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Parameter Estimates for Symmetric Stable Distributions

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Building on results of an earlier article [6], estimators are suggested for the scale parameter and characteristic exponent of symmetric stable distributions, and Monte Carlo studies of these estimators are reported. The powers of various goodness-of-fit tests of a Gaussian null hypothesis against non-Gaussian stable alternatives are also investigated. Finally, a test of the stability property of symmetric stable variables is suggested and demonstrated.

1. INTRODUCTION

Stable distributions[1] are becoming more common as data models, especially in economics where many interesting quantities can be expressed as sums of random variables,[2] (see e.g., [5, 11].) Such empirical efforts have been hampered by a lack of known closed-form densities for all but a few members of the stable class. In [6] we attempted to alleviate this problem by supplying probability tables of symmetric members of the stable class and by studying estimators of their location parameters. In this article, we suggest estimators for the two remaining parameters of symmetric stable distributions, the scale parameter and the characteristic exponent.[3] In addition, we examine two test procedures that may often be useful in data analysis: (a) goodness-of-fit tests of normality against non-normal stable alternatives, and (b) tests of the property of stability.

2. AN ESTIMATOR OF THE SCALE PARAMETER, c

An estimate of c can be obtained from sample fractiles. If the appropriate fractiles are chosen, the estimate will be only slightly dependent on α, the characteristic exponent.

In particular, the .72 fractile of a standardized (i.e., $\delta=0$, $c=1$) symmetric stable distribution is in the interval $.827 \pm .003$ for $1 \leq \alpha \leq 2$ (see [6, Table 3]). Thus, given a random sample of N observations, a sensible estimator of c is

$$\hat{c} = (1/2(.827))[\hat{x}_{.72} - \hat{x}_{.28}], \tag{2.1}$$

where $\hat{x}_f$ refers to the $(f)(N+1)$st order statistic, which is used to estimate $x_{.28}$ and $x_{.72}$, the .28 and .72 fractiles of the distribution of x.

This estimator has an asymptotic bias of less than .4 percent [6, pp. 822–3]. Being a linear combination of order statistics, it is asymptotically normally distributed with variance [2, p. 369]

$$\sigma^2(\hat{c}) \approx \frac{2(.28)(.72 - .28)}{N[p(\alpha, .72)]^2}\left(\frac{1}{1.654}\right)^2, \tag{2.2}$$

where $p(\alpha, f)$ is the density of the distribution of x at the f fractile. Since symmetry is assumed, the distribution of c is independent of the location parameter of the underlying random variable, x. The scale of x affects the asymptotic variance of $\hat{c}$ through the density $p(\alpha, .72)$ which appears in the denominator of (2.2). For a non-standardized symmetric stable distribution (i.e., $c \neq 1$), $\sigma^2(\hat{c})$ is, of course, c^2 times the value of $\sigma^2(\hat{c})$ for $c=1$.

Table 1 reports the results of a Monte Carlo study[4] of the properties of $\hat{c}$. N is the sample size and n the number of replications; $E(\hat{c})$ is the asymptotic expectation of $\hat{c}$, obtained by substituting the true fractiles $x_{.72}$ and $x_{.28}$ into (2.1); $\sigma(\hat{c})$ is the asymptotic standard deviation, obtained from (2.2); $\bar{\hat{c}}$, $\hat{c}_{med}$ and $\hat{\sigma}$ are the mean, median and standard deviation of the Monte Carlo sample values of $\hat{c}$; and SR is the studentized range (i.e., the range of the Monte Carlo distribution of $\hat{c}$ divided by its standard deviation $\hat{\sigma}$).

Differences between $\bar{\hat{c}}$ and the asymptotic expectations $E(\hat{c})$ in Table 1 (which differences are largest for small α and N) are due to finite sample bias of sample fractiles as well as to sampling variation. As one would expect,

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[1] References [6, 7, 8, 10] provide compact treatments of stable distributions.

[2] Stable distributions are the only possible limiting distributions for sums of independent identically-distributed random variables, [7, pp. 168, 302–6].

[3] For symmetric stable distributions, the log characteristic function is $i\delta t - |ct|^{\alpha}$, where t is an argument, $i=\sqrt{-1}$ and the parameters are: δ, location; c, scale; and α, characteristic exponent.

[4] Monte Carlo samples were obtained as follows. First 59,900 "cumulative probabilities" (i.e., random numbers from the uniform distribution $U(0, 1)$) were generated. For each randomly chosen cumulative probability, the numerical inverse function of [6] was used to obtain the seven standardized ($\delta=0$, $c=1$), symmetric stable random deviates corresponding to $\alpha=1.0$, 1.1, 1.3, 1.5, 1.7, 1.9, 2.0. The result is seven samples of 59,900 observations, but the samples are not independent since the ith observation in each corresponds to the same cumulative probability. This is actually an advantage since, as we shall see, it facilitates comparisons of the sampling properties of a given estimator for different values of α. For each α the random numbers were grouped on magnetic tape into five blocks of 11,980 each. The number of replications n is simply one less than five times the largest integer less than or equal to $11{,}980/N$.

The Monte Carlo results will depend on whether the 59,900 cumulative probabilities conform well to $U(0, 1)$. The population mean, variance, second moment and third moment for $U(0, 1)$ are, respectively, .5, .0833, .3333, and .25; the sample values are .500016, .082991, .333007, and .249494. Finally, a chi-square test of the sample cumulative probabilities against $U(0, 1)$, using twenty equal subintervals of length .05, yielded the value 19.027—just slightly above the median of the chi-square distribution with 19 d.f. We conclude that the sample conforms extremely well to $U(0, 1)$.

