates for multi-period cash flows within the CAPM framework. The assumption that cash flow claims have constant betas is shown to imply that cash flows are non-linear functions of the value of the market portfolio, and the valuation of cash flows that are linear functions of the market portfolio is analyzed.

## THE TERM STRUCTURE OF DISCOUNT RATES

The Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) provide the theoretical basis for the estimation of discount rates for real investment projects. Three practical problems arise in applying the results of these models either to estimate discount rates for real projects or for estimating the cost of capital for use in utility rate regulation. The first is the determination of the appropriate beta coefficient(s); the second is the determination of the appropriate risk free interest rate, and third is an appropriate estimate of the risk premium(s). In this paper our primary focus is on the last two problems.

The CAPM and the APT are *single period* models of market equilibrium, and the difficulties arise in applying them in a reasonable fashion in a multi-period context[1]. Consider first the issue of what proxy to use for the riskless interest rate. Both the CAPM and the APT imply that the riskless rate is the one period, or instantaneously, riskless rate which, in applied work, is typically proxied by the rate on a one month Treasury Bill. However, it is unlikely that the riskless rate will remain constant over the life of an investment project, and expectations based theories of the term structure suggest that a steeply sloping yield curve implies that the Bill rate is expected to change. One ad hoc solution that is sometimes employed is to use the forward rates over the life of the project to construct a series of implied riskless rates for each future period. An obvious difficulty with this solution is that forward rates

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[1] Constantinides (1982) gives sufficient conditions for the CAPM to hold in an intertemporal context with stochastic interest rates.

Connor and Korajcyk (1989) develop an intertemporal version of the APT. However, it does not allow for stochastic interest rates. For an early application of asset pricing theory to intertemporal asset valuation see Brennan (1972). Cox Ingersoll and Ross (1984) develop a complete intertemporal asset pricing theory.

embody liquidity premia as well as expected future spot rates. Perhaps more important in applying the CAPM is the relation between the interest rate and the market risk premium. It is conventional to estimate the market risk premium as the long run average excess of the market return over the Treasury Bill return, which typically yields a figure of about 8-9%. However, there is now extensive evidence that the market risk premium is not independent of the level of the short term interest rate[2], and that when short term rates are high the market risk premium is low. So pronounced in fact is this inverse relation between interest rates and the market risk premium that an increase in interest rates not only reduces the market risk premium but also reduces the expected return on common stocks gross of the riskless rate.

A related debate concerning the market risk premium is the question of whether the arithmetic mean or the geometric mean of past (excess) market returns should be used to estimate the market risk premium. On the one hand, most textbooks suggest use of the arithmetic mean[3] while Blume (1974) and Cooper(1993) argue that it is appropriate to use a weighted average of the arithmetic and geometric means to take account of error in estimation of the means. However, both of these procedures implicitly rely on the assumption that (excess) returns on the market portfolio are serially independent. While this random walk assumption was once accepted as a reasonable approximation, there is now extensive evidence of

[2] See Lintner (1975), Fama and Schwert (1977).

[3] Brealey and Myers (1991) offer a justification of this procedure in the Instructor's Manual. However, it assumes that the mean of the market return is a known parameter, and that realized returns are serially independent. Levy and Sarnat (1986) and Copeland, Koller and Mullina (1990) advocate the use of the geometric mean.

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'temporary components' which induce negative autocorrelation[4] in stock returns; this autocorrelation must be taken into account if the long run expected return on the market is to be correctly assessed.[5]

In this paper we present an empirically based, but internally consistent, dynamic model of the behavior of interest rates and the market risk premium that allows for determination of a term structure of discount rates using the capital asset pricing model. The approach is applicable with obvious changes to multi-period applications of the APT. In Section I the logic of multi-period discounting in a capital budgeting context for a cash flow with a constant beta coefficient is developed. Section II presents a model of expected returns on bonds and stocks which takes account of recent empirical findings concerning the predictability of bond and stock returns. Section III demonstrates how to compute the discount function for different values of the beta coefficient, and shows how they vary with the level of interest rates and the dividend yield on the market portfolio. Section IV considers the implications of the constant beta coefficient assumption and suggests that a more reasonable assumption about the stochastic characteristics of cash flows may sometimes involve a stochastically varying beta coefficient.

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[4] See Lo and Mackinlay (1988), Fama and French (1988).

[5] The expected value of a product of random variables is equal to the product of the expected values only if they are uncorrelated.

## I

## THE CONSTANT BETA CASE

For simplicity we shall assume that the CAPM is being used to determine the discount rate, and we shall assume in addition that the beta of the cash flow claim is a known constant, $\beta$. Invoking the principle of value-additivity, it will be sufficient for us to consider the valuation of a single cash flow payable at time T. Let $V_t(X_T)$ denote the value at time t of the claim which pays the random amount $X_T$ at time T. Under the foregoing assumptions, the value of the claim is determined by the boundary condition at maturity:

$$V_T(X_T) = X_T \tag{1}$$

and the ex-post version of the CAPM equation which relates the realized rate of return on the cash flow claim to the realized market return:

$$\begin{aligned} \frac{V_{t+1}(X_T)}{V_t(X_t)} &= 1 + r_{t+1}(1 - \beta) + \beta R_{Mt+1} + \eta_{t+1} \\ &= 1 + R^{\beta}_{t+1} + \eta_{t-1} \end{aligned} \tag{2}$$

where $r_t$ is the riskless rate for period t, $R_{Mt}$ is the return on the market portfolio, and $\eta_t$ is a random error term with mean zero, which is uncorrelated with the return on the market portfolio. $R^{\beta}_t$ is the realized return in period t on a portfolio, the 'beta portfolio', which consists of a fractional investment $\beta$ in the market portfolio and the balance in the riskless security. Define $M_t(\beta)$ as the value at time t of \$1 invested in the beta portfolio at time 0. Then, iterating equation (2), and using the boundary condition (1), we can relate $X_T$, the cash flow realization at time T, to the initial value of the cash flow claim, the time series of returns on the beta portfolio, and the error term, as follows:

$$X_T = V_T = V_0 \prod_{\tau=1}^{\tau=T} [1 + R^{\beta}_{\tau}] + V_0 \sum_{\tau=1}^{\tau=T} \eta_{\tau} \prod_{s=1, s\neq\tau}^{T} [1 + R^{\beta}_{s} + \eta_{s}]$$
$$= V_0 M_T(\beta) + V_0 \sum_{\tau=1}^{\tau=T} \eta_{\tau} \prod_{s=1, s\neq\tau}^{T} [1 + R^{\beta}_{s} + \eta_{s}] \tag{3}$$

