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VALUATION OF CORPORATE CLAIMS

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Optimal Financial Policy and Firm Valuation

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SINCE THE CLASSIC WORK of Miller and Modigliani (1961) laid down the principles for the valuation of firms under conditions of certainty some two decades ago academic interest in the problem of valuing the individual firm has waned. Yet, in addition to the obvious importance of this problem for security analysts, investors, and acquirers of corporations, the issue of firm valuation is fundamental to much of the theory of corporate finance: what determines the risk of the firm; how the rate of return required by investors may be inferred from capital market data; the influence of financing policy on firm value, and other issues of central concern to the theory of financial management.

This almost total neglect of firm valuation is attributable perhaps to the overwhelming influence of the research programme initiated by the development of the capital asset pricing model only three years after the publication of the above-mentioned paper by Miller and Modigliani. In its primary form, the capital asset pricing model is a statement about the equilibrium rates of return on securities, rather than about the equilibrium values of securities. It is straightforward to transform the model into a statement about the dependence of current equilibrium values upon an exogenously specified distribution of end of period values. However, in this form the model is inherently unsatisfactory since it fails to take account of the fact that in reality the distribution of end of period firm values is not exogenously given but is determined as part of a subsequent capital market equilibrium. Attempts to iterate the single period capital asset pricing model over several periods in order to avoid the need to take the distribution future security values as exogenous have been unsuccessful for the most part, since, not being able to derive, they have been forced to assume, both the requisite normality of the distribution of subsequent security prices, and the legitimacy of ignoring endogenously determined shifts in the individual investor's investment

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opportunity set or the market price of risk.[1] The only rigorously derived mean-variance capital asset pricing model which expresses equilibrium prices in terms of the characteristics of future cash flows, without making unwarranted assumptions about the endogenously determined distributions of intermediate security prices, is that of Stapleton and Subrahmanyam (1978)[2]; however, the complexity of the resulting valuation expressions renders them nugatory for practical purposes.

There have of course been other rigorously derived valuation models. Prior to the path breaking work of Cox, Ingersoll and Ross (1978) which is referred to below and which provides the basis for our analysis, the most notable contributions were the (effectively) complete markets models of Rubinstein (1976) and of Breeden and Litzenberger (1978). These models were concerned almost exclusively with the valuation of individual cash flows rather than of firms; and, while it is undoubtedly true that the value of a firm is nothing but the value of the cash flows it will yield, a useful model of firm value must rest on a description of the firm which is at once richer and more parsimonious than a mere summation of the values of a series of time dated cash flows, whose stochastic characteristics must be individually enumerated. After all, the cash flows from a firm will depend upon the investment and financing opportunities which are open to the firm and the managerial policies towards these opportunities. Indeed desirable characteristics of a useful firm valuation model would include the following:[3]

1) multi-period valuation within a consistent theoretical framework which takes account of risk;
2) a parsimonious description of the firm in terms of variables which are quantifiable and have clear empirical counterparts;
3) a description of the investment opportunities facing the firm;
4) a description of the financing alternatives available to the firm which is consistent with the rational pricing of securities, and takes account of constraints imposed on financing strategies by prior contracts and bond indentures;
5) a role for management decisions so that the effects of alternative financing and investment strategies may be assessed:
6) a simple closed form valuation expression.

The model developed here possesses all of the above characteristics except the last; and while the lack of a closed form valuation expression precludes standard comparative static analysis, the model lends itself readily to simple numerical analysis. The basis of the model is the Cox, Ingersoll, Ross (1978) partial

[1] Fama (1977) has criticized Bogue and Roll (1974) for a multi-period application of the CAPM which implicitly violates the conditions for the single period CAPM to hold. Fama's own model avoids this difficulty, as does that of Myers and Turnbull (1977), by assuming a deterministic market price of risk. Unfortunately since this parameter is determined endogenously in equilibrium this is assuming away the problem. Brennan (1973) appears to have understood the issue and presented a rudimentary resolution.

[2] This model has been criticized by Bhattacharya (1981) for taking the interest rate as exogenous. However it can be rescued from this criticism by treating it as a pure exchange model in which aggregate consumption follows a random walk.

[3] We are explicitly ignoring agency, moral hazard, and other problems arising from information asymmetries.

differential equation for the value of an asset, interpreted specifically for the problem at hand. This equation is itself the result of combining the continuous time capital asset pricing model with the assumption that the pricing function for securities is consistent with rational expectations.