Table 1. SUMMARY STATISTICS OF MONTE CARLO DISTRIBUTIONS OF $\hat{c}$

α	$E(\hat{c})$	$\bar{\hat{c}}$	$\hat{c}_{med}$	$\hat{\sigma}$	$\sigma(\hat{c})$	SR	$\bar{\hat{c}}$	$\hat{c}_{med}$	$\hat{\sigma}$	$\sigma(\hat{c})$	SR	$\bar{\hat{c}}$	$\hat{c}_{med}$	$\hat{\sigma}$	$\sigma(\hat{c})$	SR
		N = 24 n = 2494					N = 49 n = 1219					N = 74 n = 804				
2.0	.997	1.038	1.023	.259	.257	6.78	1.013	1.010	.178	.180	7.09	1.003	.996	.141	.147	6.57
1.9	.999	1.044	1.029	.263	.259	6.82	1.017	1.014	.180	.182	7.10	1.007	1.000	.142	.148	6.57
1.7	1.002	1.057	1.038	.273	.265	6.92	1.024	1.020	.185	.185	7.14	1.012	1.004	.146	.151	6.56
1.5	1.004	1.074	1.051	.291	.272	7.19	1.032	1.024	.194	.191	7.19	1.017	1.007	.152	.155	6.58
1.3	1.003	1.102	1.064	.324	.284	7.79[a]	1.042	1.031	.208	.199	7.25[a]	1.023	1.011	.161	.162	6.70
1.1	1.001	1.151	1.089	.390	.306	8.53[a]	1.060	1.040	.236	.214	7.31[a]	1.033	1.013	.178	.175	6.88[a]
1.0	1.000	1.191	1.110	.446	.324	8.96[a]	1.076	1.051	.259	.227	7.32[a]	1.041	1.017	.192	.185	6.97[a]
		N = 99 n = 604					N = 199 n = 299					N = 299 n = 199				
2.0	.997	1.000	.998	.124	.127	6.56	.996	.995	.091	.089	5.91	.994	.996	.072	.073	5.84
1.9	.999	1.003	1.001	.125	.128	6.57	.999	.997	.092	.090	5.91	.997	.999	.073	.074	5.85
1.7	1.002	1.008	1.004	.128	.131	6.60	1.002	1.001	.094	.092	5.90	1.001	1.002	.074	.075	5.85
1.5	1.004	1.013	1.009	.133	.134	6.64	1.005	1.002	.097	.095	5.89	1.003	1.004	.077	.077	5.84
1.3	1.003	1.017	1.013	.140	.140	6.71	1.007	1.005	.102	.099	5.88	1.004	1.004	.080	.081	5.85
1.1	1.001	1.023	1.015	.154	.151	6.79[a]	1.009	1.005	.111	.107	5.83	1.004	1.003	.087	.087	5.85
1.0	1.000	1.030	1.018	.165	.160	6.85[a]	1.012	1.007	.119	.113	5.80	1.006	1.004	.092	.092	5.59
		N = 399 n = 149					N = 499 n = 119					N = 599 n = 99				
2.0	.997	.994	.994	.061	.063	5.49	.993	.989	.053	.056	5.29	.994	.995	.051	.052	5.20
1.9	.999	.996	.997	.061	.064	5.48	.996	.992	.053	.057	5.29	.997	.997	.051	.052	5.20
1.7	1.002	1.000	1.000	.062	.065	5.48	.999	.995	.054	.058	5.28	1.000	1.000	.052	.053	5.19
1.5	1.004	1.002	1.001	.064	.067	5.47	1.001	.997	.056	.060	5.29	1.002	1.002	.053	.055	5.19
1.3	1.003	1.002	1.001	.067	.070	5.47	1.001	.998	.059	.063	5.30	1.002	1.002	.056	.057	5.17
1.1	1.001	1.002	.999	.072	.075	5.46	1.001	.996	.063	.067	5.31	1.001	.999	.060	.061	5.17
1.0	1.000	1.003	.999	.077	.080	5.46	1.001	.998	.067	.071	5.30	1.001	1.000	.063	.065	5.16

[a] Significantly different, at .90 level, from SR for the normal distribution. For the sampling distribution of SR, see [3].

$\bar{\hat{c}}-E(\hat{c})$ declines as N increases and in all cases[5] $\bar{\hat{c}}$ is within the interval $E(\hat{c})\pm 2\sigma(\hat{c})/\sqrt{N}$.

Moreover, except when both α and N are small, the standard deviations $\hat{\sigma}$ of the Monte Carlo distributions of $\hat{c}$ are close to the asymptotic values $\sigma(\hat{c})$, and the studentized ranges SR indicate close fits to normality.[6] In short, even in small samples the distributions of $\hat{c}$ conform closely to their asymptotic counterparts.

When a sample is known to be drawn from a Gaussian distribution ($\alpha=2$), the sample standard deviation $\hat{s}$ is a substantially more efficient estimator of c than $\hat{c}$. When $\alpha=2$, $c^2=s^2/2$, where s^2 is the population variance. Thus $s^2=2$ for the standardized symmetric stable distribution with $\alpha=2$. Since in the Gaussian case the asymptotic variance of $\hat{s}$ is $s^2/2N$, the ratio of the asymptotic variances of the two estimators of c is $\mathrm{Var}(\hat{s}/\sqrt{2})/\mathrm{Var}(\hat{c}) =.314$.

But this loss of efficiency may not be a large price to pay for the greater robustness of $\hat{c}$ when α is unknown. When $\alpha<2$, the sampling dispersion of $\hat{s}/\sqrt{2}$ increases rapidly as α decreases. For example, the mean absolute deviations of the Monte Carlo distributions of $\hat{s}/\sqrt{2}$ and $\hat{c}$ for $N=599$ were as follows:

measure	α 2.0	1.9	1.7	1.5	1.3	1.1	1.0
MAD ($\hat{s}/\sqrt{2}$)	.0221	.196	1.12	4.26	12.9	36.9	68.4
MAD ($\hat{c}$)	.0413	.0416	.0424	.0436	.0455	.0488	.0515

The mean absolute deviation (MAD) itself has been suggested as an estimate of dispersion when the sample may have arisen from a stable non-Gaussian process. But Monte Carlo investigations of the MAD indicate that it is less efficient than $\hat{c}$ except when α is close to 2. Moreover, the expected MAD is dependent on α. For example, sample means and standard deviations of the MAD and of $\hat{c}$ obtained from 101 Monte Carlo samples of size $N=299$ were as follows:

measure	α 2.0	1.9	1.7	1.5	1.3	1.1	1.0
$\bar{\hat{c}}$	.996	.998	1.002	1.004	1.005	1.005	1.006
$\hat{\sigma}(\hat{c})$	.067	.068	.069	.071	.075	.081	.086
$\overline{\mathrm{MAD}}$	1.123	1.179	1.360	1.747	2.655	5.017	7.723
$\hat{\sigma}$(MAD)	.043	.057	.140	.501	1.688	4.776	8.941

3. ESTIMATORS FOR THE CHARACTERISTIC EXPONENT α

The characteristic exponent α determines the "type" of a symmetric stable distribution. Stable distributions are more "thick-tailed" the smaller the α. With standardized distributions, for example, the .95 fractile decreases monotonically from 6.31 for $\alpha=1$ to 2.33 for $\alpha=2$.

[5] These cases are not independent, however, because of the Monte Carlo generating scheme. See Footnote 4.

[6] The close fits to normality of the Monte Carlo distributions of $\hat{c}$ also have been substantiated by chi-square tests and normal probability plots.

Table 2. *MONTE CARLO DISTRIBUTIONS OF $\hat{\alpha}_f$ FOR $f = .93, .95, .97\ .99$*[a]