Then, taking expectations in (3), and recognizing that $E_0[\eta_t] = 0$, the value of the cash flow at $t = 0$ may be expressed as:

$$V_0 = \frac{E_0[X_T]}{E_0[M_T(\beta)]} \tag{4}$$

Equation (4) has the intuitive interpretation that the ratio of the expected payoff to the value of the claim is equal to the expected payoff of \$1 invested at time 0 in the corresponding beta portfolio. Thus, under the CAPM in the constant beta case, the value of a claim to a cash flow due in T periods is obtained by discounting the expected cash flow by the $T^{th}$ root of the expected compound return on the appropriate beta portfolio, $k_T(\beta)$:

$$k_T(\beta) = \sqrt[T]{E_0\left\{\prod_{t=1}^{t=T} (1 + [R^{\beta}_t])\right\}} - 1$$
$$\equiv \sqrt[T]{E_0[M_T(\beta)]} - 1 \tag{5}$$

Note that this procedure takes account of any serial correlation in market returns. If the expected return on the market is a constant, $E[R_M]$, the returns are serially uncorrelated, and the risk free rate is a constant, r, then (5) reduces to the familiar expression:

$$V_0(X_T) = \frac{1}{(1+k)^T} E_0(X_T)$$

the discount rate k is given by

$$k = r(1 - \beta) + \beta E[R_M]$$

However, if these conditions are not met, then the process which generates market returns must be modelled explicitly in order to estimate $E_0[M_T(\beta)]$ from which the discount rate is computed using equation (5). In the next section we present a model of expected returns on the market portfolio which will allow us to estimate $E_0[M_T(\beta)]$.

## II

## A MODEL OF EXPECTED RETURNS

As an example of a model of expected returns[6] we assume that the expected rate of return on the market portfolio is a linear function of the instantaneous riskless interest rate, $r$, the dividend yield on the market portfolio, $\delta$, and the long term interest rate, $l$, which is taken as the yield on a putative consol bond. The short term interest rate, $r$, is selected as a predictor or state variable because there is extensive evidence that the level of the short rate predicts the expected return on common stock[7]. The second most powerful predictor of stock returns is the dividend yield on common stocks[8], $\delta$, and therefore this is included as the second state variable. Finally, we include the yield on a consol bond, $l$, as the third state variable, because of prior evidence that expected changes in the short rate are related to the current value of the long rate[9]. It is possible to extend the model to include additional state variables such a the junk bond yield spread[10]. However, attention is restricted to three variables in the interest of parsimony.

Denoting the instantaneous rate of return on the market portfolio by $dM/M$,

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[6] Brennan, Schwartz and Lagnado(1993) use this model.

[7] An early study drawing attention to the importance of this variable is Lintner (1975). A more recent study is Keim and Stambaugh (1986). Attempts to account for this empirical regularity include Geske and Roll (1983) and Fama (1981).

[8] See for example Fama and French (1988).

[9] See Brennan and Schwartz (1982); this is a natural implication of expectations based theories of the term structure.

[10] See Keim and Stambaugh (1986) for evidence that the junk bond yield spread predicts stock returns.

the joint stochastic process for the state variables and the market return is assumed to be of the form:

$$\frac{dM}{M} = \mu_M dt + \sigma_M dz_M \tag{6}$$

$$dr = \mu_r dt + \sigma_r dz_r \tag{7}$$

$$dl = \mu_l dt + \sigma_l dz_l \tag{8}$$

$$d\delta = \mu_\delta dt + \sigma_\delta dz_\delta \tag{9}$$

where $dz_M$, $dz_r$, $dz_l$, and $dz_\delta$ are increments to possibly correlated Brownian motions. In order to estimate the joint stochastic process (6)-(9) for the state variables and the stock return, it is necessary to specify the functional forms of the drift and diffusion coefficients. The basic assumption we make is that the expected return on the market portfolio and the consol bond, and the drifts of the dividend yield and short rate, are linear functions of the three state variables, r, l, and $\delta$, while the volatility of each state variable is assumed to be proportional to its current level, and the volatility of the stock rate of return is taken as constant. The assumption that the expected return on the consol bond is a linear function of the state variables implies that the drift of the long rate is a non-linear function of the state variables, being equal to the product of l and a linear function of the state variables[11]. This specification implies that the

[11] This can be seen by using Ito's Lemma to compute the expected return on a consol bond, noting that its price is $l^{-1}$.

joint stochastic process may be written as[12]:

$$\frac{dM}{M} = (a_{M1} + a_{M2}\delta + a_{M3}r + a_{M4}l)dt + \sigma_M dz_M \tag{10}$$

$$dr = (a_{r1} + a_{r2}\delta + a_{r3}r + a_{r4}l)dt + r\sigma_r dz_r \tag{11}$$

$$dl = l(a_{l1} + a_{l2}\delta + a_{l3}r + a_{l4}l)dt + l\sigma_l dz_l \tag{12}$$

$$d\delta = (a_{\delta 1} + a_{\delta 2}\delta + a_{\delta 3}r + a_{\delta 4}l)dt + \delta\sigma_\delta dz_\delta \tag{13}$$

The dividend yield is defined as the sum of the past 12 months' dividends divided by the current level of the stock index, M. The specification (13) must therefore be regarded as an approximation since the stochastic process for lagged dividends is not modelled explicitly. Given the relative stability of the lagged dividend process, the major short run influence on the dividend yield will be the price change in the stock index, so that the stochastic increment to the dividend yield will have a strong negative correlation with the return on the stock, since most of the stock return is accounted for by price changes.

The joint stochastic process was estimated on monthly data for the period January 1972 to December 1992 using a discrete approximation to the continuous process. The stock return was taken as the rate of return on the CRSP

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[12] Purists will note that this model specification may admit negative interest rates. This is easily avoided in practice by adding terms to the drifts of the long and short rates of the form h/l and h/r where h is an arbitrarily small positive number. As Hogan (1993) points out, there is also a possibility of arbitrage opportunities arising if the long rate gets too far above the short rate because a sufficiently high yield on the long bond (relative to the short rate) will allow an investor to buy the long bond and repay his borrowings at the short rate with probability one in a finite time. This can be ruled out by introducing a term in the short rate drift that will accelerate the short rate towards the long rate if the gap between them is large.