The particular valuation model presented in this paper represents a natural extension of two earlier models developed by the authors, which were ostensibly concerned with regulation and its consequences for valuation[4]. The first paper dealt with the valuation of a regulated firm whose investment policy was exogenously determined; in the second paper managerial discretion was introduced so that investment policy was endogenous. In this paper the firm is assumed to choose not only its investment policy but also its debt financing policy from a set of feasible policies, which is determined by its investment opportunities, the capital market equilibrium, and the provisions of its bond indenture.

In Section I we describe briefly the nature of the assumed capital market equilibrium and derive a general partial differential equation which governs the values of all financial claims. In Section II this partial differential equation is applied to the debt and equity claims of the firm under consideration by specifying which state variables are relevant and describing their stochastic evolution. Sufficient conditions are given for the state of the firm at any point in time to be describable in terms of three accounting variables: the book values of assets and of debt, and the rate of return on assets. The first two state variables are controllable by the firm, since they are determined by investment and financing decisions. The return on assets is partially controllable depending both on the investment policy of the firm and on stochastic changes in profitability. The firm's investment and financing decisions are constrained, not only by the consistency requirement imposed by the sources and uses of funds constraint, but also by the provisions of the bond indenture; the bond indenture also provides boundary conditions for the values of debt and equity claims. In Section III the model is used to analyze the financing and investment strategies of a hypothetical firm.

I. Capital Market Equilibrium and the Fundamental Valuation Equation

Following Merton (1971), it is assumed that each individual in the economy possesses a time-additive concave von Neumann Morgenstern utility function defined over the rate of consumption of a single good; that there are no investor taxes or transactions costs; that trading takes place continuously, and that the perfectly competitive capital market is always in equilibrium. In addition it is assumed that the conditions for aggregation are satisfied[5] so that for purposes of valuation the state of the economy at any time may be characterized in terms of aggregate wealth, W, and an s-dimensional vector of state variables, X. The behaviour of the state variables, X, is governed by the system of stochastic

[4] Brennan and Schwartz (1982a, 1982b).

[5] These conditions guarantee that equilibrium prices and rates of return are independent of the (endogenous and stochastic) distribution of wealth across individuals. This permits us to ignore the distribution of wealth in our description of the economy. For further discussion of the concept of aggregation in financial models see Rubinstein (1974), Brennan and Kraus (1978) and Milne (1979).

differential equations

$$dX_k = \mu_k(X, t)\, dt + \eta_k(X, t)\, dz_k \quad (k = 1, \ldots, s) \tag{1}$$

where t denotes calendar time and dz_k is a standard Gauss-Wiener process with $dz_j \cdot dz_k = \rho_{jk}\, dt$. The derived utility of wealth function for an individual $l(l = 1, \cdot, m)$ may be written as a function of his wealth, w_l, the state variables, X, and time t: $J^l(w_l, X, t)$. Then, as Merton has shown, the equilibrium rates of return on individual securities will satisfy the intertemporal capital asset pricing model:

$$\alpha_i - r = (W/A)\sigma_{iw} + \textstyle\sum_{k=1}^{s} (H^k/A)\sigma_{ik} \quad (i = 1, \ldots, n) \tag{2}$$

where

- α_i expected instantaneous rate of return on security $i(i = 1, \cdot, n)$
- r the instantaneously riskless interest rate
- σ_{iw} the covariance of the instantaneous rate of return on security i with the return on aggregate wealth
- σ_{ik} the covariance of the instantaneous rate of return on security i with the instantaneous rate of change in state variable $k(k = 1, \ldots, s)$
- W $= \sum_{l=1}^{m} w_l$ aggregate wealth in the economy
- A $= -\sum_{l=1}^{m} J^l_w/J^l_{ww}$ aggregate risk tolerance in the economy
- H^k $= -\sum_{l=1}^{m} J^l_{wx_k}/J^l_{ww}$ $(k = 1, \cdot, s)$

Equation (2) decomposes the instantaneous risk premium on a security into one component which is proportional to the covariance of its rate of return with the return on aggregate wealth, and s components which are proportional to the covariance of the rate of return with the state variables. The assumption of risk aversion ensures that the coefficient of σ_{iw} is positive; however, the coefficients of the remaining covariances cannot be signed without introducing additional structure which is unnecessary for the purpose of this paper. The assumption of aggregation ensures that the coefficeints A and $H^k(k = 1, \ldots, s)$ do not depend upon the distribution of wealth.