Parameter (fractile)	2.0				1.9				1.7				1.5				1.3				1.1				1.0			
	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$	$\hat{\alpha}_{.93}$	$\hat{\alpha}_{.95}$	$\hat{\alpha}_{.97}$	$\hat{\alpha}_{.99}$
	N = 99, *n* = 606																											
$\bar{\hat{\alpha}}$	1.83	1.85	1.88	1.90	1.79	1.81	1.82	1.76	1.62	1.66	1.64	1.51	1.50	1.47	1.44	1.31	1.29	1.26	1.24	1.12	1.08	1.06	1.05	.95	.98	.97	.95	.86
$s(\hat{\alpha})$	.201	.173	.138	.093	.215	.189	.163	.208	.244	.218	.201	.257	.240	.219	.205	.261	.213	.185	.189	.203	.160	.157	.169	.216	.149	.144	.157	.198
.010	1.33	1.44	1.52	1.66	1.31	1.39	1.43	1.00	1.21	1.23	1.32	.75	1.07	1.03	1.02	.63	.91	.90	.83	.62	.78	.74	.70	.65	.70	.67	.63	.43
.050	1.45	1.52	1.62	1.74	1.41	1.48	1.53	1.38	1.31	1.34	1.31	1.06	1.17	1.18	1.11	.67	1.02	1.00	.94	.74	.86	.85	.78	.61	.77	.75	.70	.56
.099	1.51	1.58	1.67	1.78	1.47	1.53	1.58	1.50	1.36	1.39	1.38	1.13	1.22	1.23	1.14	.96	1.06	1.06	.99	.80	.89	.89	.83	.67	.81	.80	.74	.60
.248	1.66	1.72	1.79	1.85	1.61	1.65	1.70	1.64	1.47	1.49	1.51	1.34	1.31	1.32	1.31	1.12	1.14	1.14	1.12	.95	.96	.95	.93	.79	.87	.86	.84	.71
$\hat{\alpha}_{med}$	1.92	1.93	1.96	1.93	1.84	1.84	1.84	1.80	1.64	1.64	1.63	1.55	1.45	1.43	1.43	1.33	1.36	1.26	1.24	1.13	1.07	1.05	1.04	.94	.97	.95	.94	.85
.752	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.91	1.97	1.85	1.78	1.70	1.66	1.60	1.56	1.51	1.39	1.36	1.36	1.30	1.17	1.16	1.14	1.10	1.05	1.03	1.04	.99
.901	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.97	1.83	1.91	1.79	1.70	1.65	1.57	1.50	1.48	1.44	1.31	1.27	1.26	1.23	1.19	1.15	1.16	1.13
.950	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.88	2.00	1.94	1.82	1.70	1.69	1.61	1.57	1.51	1.38	1.34	1.34	1.29	1.25	1.23	1.22	1.19
.990	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.99	2.00	2.00	2.00	1.80	2.00	1.84	1.73	1.66	1.55	1.51	1.48	1.44	1.39	1.39	1.35	1.33
M.S.E. × 10	.69	.51	.33	.17	.58	.46	.33	.63	.60	.49	.44	1.02	.61	.47	.46	1.04	.46	.36	.39	1.02	.34	.36	.31	.69	.23	.22	.23	.59
	N = 199, *n* = 299																											
$\bar{\hat{\alpha}}$	1.87	1.89	1.91	1.94	1.86	1.84	1.85	1.84	1.70	1.69	1.67	1.61	1.51	1.48	1.47	1.40	1.30	1.28	1.27	1.21	1.09	1.08	1.07	1.03	.99	.98	.97	.92
$s(\hat{\alpha})$	.153	.129	.102	.070	.171	.161	.132	.135	.195	.161	.141	.165	.173	.141	.132	.175	.135	.121	.122	.168	.111	.105	.110	.151	.101	.096	.103	.160
.01	1.50	1.57	1.66	1.74	1.46	1.52	1.57	1.41	1.37	1.16	1.39	1.13	1.22	1.22	1.17	.89	1.06	1.04	.98	.75	.86	.86	.80	.64	.78	.78	.73	.58
.05	1.56	1.64	1.70	1.80	1.52	1.59	1.64	1.60	1.40	1.44	1.47	1.29	1.26	1.26	1.26	1.05	1.10	1.09	1.03	.91	.93	.91	.86	.76	.84	.83	.80	.68
.10	1.63	1.68	1.76	1.81	1.57	1.62	1.69	1.67	1.45	1.49	1.51	1.38	1.30	1.33	1.31	1.13	1.13	1.16	1.12	.98	.96	.95	.93	.81	.87	.86	.84	.74
.25	1.76	1.79	1.84	1.89	1.68	1.72	1.76	1.77	1.55	1.56	1.57	1.50	1.39	1.39	1.36	1.23	1.20	1.20	1.19	1.10	1.02	1.01	1.00	.93	.92	.91	.90	.83
$\hat{\alpha}_{med}$	1.98	1.93	1.95	1.97	1.89	1.81	1.86	1.85	1.69	1.66	1.66	1.61	1.49	1.47	1.46	1.42	1.28	1.31	1.26	1.21	1.09	1.07	1.06	1.02	.99	.97	.96	.93
.75	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.93	1.86	1.80	1.75	1.73	1.62	1.57	1.54	1.54	1.36	1.35	1.34	1.33	1.17	1.15	1.14	1.15	1.06	1.05	1.03	1.03
.90	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.87	1.78	1.75	1.67	1.64	1.62	1.47	1.43	1.42	1.41	1.23	1.23	1.21	1.20	1.11	1.11	1.11	1.10
.95	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.94	1.89	1.83	1.73	1.68	1.65	1.51	1.48	1.47	1.46	1.28	1.25	1.26	1.26	1.16	1.14	1.15	1.16
.99	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.96	2.00	1.94	1.83	1.76	1.66	1.60	1.56	1.57	1.36	1.37	1.33	1.34	1.24	1.25	1.21	1.35
M.S.E. × 10	.43	.34	.19	.09	.13	.25	.17	.19	.38	.38	.31	.35	.30	.31	.18	.41	.18	.15	.16	.36	.15	.11	.13	.29	.16	.10	.12	.24
	N = 299, *n* = 154																											
$\bar{\hat{\alpha}}$	1.90	1.91	1.93	1.95	1.87	1.86	1.87	1.86	1.71	1.70	1.66	1.65	1.53	1.50	1.48	1.46	1.29	1.28	1.28	1.26	1.09	1.09	1.08	1.05	.99	.99	.98	.95
$s(\hat{\alpha})$	.145	.121	.091	.057	.153	.137	.111	.100	.176	.149	.130	.135	.143	.130	.108	.144	.107	.106	.100	.140	.086	.091	.091	.127	.074	.084	.085	.117
.01	1.47	1.54	1.62	1.38	1.43	1.48	1.56	1.45	1.33	1.35	1.42	1.13	1.19	1.19	1.25	.44	1.04	1.07	1.06	.79	.86	.86	.89	.65	.60	.38	.80	.59
.05	1.62	1.65	1.75	1.83	1.53	1.60	1.68	1.69	1.45	1.48	1.51	1.41	1.26	1.32	1.31	1.20	1.17	1.11	1.12	1.01	.94	.95	.93	.84	.85	.66	.84	.76
.10	1.65	1.71	1.77	1.87	1.61	1.65	1.71	1.74	1.48	1.52	1.54	1.44	1.33	1.34	1.34	1.26	1.16	1.16	1.25	1.03	.98	.97	.96	.89	.89	.88	.83	.81
.25	1.80	1.82	1.87	1.92	1.74	1.75	1.79	1.81	1.58	1.59	1.60	1.51	1.40	1.41	1.41	1.36	1.22	1.22	1.21	1.16	1.03	1.03	1.01	.97	.93	.96	.93	.88
$\hat{\alpha}_{med}$	1.98	1.99	1.98	1.97	1.89	1.90	1.86	1.86	1.70	1.69	1.69	1.65	1.50	1.50	1.48	1.44	1.30	1.29	1.27	1.24	1.10	1.09	1.07	1.04	1.00	.99	.98	.95
.75	2.00	2.00	2.00	2.00	2.00	2.00	1.95	1.93	1.83	1.81	1.76	1.73	1.58	1.57	1.55	1.54	1.37	1.36	1.34	1.34	1.16	1.15	1.14	1.14	1.05	1.05	1.04	1.06
.90	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.91	1.84	1.81	1.70	1.66	1.63	1.63	1.44	1.42	1.42	1.42	1.20	1.21	1.20	1.21	1.10	1.10	1.10	1.10
.95	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.91	1.84	1.78	1.74	1.66	1.65	1.49	1.48	1.46	1.46	1.24	1.24	1.24	1.25	1.13	1.14	1.13	1.14
.99	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.88	1.95	1.93	1.75	1.70	1.56	1.59	1.51	1.53	1.38	1.32	1.29	1.31	1.16	1.20	1.18	1.22
M.S.E. × 10	.30	.23	.13	.06	.27	.20	.13	.13	.29	.02	.15	.11	.21	.17	.13	.24	.12	.11	.10	.23	.08	.08	.09	.19	.06	.07	.08	.16