Value Weighted Index. The short rate is the annualized yield on a 1 month Treasury Bill which was taken from the CRSP Government Bond Files. The long rate was taken as the yield to maturity on the longest maturity taxable non-callable bond US Government Bond (excluding flower bonds) for which data were available on the CRSP Government Bond File. The dividend yield was defined as the sum of the past 12 months' dividends on the CRSP Value Weighted Market Index divided by the current value of the index.

The system of equations (10)-(13) was estimated by non-linear seemingly unrelated regression using TSP. Table 1 reports the regression estimates and Table 2 contains the estimated correlations of the innovations. As previous investigators have found, the expected return on common stocks is negatively related to the current level of the short rate and positively related to the level of the dividend yield, but it is not significantly related to the long rate. In interpreting the coefficients it should be recognized that yields are measured on an annual basis whereas the return on the market is a monthly return; thus, *ceteris paribus*, a 1% increase in the short rate is associated with a 0.513% reduction in the return on the market *per month*. Clearly, the relation between the level of the short rate and the dividend yield is very important in determining the expected return on the market. As Brennan and Schwartz (1982) have found, the change in the short rate is negatively related to its current level and positively related to the level of the long rate - thus the short rate tends to adjust towards the long rate. The change in the long rate itself is the least predictable of our series, being negatively related to its current level and positively related to the short rate at marginal levels of significance. The change in the dividend yield is negatively related to its current level, so that it shows mean reversion. In addition, the change in the dividend yield is positively related to the short rate: this reflects the fact that high short rates are associated with low stock returns. As anticipated, the innovation in the dividend yield is very highly negatively correlated with the innovation in stock returns; it is also negatively correlated with the innovation in the long rate, which is negatively correlated with the innovation in stock returns. The remaining innovations have very low correlations. This model is consistent with the mean reversion in stock prices that has been reported by several authors[13]; thus a decline in stock prices tends to be associated with an increase in the dividend yield on the market; this, in turn is associated with higher expected returns on stocks in the future.

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[13] See Fama and French (1988).

## III

## COMPUTING THE DISCOUNT RATE FUNCTION

By calculating $k_T(1)$, the discount rate applicable to a cash flow with unit beta due in T periods, for a range of values of T, we are able to compute a whole discount rate function. This is analogous to the term structure of interest rates except that it applies to the expected returns on claims to risky cash flows which have beta equal to unity. We refer to this as the term structure of discount rates. In order to calculate $k_T(1)$ we must evaluate $E_0[M_T]$ using the model of expected returns that we presented in the previous section. However, since the stochastic process for the state variables (10) - (13) is non-linear, $E_0(M_T)$ must be estimated by Monte Carlo simulation. Given the initial values of the state variables r, l, $\delta$, future values of the state variables are simulated monthly for T years and $M_T$, the value at time T of $1 invested in the market portfolio at t = 0, is computed. Averaging these values over a large number of simulations provides an estimate of $E_0(M_T)$. Then $k_T(1)$ is calculated from equation (5).

Figure 1 shows the term structures of discount rates implied by the model for various dates between 1972 and 1991. Just like the term structure of interest rates, the term structure of discount rates shows considerable variability in both level and shape over time; indeed the structures sometimes exhibit the single humped shapes that are characteristic of the term structure of default free interest rates. It is interesting to note that, with the exception of January 1982, the term structure of discount rates tends to converge for long maturities in the region of 13.5-15%. In January 1982 the 30 year discount rate was 17.7% - at this time the yield on long term bonds was

13.7%; thus the implied long run market risk premium was of the order of 4%.

Figure 2 differs from Figure 1 in that the short term interest rate prevailing on the date corresponding to the date of the term structure is subtracted from all discount rates. The variation in the 'implied market risk premia' provides an indication of the extent to which the cost of capital varies independently of the short rate. For long maturities this can be attributed in part to variation in the corresponding long-term spot interest rates. However, for short maturities most of the variation in interest rates is already accounted for by subtracting out the current short rate. The variation in the risk premium for even a one year horizon is striking, ranging from 0.5% in 1991 to 15.2% in 1977. Even allowing for possible model mis-specification and estimation error, it seems unlikely that the current practice of adding a single unconditional mean estimate of the risk premium to the riskless rate can yield an accurate measure of investors' required rates of return.

Figures 3, 4, and 5 plot the term structures of discount rates for three different dates and for different values of the beta coefficients, calculated using equation (5) and the definition of $M_T(\beta)$; as one would expect, the discount rate is increasing in the beta - at the long end, a unit increase in beta is associated with an increase in the discount rate of about 9% for all three dates. However, while the term structures of estimated discount rates for different betas are roughly parallel on two of the dates, in January 1972 they show substantial divergence; there is virtually no premium for beta risk at the short end.

## IV

## CONSTANT BETAS AND LINEARLY DEPENDENT CASH FLOWS

Thus far we have adopted without discussion the usual assumption that the beta of the cash flow claim is an intertemporal constant. It might appear that if a cash flow claim has a constant beta coefficient and the CAPM holds, then the conditional expectation of the cash flow will be a linear function of the return on the market, at least when the riskless rate is constant. However, it is shown in the Appendix that if the continuous time version of the CAPM holds, the riskless rate is constant, and the cash flow claim has a constant beta, then, conditional on the value at time T of $1 invested in the market portfolio at $t = 0$, $M_T(1)$, the expected cash flow is given by:

$$E_0[X_T | M_T(1)] = k[M_T(1)]^{\beta} \tag{16}$$

where k is a constant. Thus the assumption of a constant beta coefficient for the cash flow claim implies that, except in the special case $\beta = 1$, the cash flow itself is a non-linear function of the level of the market portfolio, being either a convex ($\beta > 1$) or a concave ($\beta < 1$) function of the level of the market portfolio; moreover, the cash flow (unlike the instantaneous return) cannot be decomposed into a riskless and risky component - instead, the whole of the cash flow is implicitly assumed to be risky.