Cox, Ingersoll, and Ross (1978) have shown that if investors possess rational expectations[6] so that they know the functions mapping the state variables, X, W, and t into asset prices, then the equilibrium condition (2) implies a fundamental partial differential equation which must be satisfied by the value of all financial assets.

Thus, defining $F_i \equiv F_i(W, X, t)$ as the market value of asset i, the instantaneous change in its value is obtained from Ito's Lemma[7] as

$$\begin{aligned} dF = [&\textstyle\sum_{j=1}^{s} F_{x_k} j_k + F_w(W\alpha_w - C) + F_t \\ &+ \tfrac{1}{2} \textstyle\sum_{k=1}^{s} \sum_{k'=1}^{s} F_{x_k x_{k'}} \rho_{kk'} \eta_k \eta_{k'} \\ &+ \tfrac{1}{2} \textstyle\sum_{k=1}^{s} F_{x_k w} \rho_{kw} \eta_k \sigma_w W + \tfrac{1}{2} F_{ww} \sigma_w^2 W^2]\, dt \\ &+ \textstyle\sum_{k=1}^{s} F_{x_k} \eta_k\, dz_k + F_w \sigma_w W\, dz_w \end{aligned} \tag{3}$$

[6] See Lucas (1978).

[7] See McKean (1968); the subscript i is omitted for the sake of clarity and the partial derivatives of F are denoted by the appropriate subscripts.

In this expression α_w is the instantaneous expected rate of return on aggregate wealth and C is the rate of aggregate consumption, so that $(W\alpha_w - C)$ is the instantaneous expected change in aggregate wealth. $\sigma_w W\, dz_w$ is the unanticipated change in aggregate wealth, where dz_w is a standard Gauss Wiener process, and σ_w is the standard deviation of the instantaneous rate of change (which may depend on X).

Reverting to equation (2), the expected instantaneous rate of return on asset i, α_i, is the sum of the instantaneous payout rate on the asset, which we write as $\delta_i(W, X, t)$, and the expected price change, divided by the current value of the asset, F:

$$\alpha_i = [\cdot]F^{-1} + \delta F^{-1} \tag{4}$$

where $[\cdot]$ is the coefficient of dt in equation (3).

It also follows from equation (3) that σ_{iw}, the covariance between the rates of return on asset i and on aggregate wealth, is given by

$$\sigma_{iw} = F^{-1}[\textstyle\sum_{k=1}^{s} F_{x_k}\eta_k\rho_{kw}\sigma_w + F_w\sigma_w^2 W^2] \tag{5}$$

Finally, the covariances between the rate of return on the asset and the state variables are

$$\sigma_{ik} = F^{-1}[F_{x_k}\eta_k^2 + F_w W\rho_{kw}\eta_k\sigma_w] \quad (k = 1, \ldots, s) \tag{6}$$

Substituting the expressions for α_i, σ_{iw} and σ_{ik} from equations (4), (5), and (6) in the equilibrium condition (2) yields

$$\begin{aligned} &\textstyle\sum_{k=1}^{s} F_{x_k}[\mu_k - \lambda_w\eta_k\rho_{kw}\sigma_w - \lambda_k\eta_k^2] + F_w[rW - C] + F_t \\ &+ \tfrac{1}{2}\textstyle\sum_{k=1}^{s}\sum_{k'=1}^{s} F_{x_k x_{k'}}\rho_{kk'}\,\eta_k\eta_{k'} + \sum_{k=1}^{s} F_{wx_k}\rho_{kw}\eta_k\sigma_w W \\ &+ \tfrac{1}{2} F_{ww}\sigma_w^2 W^2 - rF + \delta = 0 \end{aligned} \tag{7}$$

where $\lambda_w = W/A$ and $\lambda_k = H^k/A$.

In deriving this equation we have used the result, obtainable by multiplying the equilibrium conditions (2) by the market value of asset i and summing over i, that

$$\alpha_w - r = \lambda_w\sigma_w^2 + \textstyle\sum_{k=1}^{s} \lambda_k\eta_k\rho_{kw}\sigma_w \tag{8}$$

Equation (7), which is the basis of our firm valuation model, is equivalent to equation (25) of Cox, Ingersoll and Ross (1978) who have described it as "the fundamental valuation equation for contingent claims". When appropriate boundary conditions are appended, this equation determines the value of any security. In the following section we specialize and simplify it in order to develop a tractable valuation model for the individual firm.