	N = 399, *n* = 149																											
$\bar{\hat{\alpha}}$	1.90	1.91	1.94	1.96	1.86	1.86	1.86	1.87	1.71	1.70	1.69	1.66	1.51	1.49	1.49	1.61	1.30	1.29	1.29	1.26	1.10	1.09	1.09	1.06	1.00	.99	.99	.96
$s(\hat{\alpha})$	.125	.105	.077	.050	.141	.133	.098	.085	.158	.130	.103	.111	.136	.101	.089	.121	.096	.065	.083	.119	.076	.074	.076	.107	.071	.068	.070	.100
.013	1.56	1.63	1.73	1.81	1.53	1.53	1.66	1.59	1.40	1.45	1.49	1.39	1.27	1.27	1.31	1.08	1.11	1.09	1.12	.90	.93	.93	.93	.75	.85	.83	.85	.68
.047	1.65	1.69	1.79	1.86	1.59	1.61	1.72	1.73	1.47	1.51	1.55	1.43	1.33	1.34	1.35	1.26	1.17	1.15	1.16	1.06	.99	.96	.97	.88	.89	.87	.88	.80
.100	1.71	1.75	1.82	1.88	1.66	1.68	1.75	1.77	1.55	1.54	1.58	1.51	1.36	1.37	1.38	1.39	1.18	1.19	1.19	1.10	1.00	1.00	.99	.92	.91	.91	.90	.83
.247	1.80	1.84	1.88	1.93	1.74	1.77	1.79	1.83	1.59	1.60	1.61	1.59	1.41	1.41	1.42	1.37	1.22	1.22	1.23	1.18	1.03	1.03	1.03	.99	.94	.94	.94	.90
$\hat{\alpha}_{med}$	1.98	1.96	1.97	1.98	1.90	1.88	1.86	1.87	1.69	1.69	1.68	1.66	1.50	1.49	1.49	1.46	1.29	1.28	1.29	1.28	1.10	1.08	1.09	1.06	1.00	.99	.99	.97
.753	2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.93	1.83	1.84	1.76	1.74	1.60	1.57	1.54	1.54	1.36	1.36	1.34	1.34	1.15	1.15	1.14	1.14	1.04	1.05	1.03	1.04
.900	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.96	1.90	1.84	1.79	1.66	1.63	1.62	1.61	1.42	1.40	1.41	1.41	1.20	1.19	1.20	1.20	1.09	1.08	1.09	1.10
.953	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.95	1.91	1.86	1.70	1.66	1.65	1.63	1.48	1.43	1.43	1.43	1.23	1.21	1.22	1.22	1.12	1.10	1.12	1.11
.987	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.90	1.84	1.72	1.72	1.72	1.52	1.46	1.51	1.51	1.26	1.26	1.28	1.31	1.14	1.15	1.17	1.20
M.S.E. × 10	.16	.14	.10	.05	.22	.17	.10	.08	.25	.17	.11	.14	.16	.10	.08	.13	.09	.07	.07	.16	.06	.06	.06	.13	.05	.05	.05	.12
	N = 499, *n* = 119																											
$\bar{\hat{\alpha}}$	1.91	1.92	1.94	1.96	1.87	1.87	1.88	1.88	1.71	1.70	1.69	1.66	1.51	1.49	1.49	1.46	1.30	1.29	1.29	1.26	1.10	1.09	1.09	1.07	1.00	.99	.99	.97
$s(\hat{\alpha})$	.114	.092	.069	.048	.129	.110	.090	.075	.136	.112	.089	.100	.100	.081	.077	.111	.076	.070	.071	.110	.062	.060	.066	.099	.057	.056	.061	.092
.006	1.61	1.68	1.72	1.80	1.50	1.61	1.67	1.58	1.45	1.48	1.52	1.28	1.30	1.32	1.33	1.05	1.12	1.15	1.13	.89	.94	.97	.93	.75	.86	.88	.86	.68
.050	1.66	1.73	1.80	1.86	1.61	1.68	1.74	1.75	1.48	1.54	1.55	1.48	1.34	1.36	1.36	1.28	1.18	1.19	1.17	1.03	.99	1.00	.98	.90	.90	.91	.89	.82
.100	1.73	1.77	1.84	1.90	1.67	1.71	1.76	1.77	1.53	1.56	1.59	1.52	1.37	1.39	1.40	1.31	1.19	1.20	1.26	1.11	1.01	1.01	1.01	.93	.93	.92	.92	.86
.250	1.83	1.85	1.90	1.93	1.78	1.78	1.81	1.83	1.61	1.60	1.63	1.66	1.44	1.42	1.43	1.39	1.25	1.23	1.24	1.19	1.06	1.04	1.04	1.00	.96	.94	.94	.91
$\hat{\alpha}_{med}$	1.99	1.96	1.96	1.98	1.90	1.88	1.88	1.88	1.71	1.69	1.68	1.67	1.50	1.48	1.48	1.46	1.30	1.29	1.28	1.26	1.09	1.09	1.08	1.08	1.00	.99	.98	.97
.750	2.00	2.00	2.00	2.00	2.00	2.00	1.97	1.92	1.80	1.77	1.74	1.74	1.57	1.55	1.55	1.55	1.35	1.34	1.34	1.35	1.14	1.13	1.14	1.14	1.04	1.03	1.03	1.04
.900	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.92	1.87	1.82	1.79	1.66	1.62	1.59	1.60	1.41	1.39	1.38	1.41	1.16	1.17	1.17	1.20	1.07	1.07	1.07	1.09
.950	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.96	1.92	1.85	1.86	1.68	1.66	1.62	1.63	1.43	1.41	1.41	1.43	1.20	1.19	1.20	1.23	1.09	1.08	1.09	1.12
.997	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.97	1.93	1.84	1.76	1.70	1.74	1.54	1.49	1.48	1.55	1.28	1.26	1.26	1.34	1.17	1.15	1.14	1.23
M.S.E. × 10	.21	.11	.06	.04	.16	.13	.09	.06	.19	.13	.08	.16	.10	.07	.06	.14	.06	.05	.05	.14	.04	.04	.05	.11	.03	.03	.04	.09
	N = 599, *n* = 99																											
$\bar{\hat{\alpha}}$	1.91	1.93	1.95	1.96	1.86	1.88	1.87	1.88	1.71	1.71	1.69	1.67	1.51	1.50	1.49	1.47	1.30	1.30	1.29	1.27	1.10	1.10	1.09	1.07	1.00	1.00	.99	.97
$s(\hat{\alpha})$	.103	.085	.068	.036	.128	.107	.088	.064	.138	.113	.082	.085	.108	.086	.073	.095	.080	.074	.068	.095	.064	.061	.061	.088	.058	.056	.057	.081
.01	1.67	1.72	1.70	1.83	1.63	1.63	1.65	1.74	1.49	1.51	1.50	1.43	1.33	1.33	1.30	1.32	1.16	1.15	1.13	1.03	.98	.98	.94	.85	.89	.88	.85	.77
.05	1.69	1.75	1.81	1.89	1.64	1.69	1.74	1.77	1.51	1.56	1.56	1.53	1.36	1.40	1.37	1.30	1.19	1.21	1.18	1.09	.99	1.02	.99	.93	.90	.92	.89	.84
.10	1.73	1.79	1.85	1.91	1.69	1.73	1.78	1.81	1.54	1.58	1.60	1.55	1.38	1.40	1.40	1.33	1.20	1.21	1.21	1.14	1.01	1.03	1.01	.98	.92	.93	.93	.87
.25	1.83	1.87	1.96	1.94	1.76	1.79	1.82	1.84	1.60	1.63	1.63	1.61	1.42	1.43	1.44	1.40	1.24	1.24	1.25	1.21	1.05	1.04	1.05	1.01	.96	.95	.95	.92
$\hat{\alpha}_{med}$	1.96	1.98	1.99	1.97	1.88	1.89	1.89	1.88	1.69	1.69	1.68	1.66	1.50	1.49	1.48	1.48	1.29	1.29	1.28	1.26	1.11	1.09	1.08	1.06	1.00	.99	.98	.96
.75	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.94	1.83	1.79	1.76	1.73	1.56	1.56	1.55	1.53	1.36	1.34	1.34	1.32	1.14	1.14	1.13	1.12	1.03	1.04	1.03	1.02
.90	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.91	1.84	1.81	1.78	1.63	1.60	1.60	1.59	1.39	1.39	1.39	1.39	1.17	1.16	1.18	1.18	1.06	1.07	1.07	1.07
.95	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.94	1.85	1.83	1.69	1.66	1.62	1.65	1.44	1.44	1.41	1.43	1.22	1.23	1.20	1.23	1.11	1.12	1.09	1.12
.99	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.87	1.90	1.85	1.78	1.62	1.73	1.53	1.49	1.42	1.49	1.27	1.26	1.12	1.34	1.15	1.15	1.11	1.23
M.S.E. × 10	.20	.07	.07	.03	.18	.13	.09	.05	.19	.13	.07	.08	.12	.07	.05	.10	.06	.05	.05	.10	.04	.04	.04	.09	.03	.03	.03	.08