An alternative assumption, which under certain circumstances may appear more reasonable to the analyst, is that the cash flow itself, $X_T$, is a linear function of the level of the market portfolio, $M_T(1)$[14], plus noise, $\varepsilon_T$, which is uncorrelated with

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[14] It is straightforward to generalize this analysis to allow the cash flow to depend on <u>past</u> levels of the market portfolio or to allow the cash flow to depend on the level of the market index excluding reinvested dividends.

the return on the market portfolio:

$$X_T = a + bM_T(1) + e_T \tag{17}$$

Figure 6 plots the conditional expectation of the cash flow as a function of the market level for three different cases: cases a) and b) correspond to the constant $\beta$ assumption with $\beta$ equal to 1.5 and 0.5 respectively; case c) corresponds to the specification (17) with $b = 0.8$.

In order to value a claim to a cash flow that is a linear function of the level of the market portfolio as in (17), define $X^*_T \equiv X_T - a$ as the risky component of the cash flow. It follows from (17) that

$$X^*_T = bM_T(1) + e_T \tag{18}$$

Then inspection of equations (3) and (4) reveals that $V_0(X^*_T)$ is given by:

$$V_0(X^*_T) = \frac{E_0[X^*_T]}{E_0[M_T(1)]} = \frac{E_0[X_T] - a}{E_0[M_T(1)]} \tag{19}$$

$V_0(a)$, the present value of a, the riskless component of the cash flow, is simply obtained by discounting at the T-period riskless interest rate, $r_T$. Therefore the present value of the total cash flow, $X_T$, may be written as:

$$\begin{aligned} V_0(X_T) &= V_0(X^*_T) + V_0(a) \\ &= \frac{E_0[X_T] - a}{E_0[M_T(1)]} + a e^{-r_T T} \\ &= \frac{E_0[X_T]}{E_0[M_T(1)]} + a\left[\frac{1}{e^{r_T T}} - \frac{1}{E_0[M_T(1)]}\right] \end{aligned} \tag{20}$$

Equation (20) shows that if the cash flow can be written as a linear function of the market level then it can be valued by identifying the component, a, that is riskless (orthogonal to the market return). Then the whole cash flow is first valued as in Section I when the beta is equal to unity, and then this value is adjusted by the difference between the present value of a discounted at the riskless return and the present value discounted at the expected compound return on the market portfolio.

Given the expected value of the cash flow, it is relatively easy to analyze the sensitivity of the present value to the assumed value of a using equation (20). Moreover, it is probably easier to estimate a than it is to estimate the assumed constant beta of a cash flow claim which, as equation (16) shows, corresponds to the elasticity of the cash flow with respect to the market return when the riskless rate is constant. For this reason we suggest that the linearly dependent cash flow model (17) is likely to lead in many cases to more precise estimates of present values than the traditional constant beta model. Clearly, it would be desirable to have some empirical evidence bearing on the approriateness of the two models, but it is difficult to see how that could be obtained.

V

## CONCLUSION

Current implementations of the CAPM to determine discount rates for capital budgeting purposes typically rely on the assumption that the market risk premium is an intertemporal constant which may be estimated from the time series average of past market returns. This assumption is at odds with recent empirical evidence of time-varying (excess) returns on stocks. In this paper we have shown how a model of time variation in expected returns may be used to estimate a discount rate for a cash flow claim with a known beta. The technique we propose is theoretically rigorous, easy to implement, and also takes account of time variation in riskless interest rates. Applying the model, we derive term structures of discount rates that depend on the current levels of the short and long term interest rates, and the dividend yield on the market portfolio.

We also show that the common assumption that a cash flow claim has a constant beta implies that the cash flow itself is a non-linear function of the level of the market portfolio plus noise. A more appropriate assumption in many cases will be that the cash flow itself is a linear function of the level of the market portfolio. Then the cash flow should be valued by decomposing it into its riskless and risky components. The former is valued by discounting at the appropriate riskless interest rate, while the latter is valued as a cash flow claim with a constant beta coefficient of unity.

## APPENDIX

We wish to show that if the continuous time version of the CAPM holds, the interest rate is constant, and a cash flow claim has a constant beta, then the cash flow realization is related to the level of the market portfolio, $M_T(1)$, as shown in equation (16). Under the stated assumptions, V, the value of the claim, follows the stochastic process:

$$\frac{dV}{V} = [(1 - \beta)r + \beta\mu_M]dt + \beta\sigma_M dz_M + d\eta \tag{A-1}$$

where r is the instantaneous riskless rate, $\mu_M$ and $\sigma_M$ are the drift and diffusion coefficients of the market portfolio value process, and $dz_M$ and $d\eta$ are increments to independent Brownian motions. Letting $V_t$ denote the value of the claim at time t, (A-1) implies that the terminal value of the claim is related to its initial value by the stochastic integral:

$$\ln\left[\frac{V_T}{V_0}\right] = \int_0^T [(1 - \beta)r(s) + \beta\mu_M(s) - \frac{1}{2}\beta^2\sigma^2_M(s) - \frac{1}{2}\sigma^2_\eta(s)]ds + \beta\sigma_M(s)dz_M(s) + d\eta(s) \tag{A-2}$$

Define $m_t \equiv [M_t(1)]$, the compounded value at time t of $1 invested in the market portfolio at $t = 0$. Then Ito's Lemma implies that:

$$\frac{dm^\beta}{m^\beta} = [\beta\mu_M + \frac{1}{2}\beta(\beta - 1)\sigma^2_M]dt + \beta\sigma_M dz_M \tag{A-3}$$

so that, recalling $\ln m_0^\beta = 0$,

$$\ln m_T^{\beta} = \int_0^T [\beta\,\mu_M(s) - \frac{1}{2}\beta\,\sigma_M^2(s)]ds + \beta\,\sigma_M dz_M(s) \qquad \text{(A-4)}$$

Combining (A-2) and (A-4), and recalling the definition of $m_t$,

$$\ln\left[\frac{V_T}{V_0}\right] = \ln M_T(1)^{\beta} + \int_0^T [(1-\beta)r(s) - \frac{1}{2}\beta(\beta-1)\sigma_M^2(s) - \frac{1}{2}\sigma_\eta^2(s)]ds + d\eta(s$$

$$\text{(A-4)}$$

Imposing the boundary condition (1), this implies that the cash flow realization may be written as:

$$\tilde{X}_T = kV_0[\tilde{M}_T(1)]^{\beta}\,\tilde{u}_T \qquad \text{(A-5)}$$

where

$$\tilde{u}_T = e^{\int_0^T -\frac{1}{2}\sigma_\eta^2(s)ds + d\eta(s)} \qquad \text{and,}$$

$$k = e^{[(1-\beta)r(s) - \frac{1}{2}\beta(\beta-1)\sigma_M^2(s)]ds}$$

$u_T$ is a random variable with mean zero, and if the interest rate and market volatility constantareor, at most, deterministic functions of time, then k is a constant. (A-5) implies that the conditional expectation of the cash flow satisfies (16).