II. Firm Valuation

The Stochastic Process for Earnings and Investment Policy

Consider a firm which invests in a single homogeneous asset whose profitability evolves stochastically over time in response to the firm's investment policy. In particular, the instantaneous net cash flow rate of the firm, s, is assumed to be given by the difference between an autonomous cash flow rate which depends only on the current state, and a controllable cash flow which depends on the firm's investment policy. The current state is described by two variables: y represents the autonomous cash flow rate per unit of scale, and z represents the current scale of the firm (for example the number of machines), so that the autonomous cash flow rate can be expressed as the product $y \cdot z$. The investment control, g, is the rate of growth of the firm, so that

$$dz = g \cdot z \cdot dt \tag{9}$$

The controllable cash flow also is assumed also to be proportional to the current scale so that the net cash flow rate is

$$s = y \cdot z - \phi(g) \cdot z \tag{10}$$

Finally, changes in the profitability variable, y, follow the controllable stochastic differential equation:

$$dy = \mu(y, g, t)\, dt + \delta(y, g, t)\, dw \tag{11}$$

Competition in the market for physical capital implies that the marginal cost of increments of capacity is constant so that $\phi(\cdot)$ is linear[8]:

$$\phi(g) = a + bg \tag{12}$$

Defining $x \equiv y - a$ and using (12), the cash flow rate in equation (10) can be written as

$$s = x \cdot z - b \cdot g \cdot z \tag{13}$$

Setting $b = 1$ an accounting system is assumed to be chosen such that $A \equiv z$ is the book value of the assets of the firm; $d \equiv a$ is the depreciation rate, and $\pi = xA$ is the instantaneous rate of accounting earnings before interest and taxes. The stochastic process for accounting earnings is then described by

$$\begin{aligned} \pi &= xA \\ dA &= gA\, dt \\ dx &= \mu(x, g, t)\, dt + \sigma(x, g, t)\, dw \end{aligned} \tag{14}$$

This system suffices to describe the behaviour of the net cash flow rate of the firm before interest and taxes but after investment, $s \equiv (x - g)A$.

The firm's assets are assumed to be financed by a single homogeneous debt series and by common stock, as illustrated in Figure 1.

The face value of the debt is denoted by M and the book value of the equity by S. The financing control, f, is defined as the rate of growth in the face value of the debt so that

$$dM = fM\, dt \tag{15}$$

The net rates of cash distributions to stockholders and bondholders, δ_E and δ_B,

[8] $\phi(\cdot)$ would be proportional to g only if no outlays are required to maintain the current scale.

		Book Values		Market Values
Assets:	A	Bonds:	M	$B(x, A, M, t)$
		Common stock and retained earnings:	S	$E(x, A, M, t)$
	A	=	$M + S$	$V(x, A, M, t)$

x: current return on assets

Firm Balance Sheet

Figure 1

are then

$$\delta_B = cM - fB \tag{16a}$$

$$\delta_E = (x - g)A - \delta_B - T \tag{16b}$$

where c is the continuous coupon rate on the bonds, B is the market value of the outstanding bonds and T is the rate of corporate tax payments. (16a) expresses the net rate of distributions to bondholders as the difference between coupon payments on outstanding bonds and the dollar rate of new bond issues, and embodies the rational expectations assumption that new issues are floated at the same price as outstanding bonds. In (16b) $(x - g)A$ is the rate of cash flow available for distribution to investors before tax, and stockholders receive what is left after payment to bondholders, δ_B, and taxes, T. Assuming full loss offset provisions the corporate tax is[9]

$$T = (xA - cM)\tau \tag{17}$$

where τ is the corporate tax rate.

Since we shall be concerned with the effects on the value of the firm of alternative investment policies, represented by the asset growth rate g, and financing policies, represented by the debt growth rate f, it is necessary to consider how the asset growth rate affects the drift in the return on assets, $\mu(x, g, t)$.

It is assumed that new investment contributes to profits instantaneously, and that the return on additional assets, ρ, follows the same stochastic process as, and is perfectly correlated with, the return on the pre-existing assets. Then if the return on assets in the absence of any net change in assets follows the stochastic process

$$dx = \mu(x, t)\, dt + \sigma(x, t)\, dw \tag{18}$$

it is simply shown that the effect of growth in assets is to change the stochastic process to

$$dx = (\mu(x, t) + (\rho(x, g, t) - x)g)\, dt + \sigma(x, t)\, dw \tag{19}$$

[9] The corporate tax may also be formulated as $T = \max[0, (xA - cM)_\tau]$ which assumes no loss offset. The Canadian and U.S. tax systems appear to be intermediate between these extremes.