[a] *N* denotes sample size; *n* denotes number of replications.

This behavior of higher fractiles suggests a simple estimator of α from order statistics. For some large f (for example, $f = .95$) first calculate

$$\hat{z}_f = \frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{c}} = (.827)\,\frac{\hat{x}_f - \hat{x}_{1-f}}{\hat{x}_{.72} - \hat{x}_{.28}} \tag{3.1}$$

from the sample. Given that x is a symmetric stable variable with exponent α, and scale c, $\hat{z}_f$ is an estimator of the f fractile of the *standardized* symmetric stable distribution with exponent α. Thus an estimate of α can be obtained by searching a "table" of standardized symmetric stable c.d.f.'s for the value, call it $\hat{\alpha}_f$, whose f fractile matches $\hat{z}_f$ most closely. Formally,

$$\hat{\alpha}_f = G(f, \hat{z}_f) \tag{3.2}$$

where G is a function that uniquely maps the fractile z_f and the cumulative probability f into α, and $\hat{z}_f$ is the sample fractile given by (3.1).

An "optimal" value of f in (3.2) apparently cannot be determined analytically. An incentive to choose large values of f is that differences between fractiles for different values of α increase with f (see [6], Table 3). On the other hand, the sampling dispersion of a fractile estimate is inversely proportional to the density of the underlying distribution at the chosen fractile, and the density will decline as we choose fractiles further into the tails of the distribution. Monte Carlo experiments remain as yet the only approach to this problem.

Tables 2 and 3 report Monte Carlo sampling distributions of $\hat{\alpha}_f$ for five values of f in the interval $.93 \leq f \leq .99$ and for nine sample sizes N. The tables show the means $(\bar{\hat{\alpha}})$, standard deviations $(s(\hat{\alpha}))$ mean square errors (M.S.E.), medians $(\hat{\alpha}_{med})$, and other selected fractiles.

Table 3. MONTE CARLO DISTRIBUTIONS OF $\hat{\alpha}_{.95}$ FOR $N=24$, $n=2494$; $N=49$, $n=1219$; $N=74$, $n=804$

Parameter or fractile	α = 2.0			1.9			1.7			1.5			1.3			1.1			1.0		
N	24	49	74	24	49	74	24	49	74	24	49	74	24	49	74	24	49	74	24	49	74
$\bar{\hat{\alpha}}$	1.77	1.83	1.85	1.67	1.77	1.80	1.48	1.60	1.64	1.30	1.41	1.44	1.11	1.21	1.24	.94	1.01	1.04	.86	.92	.95
$s(\hat{\alpha})$	.250	.195	.172	.309	.221	.194	.361	.269	.229	.367	.274	.234	.347	.253	.218	.315	.220	.191	.289	.203	.177
.01	1.16	1.32	1.42	.88	1.24	1.34	.66	1.02	1.14	.54	.84	.96	.45	.69	.79	.38	.57	.65	.34	.51	.58
.05	1.31	1.47	1.53	1.11	1.38	1.46	.88	1.18	1.28	.71	.99	1.09	.60	.82	.92	.50	.68	.77	.45	.62	.69
.10	1.39	1.53	1.59	1.25	1.45	1.52	1.01	1.26	1.35	.83	1.07	1.15	.70	.90	.98	.58	.75	.82	.52	.68	.74
.25	1.57	1.68	1.72	1.44	1.59	1.64	1.22	1.41	1.47	1.03	1.22	1.28	.87	1.04	1.10	.72	.86	.92	.65	.78	.83
$\hat{\alpha}_{med}$	1.83	1.90	1.93	1.70	1.80	1.83	1.48	1.58	1.62	1.27	1.38	1.42	1.08	1.19	1.22	.91	.99	1.03	.83	.90	.93
.75	2.00	2.00	2.00	2.00	2.00	2.00	1.78	1.82	1.80	1.52	1.57	1.57	1.32	1.36	1.36	1.12	1.16	1.14	1.02	1.05	1.04
.90	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.86	1.80	1.79	1.57	1.53	1.53	1.35	1.30	1.30	1.23	1.19	1.18
.95	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.97	1.91	1.80	1.64	1.63	1.49	1.40	1.38	1.37	1.28	1.27
.99	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.98	1.63	1.65	1.74	1.48	1.48
M.S.E. X 10	1.15	.67	.52	1.48	.66	.48	1.79	.82	.56	1.72	.83	.58	1.57	.72	.51	1.25	.57	.40	1.03	.48	.34

Since the value of α for a stable distribution must be in the interval $0<\alpha\leq 2$, observed values of $\hat{\alpha}_f$ greater than 2 were assigned the value 2. No values of $\hat{\alpha}_f$ at the lower bound $\alpha=0$ were observed in our experiments—because we only generated data for distributions with $1\leq\alpha\leq 2$.

Concentrating on Table 2, when $\alpha\geq 1.9$, the best estimator, in terms of both low bias and standard deviation, is $\hat{\alpha}_{.99}$. This reflects the fact that when α is close to 2, one must look further into the tails of symmetric stable distributions to find noticeable differences in the values of a given fractile for different values of α (see [6, Table 3]). And large samples permit one to obtain reliable estimates of these extreme fractiles.

But if one does not have both a large sample and knowledge that the true value of α is close to 2, then $\hat{\alpha}_{.99}$ is a relatively poor estimator. In terms of both minimum bias and dispersion it is dominated by both $\hat{\alpha}_{.95}$ and $\hat{\alpha}_{.97}$ when $\alpha<1.9$, and its inferiority relative to these two estimators is more substantial the lower the value of α.

When α is close to 1.0 there is no substantial advantage in using estimates of α based on fractiles lower than the .95. For example, in Table 2, the standard deviations of $\hat{\alpha}_{.93}$ are never substantially lower than those of either $\hat{\alpha}_{.95}$ or $\hat{\alpha}_{.97}$; indeed $s(\hat{\alpha}_{.95})$ and $s(\hat{\alpha}_{.97})$ are always lower than $s(\hat{\alpha}_{.93})$ when $\alpha\geq 1.3$, and the dominance of these two estimators is stronger the higher the value of α. Though not reported, values of $f<.93$ have been also tried (e.g., $f=.90$, .85), but the sampling distributions of $\hat{\alpha}_f$ obtained were more dispersed than those for $f\geq .93$.