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| | | δ | r | l | σ |
|---|---|---|---|---|---|
| dM/M | -0.022 | 1.707 | -0.513 | -0.017 | 0.045 |
| | (1.63) | (3.95) | (3.25) | (0.08) | |
| dr | -0.001 | 0.0048 | -0.216 | 0.181 | 0.134 |
| | (0.34) | (0.05) | (-5.66) | (3.54) | |
| dI | 0.025 | -0.318 | 0.247 | -0.308 | 0.037 |
| | (1.82) | (0.79) | (1.68) | (1.55) | |
| dδ | 0.0009 | -0.059 | 0.023 | 0.0005 | 0.048 |
| | (1.62) | (2.97) | (3.27) | (0.05) | |

(t-statistics in parentheses)

The Estimated Stochastic Process for the State Variables and the Market Return
January 1972 - December 1991

**Table 1**

| | market return | r | l | δ |
|---|---|---|---|---|
| market return | 1.0 | | | |
| r | -0.037 | 1.0 | | |
| l | -0.330 | 0.330 | | |
| δ | -0.995 | 0.032 | 0.298 | 1.0 |

Correlations of State Variable and
Stock Return Innovations
January 1972-December 1991

Table 2

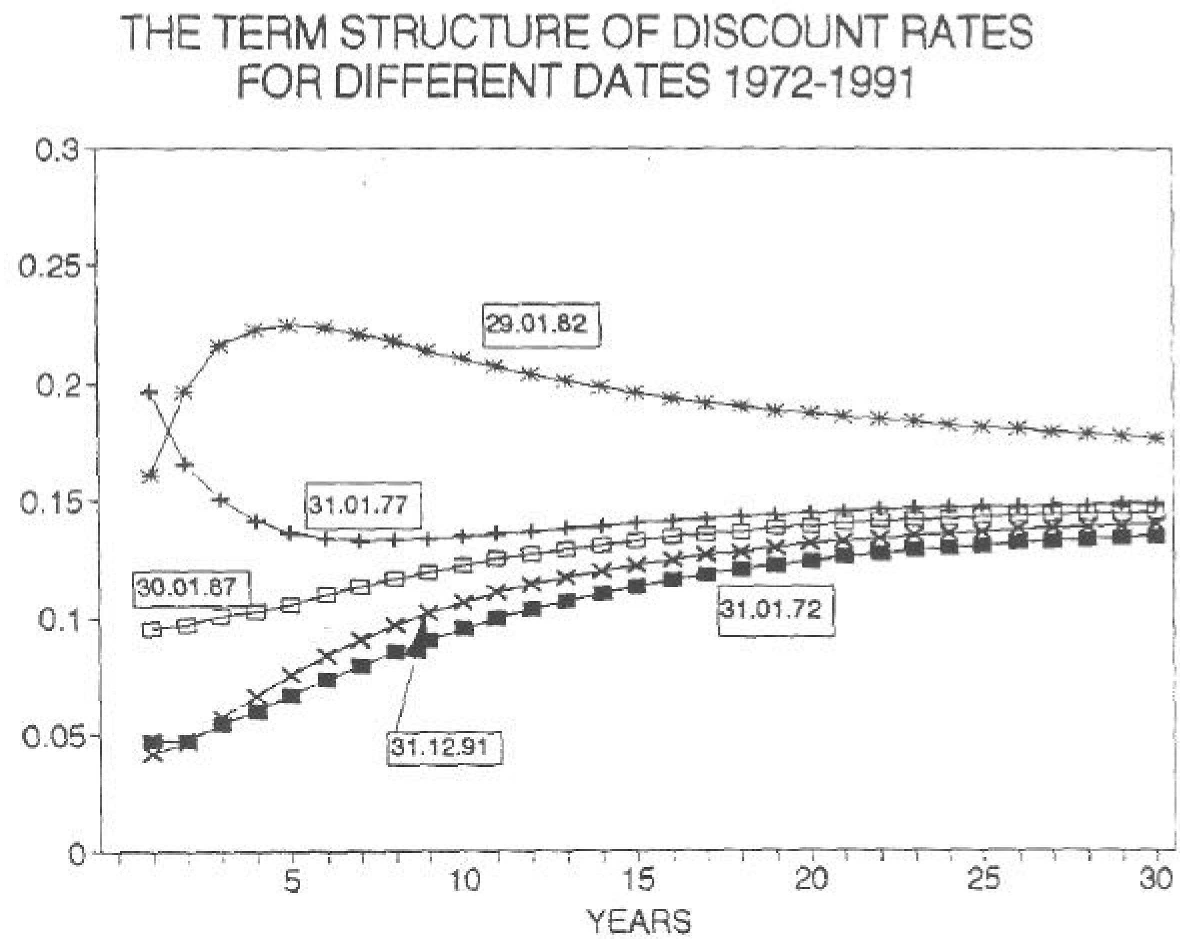

THE TERM STRUCTURE OF DISCOUNT RATES
FOR DIFFERENT DATES 1972-1991
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
YEARS
29.01.82
31.01.77
30.01.87
31.01.72
31.12.91