In (19) $\rho(x, g, t)$ is the rate of return on new assets at time t which may depend on the return on pre-existing assets as well as on the rate of growth in assets (as would be the case when there is a decreasing marginal return on capital).

Partial Differential Equation for Security Values

To simplify the valuation problem, the interest rate r[10] is taken to be an intertemporal constant, and we assume further that

$$\mu(x, g, t) - \lambda_w \sigma(x, t)\rho_{xw}\sigma_w - \lambda_x \sigma^2(x, t) = \mu(x, g, t) - \lambda\sigma(x, t) = \hat{\mu}\ (x, g, t) \quad (20)$$

Condition (20) is sufficient to ensure that none of the coefficients in the partial differential equation (7), when it it applied to the firm's securities involve aggregate wealth W or any of the state variables beyond those pertaining directly to the firm: x, A, M, t. Without such an assumption the value of the firm's securities will in general depend on all of the state variables through their effect on the market price of risk parameters λ_w and $\lambda_k (k = 1, \ldots, s)$. Sufficient conditions for (20) to be satisfied include universal logarithmic utility which implies $\lambda_w = 1$, $\lambda_k = 0$, $(k = 1, \ldots, s)$ and universal power utility with $\lambda_x = 0$: power utility ensures that λ_w is non-stochastic and λ_x will be equal to zero if x is independent of the opportunity set facing investors in the capital market.

With the foregoing assumptions, the market value of the firm's debt and equity securities will be functions only of the firm specific state variables and time which we write as $B(x, A, M, t)$ and $E(x, A, M, t)$ respectively, as can be seen in Figure 1.

The rate of growth of the face value of debt and the rate of growth in the book value of the assets, f and g, are policy controls to be chosen by the firm management from some feasible set. For given values of these controls the market values of the firm's two securities satisfy the following simplified versions of the fundamental equation (7):

$$L^{f,g}E(x, A, M, t) + \delta_E(x, A, M, f, g, t) - rE = 0 \quad (21)$$

$$L^{f,g}B(x, A, M, t) + \delta_B(x, A, M, f, g, t) - rB = 0 \quad (22)$$

$L^{f,g}K(x, A, M, t)$ is the differential operator of K associated with the controls f and g:

$$L^{f,g}K \equiv \tfrac{1}{2}\sigma^2 K_{xx} + \hat{\mu}(x, g, t)K_x + gAK_A + fMK_M + K_t \quad (23)$$

It is assumed that management selects financing and investment policies with the object of maximizing the value of the firm's outstanding shares and that this is fully anticipated by investors. Then the value maximizing policies, f, g, and the value of the firm's securities under the value maximizing strategy satisfy[11]

$$L^{\hat{f},\hat{g}}E + \delta_E - rE = 0 \quad (24)$$

$$L^{\hat{f},\hat{g}}B + \delta_B - rB = 0 \quad (25)$$

[10] A sufficient condition for this is that aggregate consumption follow a geometric random walk and that the representative investor have a time additive iso-elastic utility function.

[11] See Cox, Ingersoll and Ross (1978) Lemma 1; Fleming and Rishel, Ch. VI (1975); Merton (1971) Theorem 1.

where

$$L^{\hat{f},\hat{g}}E(x, A, M, t) + \delta_E(x, A, M, \hat{f}, \hat{g}, t)$$
$$= \max_{f,g\in D}[L^{\hat{f},\hat{g}}E(x, A, M, T) + \delta_E(x, A, M, f, g, t)] \quad (26)$$

and D is the set of feasible investment and financing policies.

Bond Indenture

To protect themselves against expropriation, bondholders insist on certain provisions in the bond indenture which limits the range of investment and financing policies open to management. These provisions are typically framed in terms of accounting numbers which do not enter most financial models. However, since our basic description of the firm is in terms of variables produced by the accounting system, we are able to model the protective covenants described below.