Tables 2 and 3 show that all $\hat{\alpha}_f$ estimators have some downward bias for all values of α when $N\leq 99$. For larger sample sizes and for $\alpha\leq 1.7$, the estimators $\hat{\alpha}_{.93}$, $\hat{\alpha}_{.95}$, and $\hat{\alpha}_{.97}$ show trivial bias but more usually none at all. In large samples the bias of these estimators when α is close to 2 results from the truncation of the sampling distributions at the value $\hat{\alpha}=2$.

But though $\hat{\alpha}_{.95}$ and $\hat{\alpha}_{.97}$ are "robust" relative to other estimators in the class defined by (3.2), they may be inefficient *vis-à-vis* other classes of estimators. $\hat{\alpha}_f$ has few competitors[7] and two we had thought most promising were tested.

First, since sample fractiles are not perfectly positively correlated, some average of $\hat{\alpha}_f$ for different values of f might have a lower sampling dispersion than α_f alone. But choosing an optimum average would itself be a difficult problem. We know that for $f<.93$, $\hat{\alpha}_f$ is not individually as reliable as for higher values of f. On the other hand, closely adjacent values of f would cause components of the average to be highly intercorrelated. One compromise is the composite estimator,

$$\hat{\alpha}_k = [\hat{\alpha}_{.93} + \hat{\alpha}_{.99}]/2.$$

But in our experiments $\hat{\alpha}_k$ turned out to be slightly more biased and to have slightly higher standard deviation than $\hat{\alpha}_{.97}$.

An estimator completely unrelated to $\hat{\alpha}_f$ was suggested in [5]. A sum of m independent, identically distributed, stable variables is stably distributed with the same characteristic exponent but with scale parameter

$$c_m = m^{1/\alpha}c_1,$$

where c_1 is the scale parameter for individual terms in the sum. Solving this expression for α and substituting estimates of c_m and c_1 gives an estimator

$$\hat{\alpha}_c = \log_e m/(\log_e \hat{c}_m - \log_e \hat{c}_1). \tag{3.3}$$

We tested $\hat{\alpha}_c$ for 55 samples of size $N=999$ and summing intervals $m=2$, 5, 10. The scale parameters estimates $\hat{c}_m$ and $\hat{c}_1$ were obtained from (2.1). That is, for each sample of $N=999$, (2.1) was used to compute $\hat{c}_1$, then to compute $\hat{c}_2$ for non-overlapping sequential sums

[7] Indeed, to obtain a feeling for the frustrations met in earlier attempts to estimate α, see [5].

Table 4. POWERS OF THE SHAPIRO-WILK STATISTIC (SW), THE STUDENTIZED RANGE (SR), AND $\alpha_{.96}$ AS GOODNESS-OF-FIT TESTS FOR SAMPLE SIZES $N=24$, 49, 74

	α																	
PR(I)	1.9			1.7			1.5			1.3			1.1			1.0		
	SW	SR	$\hat{\alpha}_{.96}$	SW	SR	$\hat{\alpha}_{.96}$	SW	SR	$\hat{\alpha}_{.96}$	SW	SR	$\hat{\alpha}_{.96}$	SW	SR	$\hat{\alpha}_{.96}$	SW	SR	$\hat{\alpha}_{.96}$
								N = 24, n = 2494										
.010	.08	.07	.06	.23	.19	.20	.40	.32	.37	.59	.46	.59	.76	.56	.79	.83	.59	.86
.019	.10	.09	.08	.28	.25	.25	.46	.42	.44	.65	.57	.66	.81	.68	.84	.87	.70	.89
.048	.14	.16	.13	.32	.36	.32	.52	.54	.54	.70	.69	.74	.84	.78	.88	.90	.81	.93
.096	.19	.22	.19	.37	.44	.40	.57	.63	.62	.74	.75	.80	.86	.83	.92	.92	.86	.95
								N = 49, n = 1219										
.010	.16	.14	.02	.43	.41	.15	.71	.66	.41	.88	.81	.71	.97	.89	.91	.98	.92	.96
.020	.18	.17	.06	.45	.47	.24	.75	.71	.54	.90	.85	.75	.97	.92	.95	.99	.94	.98
.049	.22	.25	.11	.51	.60	.32	.77	.80	.62	.92	.91	.85	.98	.96	.97	.99	.96	.99
.098	.27	.33	.17	.56	.69	.43	.81	.86	.71	.93	.94	.90	.99	.97	.98	.99	.98	1.00
								N = 74, n = 804										
.010	.21	.25	.03	.55	.63	.18	.83	.84	.51	.96	.94	.83	.99	.98	.96	.99	.99	.98
.020	.23	.26	.04	.58	.65	.21	.86	.85	.55	.96	.95	.85	.99	.98	.97	.99	.99	.99
.050	.27	.34	.10	.62	.74	.34	.88	.91	.69	.98	.97	.90	.99	.99	.98	1.00	.99	.99
.099	.31	.40	.17	.66	.78	.45	.90	.93	.78	.98	.98	.94	.99	.99	.99	1.00	.99	1.00

of two observations and then similarly for sums of five and ten. The results of our tests for each of the three summing intervals were clear cut: $\hat{\alpha}_c$ is an inefficient estimator relative to $\hat{\alpha}_{.97}$. In every case the standard deviation of $\hat{\alpha}_c$ was at least twice that of the corresponding distribution of $\hat{\alpha}_{.97}$.

In short, for $.95 \leq f \leq .97$, the simple interfractile estimator $\hat{\alpha}_f$ has sampling properties that are "robust" against variation in the true value of α. This estimator has high reliability in large samples and is more efficient than the available competitors we have tested.

4. SOME GOODNESS-OF-FIT TESTS OF NORMALITY AGAINST NON-NORMAL STABLE ALTERNATIVES

Models based on normality assumptions are commonly justified by appeal to the Central Limit Theorem, since the variables under consideration are often sums of random variables. But the Gaussian is just one member of the class of limiting distributions, so that other distributions for which the Generalized Central Limit Theorem also applies (*viz.*, non-normal stable distributions) are appealing alternatives in goodness-of-fit tests of Gaussian null hypotheses.

Tables 4 and 5 report the powers of several goodness-of-fit tests of normality against non-normal symmetric stable alternatives. The test statistics examined are the Shapiro-Wilk (SW), the studentized range (SR), and $\hat{\alpha}_f$ for $f = .96$, .99. Let $\hat{x}_j$ be the jth order statistic from a sample of size N. Then the studentized range [3] and the Shapiro-Wilk statistic [13] are defined as

$$\mathrm{SR} = (\hat{x}_N - \hat{x}_1) \Big/ \left[\frac{1}{N-1} \sum_{j=1}^{N} (\hat{x}_j - \bar{x})^2\right]^{1/2},$$

$$\mathrm{SW} = \sum_{j=1}^{I(N/2)} a_{N-j+1}(\hat{x}_{N-j+1} - \hat{x}_j) \Big/ \sum_{j=1}^{N} (\hat{x}_j - \bar{x})^2,$$

where $\bar{x}$ is the sample mean of the $\hat{x}_j$, $I(N/2)$ is the greatest integer equal to or less than $N/2$, and the weights a_{N-j+1} are the approximated coefficients suggested by Shapiro and Wilk [13, p. 596]. The statistic $\hat{\alpha}_f$ is, of course, the estimator of α defined by (3.2) and discussed in Section 3.[8]

[8] The analysis summarized in Tables 4 and 5 was also performed for the following goodness-of-fit statistics: Sherman's statistic [15]; the Kolmogorov-Smirnov [9], Cramér-Von Mises [2], and Anderson-Darling [1] tests; Durbin's [4] Modified Median, Modified Probability, and Modified Kolmogorov tests; and chi-square with 9 d.f. In each case these statistics had lower power than SR and SW for the symmetric stable alternatives considered. Of course, it is not too surprising that SR and SW performed better. They were specifically designed for a Gaussian null hypothesis while the other tests are distribution free.