Figure 1

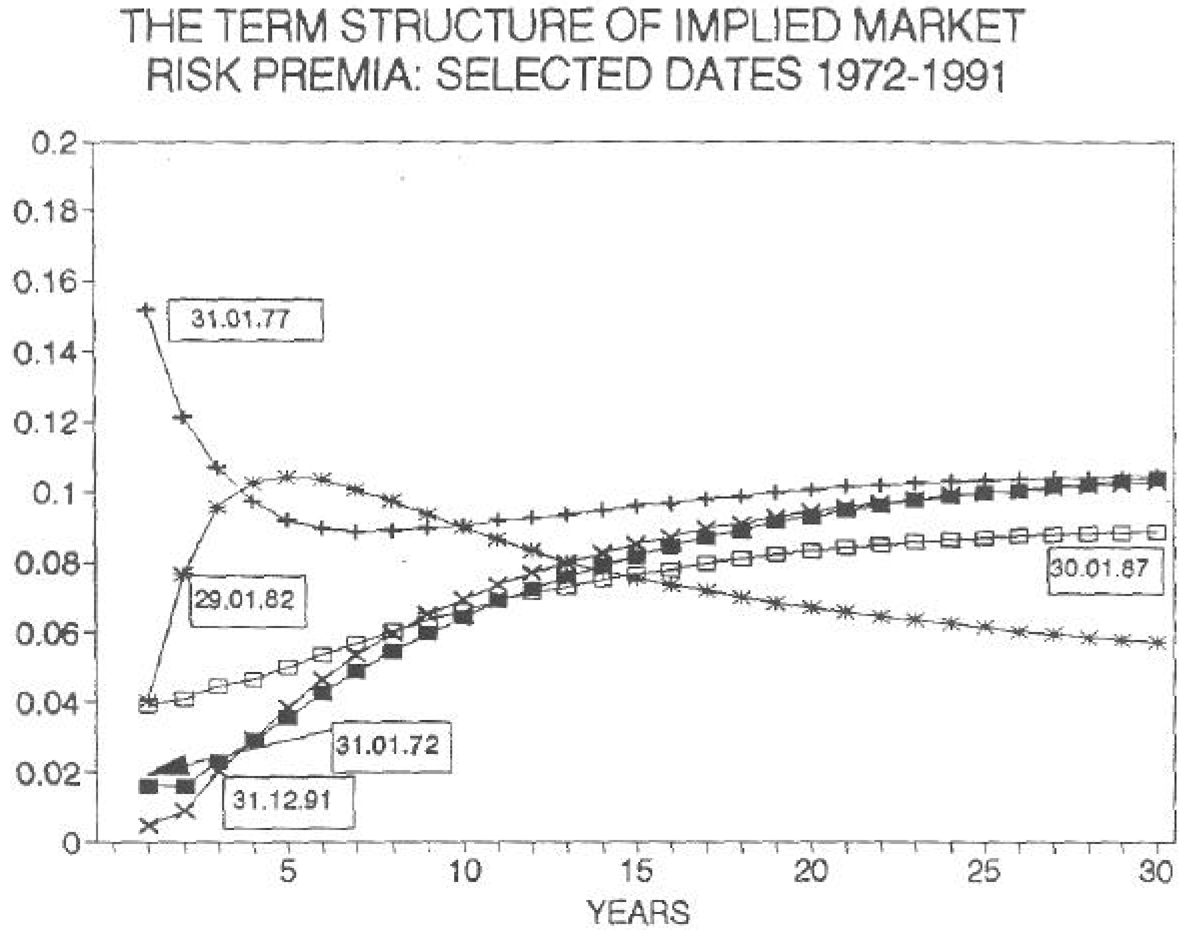

THE TERM STRUCTURE OF IMPLIED MARKET
RISK PREMIA: SELECTED DATES 1972-1991
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
5
10
15
20
25
30
YEARS
31.01.77
29.01.82
30.01.87
31.01.72
31.12.91

Figure 2

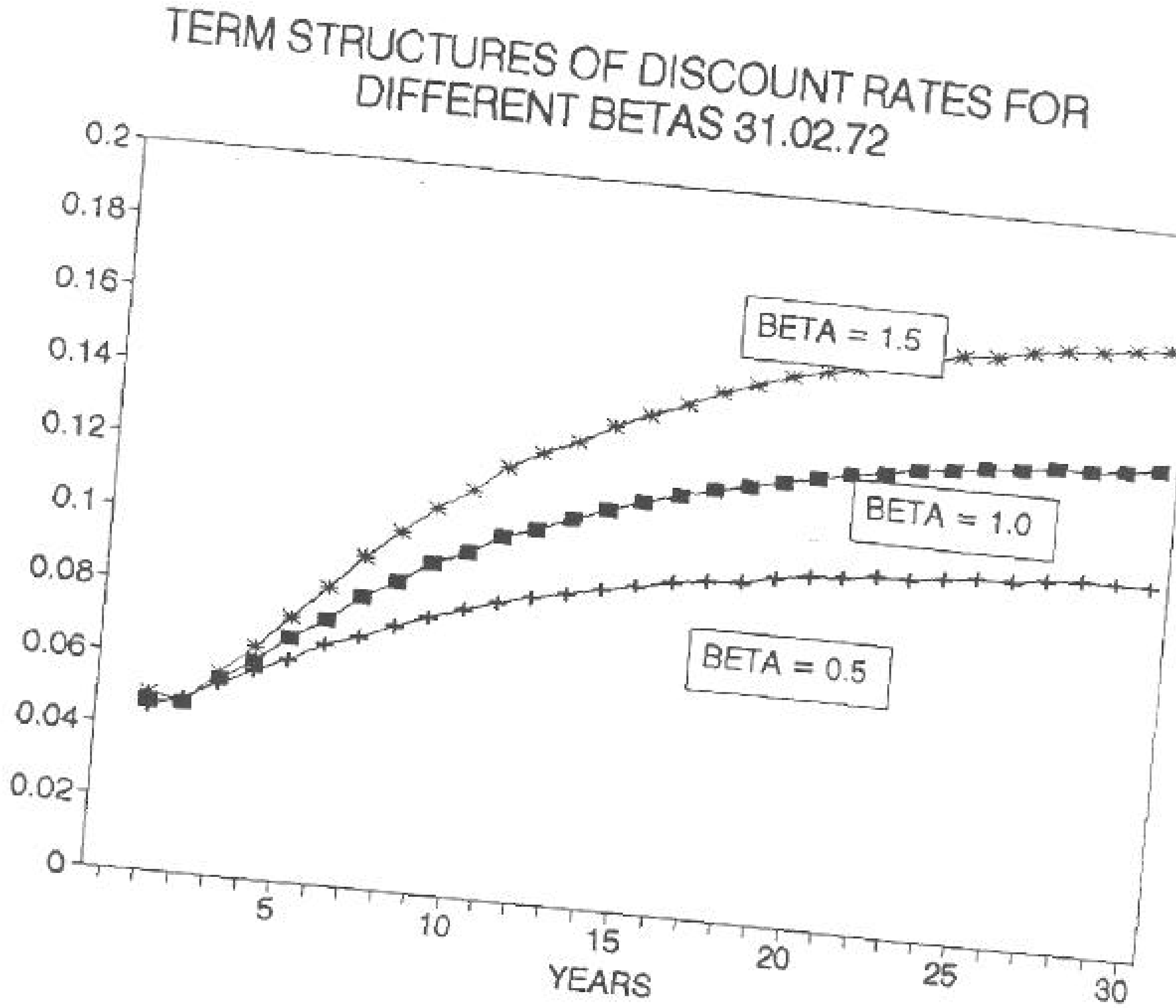

TERM STRUCTURES OF DISCOUNT RATES FOR DIFFERENT BETAS 31.02.72
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
BETA = 1.5
BETA = 1.0
BETA = 0.5
5
10
15
20
25
30
YEARS