First the firm is prohibited from selling assets since this would reduce the security of the bondholders: thus the maximum rate of disinvestment is constrained by the depreciation rate of the assets, d, so that

$$g \geq -d \quad (27)$$

The firm is also prohibited from issuing additional debt unless certain interest coverage and asset coverage tests are met. These are expressed by

$$f \leq 0 \quad \text{if } M \geq \bar{A} \quad (28)$$

$$f \leq 0 \quad \text{if } \bar{k}cM \geq xA \quad (29)$$

where $\bar{m}$ is the maximum debt ratio allowed and $\bar{k}$ is the minimum interest coverage for new debt to be floated. Note that since new debt issues are continuous there is no need to specify whether these coverage tests apply on a current or a pro-forma basis.

It is worth observing that the firm is permitted to repurchase its own debt, though it does so at market value. It would be straightforward either to prohibit debt repurchase or to make the debt callable.

It is assumed that all debt matures at the same data $t = T$ and that the firm's management issues stock at that time in order to repay the debt if it is to the advantage of the stockholders to do so. Define $F(x, A, T)$ as the value of the firm at time T assuming that the value maximizing amount of new debt is issued.[12] Then the boundary conditions at maturity are:

$$E(x, A, M, T) = \max[F(x, A, T) - M, 0] \quad (30)$$

$$B(x, A, M, T) = F(x, A, T) - E(x, A, M, T \quad (31)$$

(30) states that the stockholders receive at maturity an amount equal to the difference between the value of the firm and the face value of the debt, if this difference is positive; if the difference is negative, then the firm defaults on its debt and the stockholders receive nothing. According to (31) the maturity value

[12] The terms of the new bond issue are assumed to be determined conventionally.

of the debt is equal to the difference between the value of the firm and the value of the common stock.

The bond indenture is assumed to contain a provision that if the debt ratio M/A reaches an upper bound $\bar{\bar{m}}$ the debt becomes immediately due. Taking account of the possibility of default this gives rise to the conditions

$$E(x, A, \bar{\bar{m}}A, t) = \max[E(x, A, 0, t) - \bar{\bar{m}}A, 0] \tag{32}$$

$$B(x, A, \bar{\bar{m}}A, t) = E(x, 0, t) - E(x, A, \bar{\bar{m}}A, t) \tag{33}$$

(32) follows from the fact that when the debt is repaid the value of the equity (and the value of the firm) will be given by the zero debt case.

Bankruptcy may also occur before the debt ratio reaches the upper bound $\bar{\bar{m}}$ which produces automatic default, because current losses (and the prohibition on sale of assets) force the shareholders either to make additional equity contributions or to default. Define $\hat{x}(A, M, t)$ as the return on assets at which shareholders decide to default. Then

$$E(\hat{x}, A, M, t) = 0 \tag{34}$$

$$B(\hat{x}, A, M, t) = E(\hat{x}, A, 0, t) \tag{35}$$

These conditions reflect the assumption that on default the stockholders receive nothing and the bondholders receive the firm with no debt outstanding: the bondholders are of course free to recapitalize in the optimal manner but $E(\hat{x}, A, 0, t)$ takes account of this possibility because it is calculated on the assumption that the firm is managed optimally.

Since $\hat{x}$, the return on assets at which voluntary default occurs, is chosen by the firm to maximize the value of the equity, $E(\cdot)$ satisfies the high contact condition:[13]

$$E_x(\hat{x}, A, M, t) = 0 \tag{36}$$

Then the values of the debt and equity securities issued by the firm, and its optimal investment and financing strategies, $\hat{g}$ and $\hat{f}$, are obtained by solution of the partial differential equation system (24) and (25) subject to the optimizing condition (26) and the boundary conditions and constraints (27)–(36). There exists no analytic solution to this problem so that it must be analyzed numerically. The numerical analysis is facilitated by a reduction in the state space from four state variables to three which is made possible by the first degree homogeneity of $E(\cdot)$ and $B(\cdot)$ in A. Thus define the normalized equity and debt values by

$$e(x, m, t) \equiv A^{-1}E(x, A, M, t) \tag{37}$$

$$b(x, m, t) \equiv A^{-1}B(x, A, M, t) \tag{38}$$

and the normalized tax variable, κ, by

$$\kappa \equiv TA^{-1} = (x - cm)\tau \tag{39}$$

where $m = M/A$

Then, expressing the partial derivatives of E and B in terms of those of e and

[13] See Merton (1973) footnote 60.

b, substituting in (21) and (22) and simplifying, the optimal control problem can be re-expressed in terms of the normalized variables as

$$\max_{f,g\in D}\{\tfrac{1}{2}\sigma^2 e_{xx} + \hat{\mu}e_x + (f-g)me_m + e_t$$
$$- (r-g)e + x - g - cm - \kappa + fb\} = 0 \quad (40)$$

$$\tfrac{1}{2}\sigma^2 b_{xx} + \hat{\mu}b_x + (f+g)mb_m + b_t - (r-g)b + cm - fb = 0 \quad (41)$$

the boundary conditions and constraints (28)–(36) having been re-expressed in terms of the normalized state variables. It is this reduced system for which results are presented in the following section. Before turning to that it is necessary to point out one technical assumption.