Table 5. POWERS OF THE SHAPIRO-WILK STATISTIC (SW), THE STUDENTIZED RANGE (SR), AND $\alpha_{.99}$ AS GOODNESS-OF-FIT TESTS FOR SAMPLE SIZES $N=99, 199, \ldots, 599$

	α							α							α					
	1.9			1.7				1.9			1.7				1.9			1.7		
PR(I)	SW	SR	$\hat{\alpha}_{.99}$	SW	SR	$\hat{\alpha}_{.99}$	PR(I)	SW	SR	$\hat{\alpha}_{.99}$	SW	SR	$\hat{\alpha}_{.99}$	PR(I)	SW	SR	$\hat{\alpha}_{.99}$	SW	SR	$\hat{\alpha}_{.99}$
	N = 99, n = 604							N = 199, n = 299							N = 299, n = 199					
.010	.23	.33	.26	.63	.72	.67	.010	.35	.54	.18	.81	.91	.78	.010	.44	.66	.17	.86	.98	.82
.020	.26	.39	.30	.65	.78	.74	.020	.36	.62	.22	.82	.94	.82	.020	.45	.73	.24	.87	.99	.90
.050	.30	.42	.39	.68	.80	.81	.050	.40	.64	.34	.85	.95	.90	.050	.47	.75	.32	.92	1.00	.94
.099	.37	.48	.45	.72	.84	.85	.100	.43	.67	.44	.87	.96	.95	.100	.49	.81	.52	.92	1.00	.98
	N = 399, n = 149							N = 499, n = 119							N = 599, n = 99					
.007	.49	.73	.21	.93	1.00	.95	.008	.52	.86	.13	.97	1.00	.93	.010	.57	.90	.24	.97	1.00	.96
.020	.49	.73	.23	.93	1.00	.95	.017	.54	.86	.18	.98	1.00	.96	.020	.58	.90	.35	.97	1.00	.98
.047	.51	.81	.47	.93	1.00	.98	.050	.58	.87	.36	.98	1.00	.97	.050	.61	.91	.61	.97	1.00	.98
.100	.55	.85	.57	.94	1.00	.98	.100	.58	.88	.62	.98	1.00	.98	.100	.61	.92	.75	.97	1.00	1.00

Monte Carlo distributions of each statistic were obtained for nine sample sizes N and seven values of α, *including null distributions for* $\alpha=2$. For SW and $\hat{\alpha}_f$, departures from normality lead to low values of the test statistics. Thus, for given N and $\alpha<2$, the power for a Type I error $PR(I)$ was the proportion of times the Monte Carlo test statistic was below the $PR(I)$ fractile of the Monte Carlo null distribution. On the other hand, since non-normal stable distributions would be expected to yield large values of the studentized range, the power of SR, given $\alpha<2$, N and $PR(I)$, was the proportion of times SR exceeded the $1-PR(I)$ fractile of the Monte Carlo distribution of SR for $\alpha=2$.

As indicated in Footnote 4, the Monte Carlo data distributions for different values of α are obtained from the same underlying sample of cumulative probabilities so that when the null distributions are computed from the Monte Carlo data, the resulting test powers measure the pure effect of varying α in the non-null distribution.[9] But in fact our procedures are dictated as much by necessity as by virtue. Exact fractiles of the null distributions of test statistics reported in Tables 4 and 5 have been tabulated [3] only for the studentized range, and then only for a limited number of sample sizes and fractiles.

For the larger sample sizes (Table 5) results are reported only for $\alpha=1.9$, 1.7. Lower values of α are omitted since when $N\geq 99$, both SR and SW have almost perfect power against alternatives with $\alpha\leq 1.5$. For the sample sizes reported in Table 5, SR shows more power than SW for *all* α. But for $N=24$, 49 (Table 4), SW shows more power than SR in some cases. This is in conformance with the results of Shapiro, Wilk, and Chen [14] who do not consider the larger sample sizes. Even in the smaller samples, however, the advantage of SW over SR is marginal except for $N=24$ and $\alpha\sim 1$. Thus given the relative difficulty of obtaining the coefficient weights a_{N-j+1} required to estimate SW, the studentized range SR would seem to be a good general technique for goodness-of-fit tests of normality against non-normal stable alternatives.

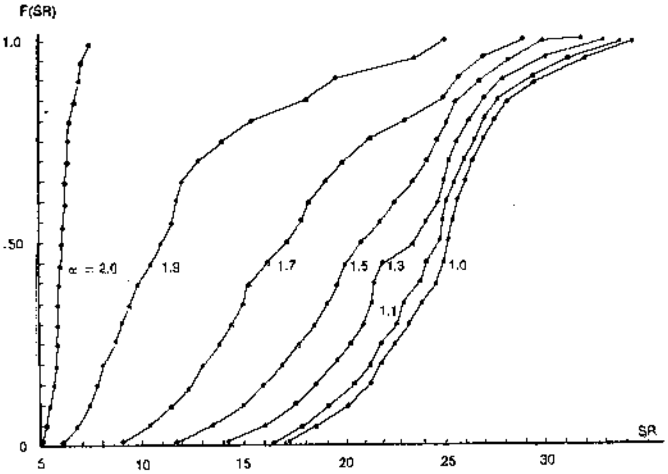
Finally, any procedure for estimating α can be used as the basis of goodness-of-fit tests of normality. Tables 4 and 5 suggest, however, that $\hat{\alpha}_f$ performs poorly as a goodness-of-fit test compared to SW and especially SR. On the other hand, although SW and SR work well in

Table 6. TESTS OF STABILITY

Distribution	N = 999	Differences for sums of 2	5	10
Stable: α = 1.7	1.698 (.066)	-.0161 (.088)	-.0097 (.165)	-.0793 (.205)
Stable: α = 1.3	1.298 (.056)	-.0151 (.056)	-.0333 (.134)	-.1074 (.174)
Mixture of Normals	1.031 (.021)	.3773 (.064)	.7425 (.112)	.8331 (.135)

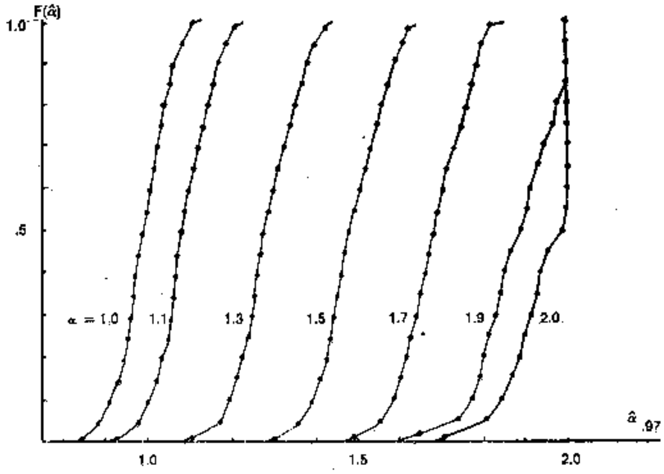
[9] This should not imply, however, that true test powers are reported in Tables 4 and 5. Even though the Monte Carlo sample sizes are large, there is still enough sampling variation to cast doubt on the accuracy of the second digit of each entry.