Figure 3

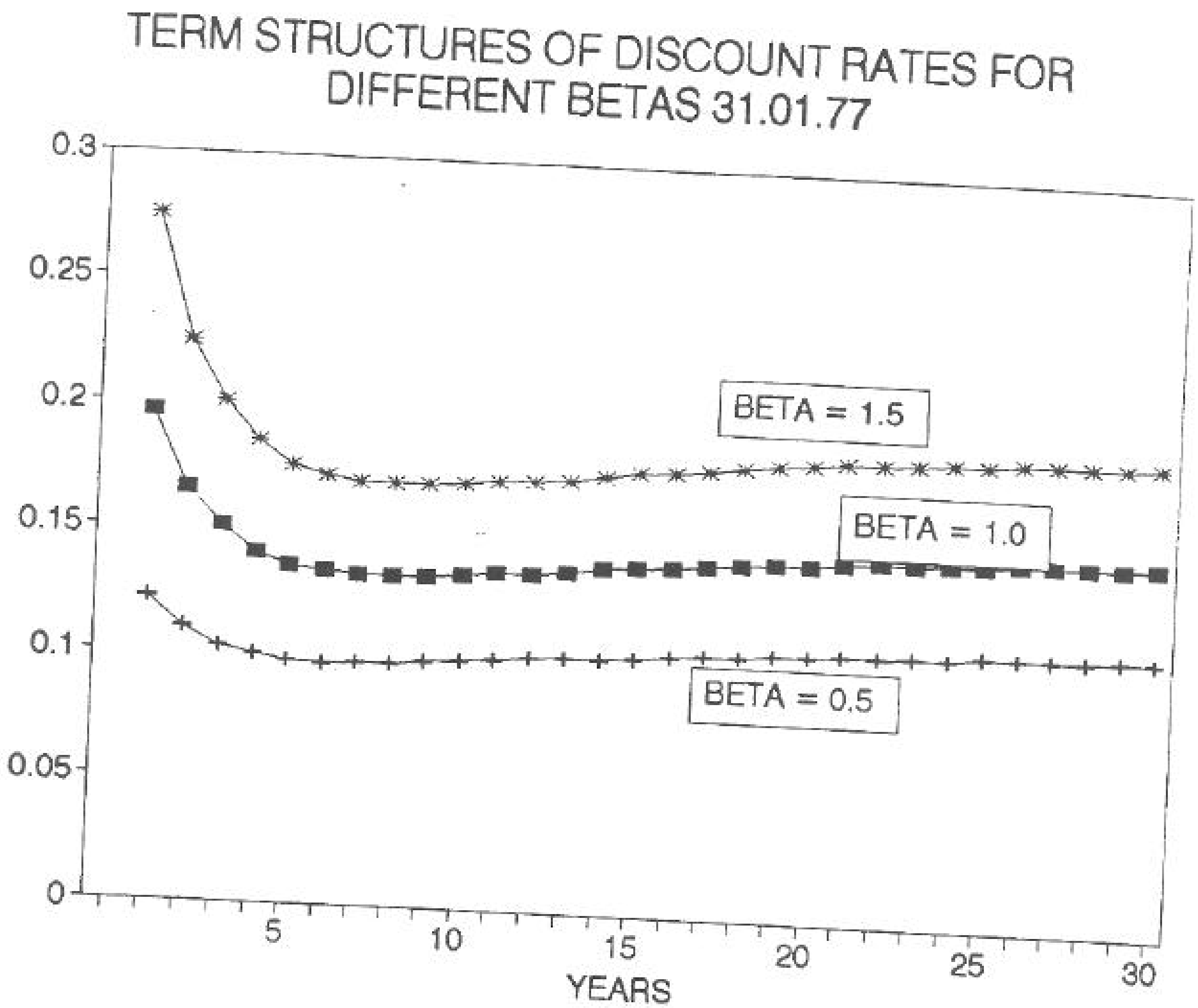

TERM STRUCTURES OF DISCOUNT RATES FOR
DIFFERENT BETAS 31.01.77
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
YEARS
BETA = 1.5
BETA = 1.0
BETA = 0.5