The homogeneity in A which made possible the reduction in state space was obtained at the cost of specifying the financing control, f, as *the rate of growth* in the amount of debt. An undesirable consequence of this is to make the zero debt state absorbing. To avoid this implicit assumption affecting the model value of the securities we assume that the value of the firm is invariant to the debt ratio in the limit as $m \to 0$:

$$\lim_{m\to 0} e_m(x, m, t) + b_m(x, m, t) = 0 \quad (42)$$

This is tantamount to assuming that the firm can move costlessly from a state of zero debt to a state of $\epsilon > 0$ as well as the reverse: since this is in fact the case, boundary condition (42) eliminates the problem caused by our proportional formulation of the financing policy control.

III. Financing, Investment Policy and Firm Value: An Example

We take as our example the firm described by the parameters given in Section A of Table 1. This firm has an outstanding issue of debt which is due in 5 years,

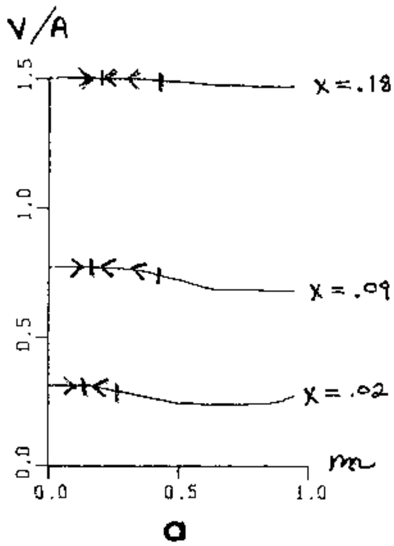
TABLE 1

A. Firm Characteristics and Market Parameters

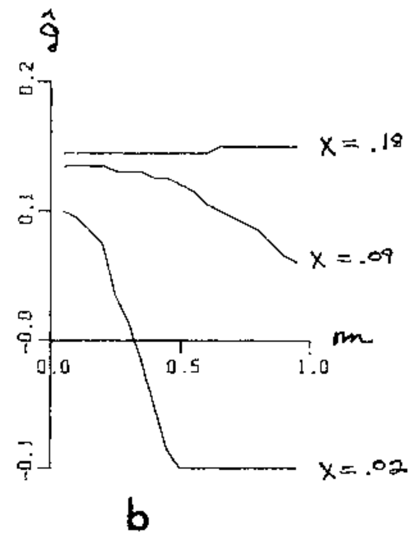
Drift:	$\mu(x)$	=	0
Risk:	σ	=	.05/year
Risk adjustment:	λ	=	.25
Size:	A	=	1000
Investment opportunities:	$\rho(x, g, t)$	=	$.25 - .5g + .1x - .01t$
	$-.10 \le g$		
Interest Rate:	r	=	.08/year

B. Bond Indentures

		(1)		(2)
Coupon Rate			.10	
Debt Issue Constraints:	$\bar{m}$	.8		.25
	$\bar{k}$	1.0		3.0
Default point:	$\bar{\bar{m}}$	1.0		.5
Debt maturity	T		5 years	



Firm Value and Leverage
(a)



Optimal Growth Rate and Leverage
(b)

Figure 2. Case (1): No Corporate Tax, Loose Covenants

and faces a downward sloping investment opportunities schedule which is subject to a downward drift over time.