Figure 1. C.D.F. OF STUDENTIZED RANGE (SR) FOR $N=599^{a}$



[a] For each distribution, only the extremes and every twentieth point are plotted

Figure 2. C.D.F. OF $\hat{\alpha}_{.97}$ FOR $N=599^{a}$



[a] For each distribution, only the extremes and every twentieth point are plotted.

distinguishing the normal distribution from other members of the stable class, they would not be very powerful (*vis-à-vis* $\hat{\alpha}_f$) in distinguishing among non-normal stable distributions (and thus in estimating α). To support this statement, we offer a single but representative example: the Monte Carlo c.d.f.'s of SR and $\hat{\alpha}_{.97}$ for $N=599$ are shown in Figures 1 and 2. The overlapping sections of the SR c.d.f.'s for adjacent values of α are much longer than those for $\hat{\alpha}_{.97}$, *except between* $\alpha=1.9$ *and* $\alpha=2.0$. These results, typical of other sample sizes and of SW, explain why SR (or SW) is the better goodness-of-fit test of normality, while $\hat{\alpha}_f$ is better in estimating the characteristic exponent of a non-normal stable distribution.

5. AN EXPERIMENT TO TEST STABILITY

Data analysis techniques based on sample moments are embedded in the social sciences. Economists, psychologists, and sociologists frequently dismiss non-normal stable distributions as data models because

a. They cannot believe that processes generating prices, breakdowns, or riots, can fail to have second moments (usually because the relevant variables are bounded); and
b. The widely observed "thick-tails" of empirical distributions could be generated, for example, by a process that is a suitable *mixture* of normals rather than by non-normal stable distributions (cf. [12]).

The statement in (a) will be considered later. Now we consider one way to check the statement in (b).

Our suggested test is based on the definition of the stable class: every sum of independent stable variates with a given characteristic exponent α has a stable distribution with the same α. Thus if we estimate α for an entire random sample and then reestimate it for non-overlapping *sums* of observations drawn from the sample, the two estimates will tend toward equality (after accounting for bias and sampling dispersion) if the sample is actually drawn from a stable distribution. If the sample is not drawn from a stable distribution but rather from some process *with finite second moment*, the sums should yield an estimate of α closer to 2 than the estimate obtained from the individual observations.

Table 6 reports an experiment with this procedure based on 55 random samples of 999 observations each for (a) two stable distributions ($\alpha=1.3$, 1.7), and (b) a process that is a Gaussian mixture in which a given drawing is, with equal chance, either from a normal distribution with mean zero and standard deviation $s=1$ or from another normal again with mean zero but with $s=9$. For each of these processes and for each of the 55 samples of size $N=999$, $\hat{\alpha}_{.97}$ was computed according to (3.2). The means and standard deviations (in parentheses) of these estimates are shown in the first column of Table 6. Each sample of 999 was also transformed first into a sample of 499 non-overlapping sums of 2 observations, then into 199 sums of five observations, and finally into 99 sums of 10 observations; and $\hat{\alpha}_{.97}$ was estimated for each of the three samples of sums. The difference between the value of $\hat{\alpha}_{.97}$ obtained from a sample of sums and $\hat{\alpha}_{.97}$ obtained from the corresponding unsummed data sample was then computed. The averages and standard deviations (in parentheses) of these differences are shown in the last three columns of Table 6.

When the true process is non-normal stable, the estimates of α drift downward slightly as the number in the sum is increased. This drift is due to the known downward bias in $\hat{\alpha}_{.97}$ (see Section 3) that increases as the sample size is decreased. On the other hand, the estimated α from the mixture of normals increases substantially as the number in the sum increases. For the samples of 999, the mean estimated α is 1.031. For sums of 2 from the same sample, the mean estimated α is $1.031+.377=1.408$, and for sums of 10 the mean estimated α jumps to 1.864. The largest of the 55 estimates for the unsummed data is 1.084. For the sums, none of the 165 estimates is below 1.25. These preliminary results suggest that our proposed procedure is likely to be of some value in distinguishing "thick-tailed" mixtures of normals from non-normal stable processes.

Finally, the numbers produced by computers are, of course, bounded (by 10^{64} in our case) so that our "stable" distributions are truncated and all their moments exist. Nevertheless, summing these truncated variables did not produce convergence toward a normal distribution, and in fact the data provide results indistinguishable from those that would be obtained from an unbounded stable process. The reason is clear: the bound is so large that the chance of hitting it is negligible. (In fact, the largest number generated in this study was 20,036.) Still the results are sufficient to dispel the (somewhat naive) notions that boundedness in itself is sufficient either to justify normality assumptions for sums of random variables or to reject models which assume that the underlying variables behave *as if* they were generated by processes whose higher-order moments do not exist.

6. SUMMARY

To review, simple interfractile range estimators of the scale and characteristic exponent of symmetric stable distributions perform well compared to available competitors. The distribution of the scale estimate $\hat{c}$ defined by (2.1) approaches its asymptotic normal distribution at small sample sizes (e.g., $N \geq 50$), so that its "large sample" properties (which can be determined analytically) will describe its sampling behavior in many applications. Unlike the mean absolute deviation (MAD), $\hat{c}$ has a sampling distribution that is fairly insensitive to the true value of α, which is usually unknown. And $\hat{c}$ is a more efficient estimate of scale than the MAD, except when α is close to 2.

Similarly, the interfractile estimator of α defined by (3.1) and (3.2) is easy to compute, seems to be reliable (at least in large samples), and is more efficient than alternative estimators we have examined.

Finally, our Monte Carlo experiments also recommend the Studentized range for goodness-of-fit tests of a Gaussian null hypothesis against non-normal stable alternatives at all sample sizes.

REFERENCES

[1] Anderson, T. W. and Darling, D. A., "A Test for Goodness of Fit," *Journal of the American Statistical Association*, 49 (December 1954), 765–9.
[2] Cramér, Harald, *Mathematical Methods of Statistics*, Princeton, N.J.: Princeton University Press, 1945.
[3] David, H. A., Hartley, H. O. and Pearson, E. S., "The Distribution of the Ratio, in a Single Normal Sample, of Range to Standard Deviation," *Biometrika*, 41 (1954), 482–93.
[4] Durbin, J., "Some Methods of Constructing Exact Tests," *Biometrika*, 48 (1961), 41–55.
[5] Fama, Eugene F., "The Behavior of Stock Market Prices," *Journal of Business*, 38 (1965), 34–105.
[6] ———, and Roll, Richard, "Some Properties of Symmetric Stable Distributions," *Journal of the American Statistical Association*, 63 (September 1968), 817–36.
[7] Feller, W., *Introduction to Probability Theory and Its Applications*, Vol. 2. New York: John Wiley and Sons, Inc., 1966.
[8] Gnedenko, B. V. and Kolmogorov, Z. N., *Limit Distributions for Sums of Independent Random Variables*, Translated by K. L. Chung, Reading, Mass.: Addison-Wesley Publishing Co., 1954.
[9] Kolmogorov, A. N., "Sulla Determinazione Empirica di Una Legge di Distribuzione," *Giornale dell' Istituto Italiano degli Attuari*, 4 (1933).
[10] Lamperti, John, *Probability*, New York: W. A. Benjamin, Inc., 1966.
[11] Mandelbrot, Benoit, "The Variation of Certain Speculative Prices," *Journal of Business*, 36 (1963), 394–419.
[12] Press, S. James, "A Compound Events Model for Security Prices," *Journal of Business*, 40 (1968), 317–35.
[13] Shapiro, S. S. and Wilk, M. B., "An Analysis of Variance Test for Normality (Complete Samples)," *Biometrika*, 52 (1965), 591–611.
[14] Shapiro, S. S., Wilk, M. B. and Chen, H. J., "A Comparative Study of Various Tests for Normality," *Journal of the American Statistical Association*, 63 (December 1968), 1343–72
[15] Sherman, B., "A Random Variable Related to the Spacing of Sample Values," *Annals of Mathematical Statistics*, 21 (1950), 339–61.