Figure 4

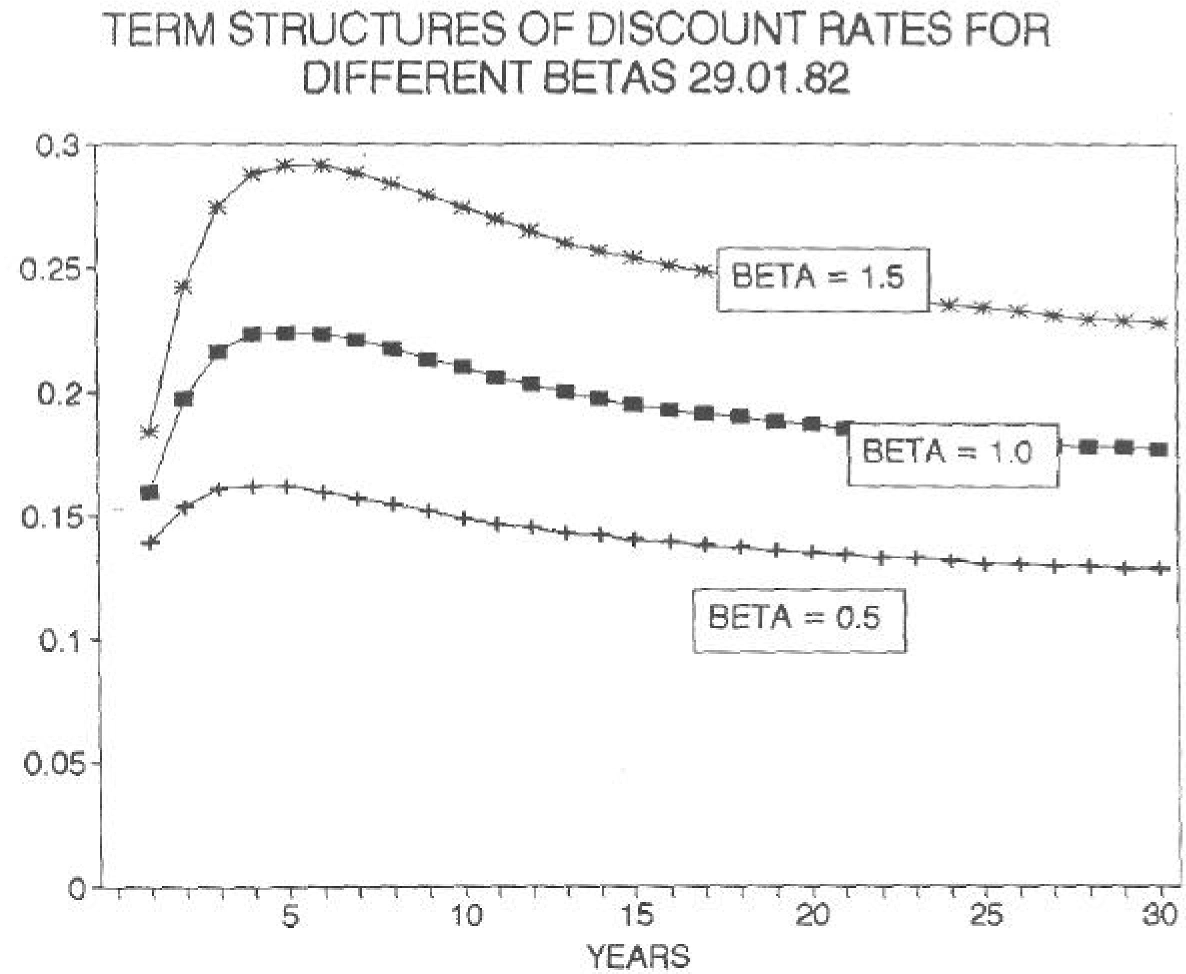
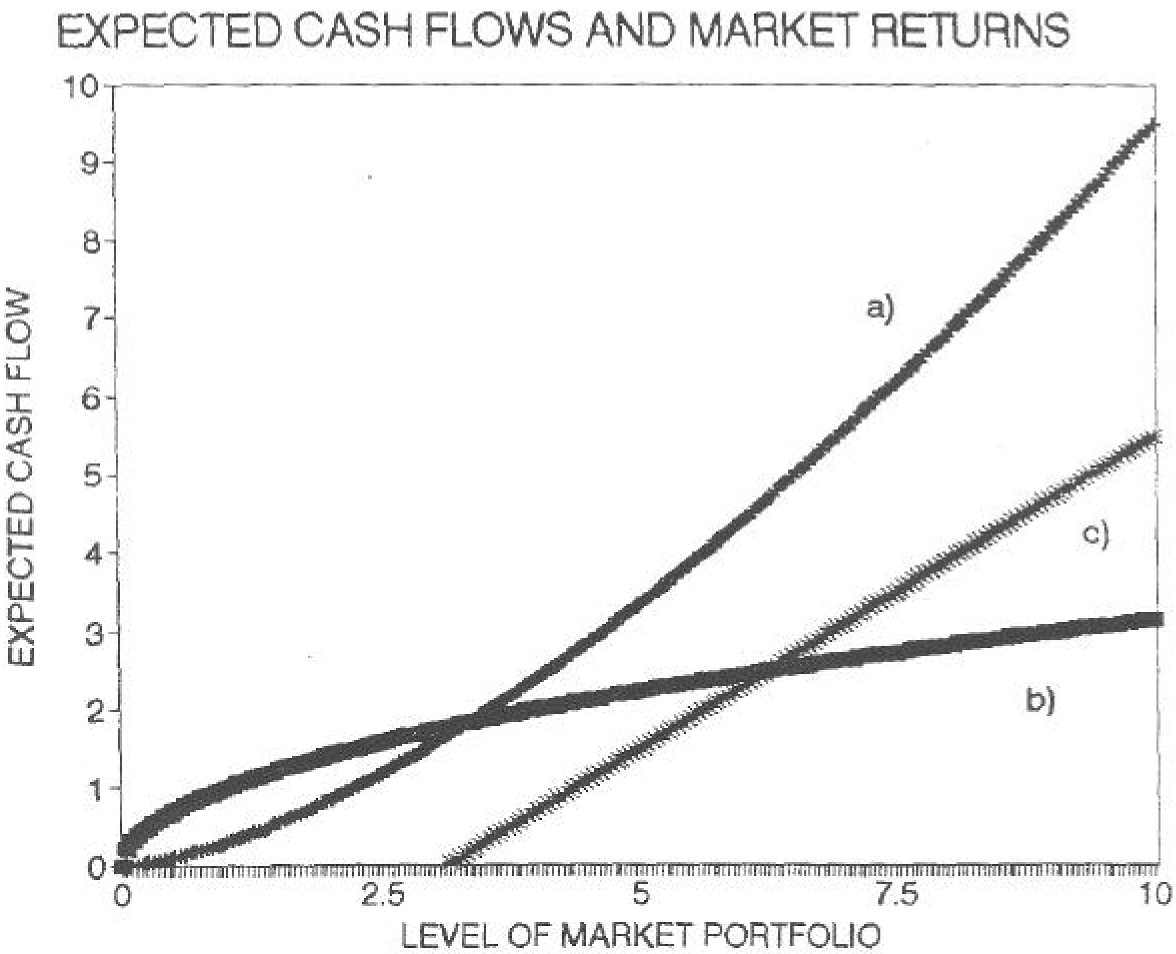

TERM STRUCTURES OF DISCOUNT RATES FOR
DIFFERENT BETAS 29.01.82
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
YEARS
BETA = 1.5
BETA = 1.0
BETA = 0.5

Figure 5




Expected Cash Flows conditional on the level of the Market Portfolio

a) Constant Beta : $\beta = 1.5$
b) Constant Beta : $\beta = 0.5$
c) Linear cash flow : $D = 0.8$

Figure 6