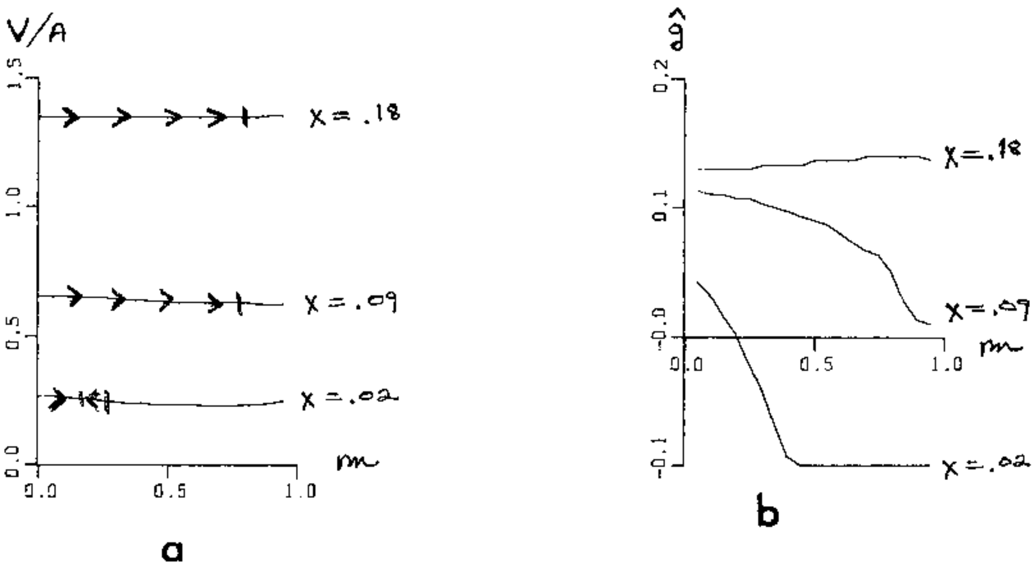
Figure 2a shows the normalized value of the firm as a function of the book debt ratio, m, for three different profitability rates, x, while Figure 2b shows the corresponding optimal instantaneous asset growth rates. In this first example, Case (1), the corporate tax rate is assumed to be zero and the terms of the bond indenture are those given in column (1) of Section B of Table 1: these covenants are quite loose, the firm being allowed to issue debt so long as the interest coverage is unity and the asset coverage is 1.25, and default occurring only when the book value of the equity falls to zero or an interest payment is omitted.

The influence of financial structure on investment policy is readily apparent in Figure 2b. When the firm is relatively unprofitable ($x = 9\%$ or 2%), the optimal asset growth declines rapidly with leverage: for $x = 2\%$ gross investment is reduced to zero for capital structures with more than 50% debt. By contrast, when the firm is highly profitable ($x = 18\%$) the adverse incentives created by the outstanding debt cause the firm to increase its investment rate above the rate that would otherwise be optimal.

The costs of the agency problems created by debt are apparent in the downward slope of the value schedules in Figure 2a. For $x = 2\%$ or 9% the decrease in firm value as the debt ratio moves from zero to 65% is of the order of 8.5%; of course when the firm is highly profitable and the prospect of bankrupty relatively remote the agency costs are much less. It is of interest to note that the value schedule turns up for extreme debt levels when $x = 2\%$. The reason for this is that equity holders do not find it worthwhile to put up funds to avert bankruptcy ($g = -10\%$) and when the firm does go bankrupt it is assumed to be recapitalized in the optimal fashion so that further agency costs are avoided.

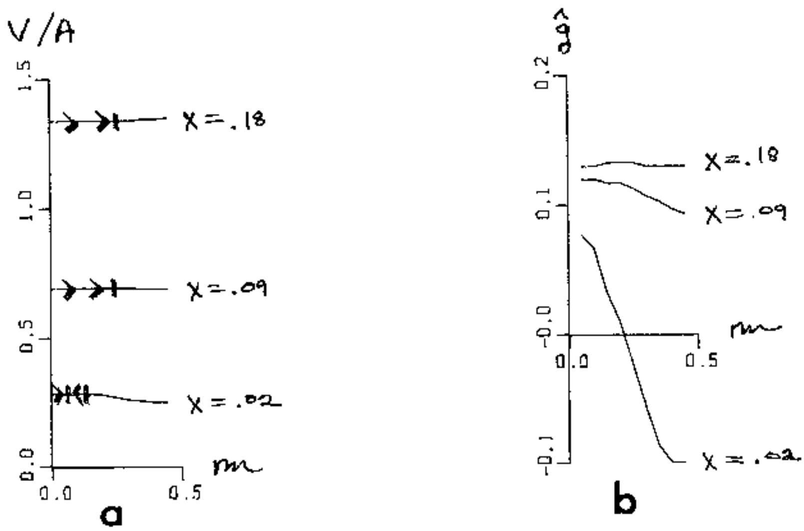
It is clear that if there are agency costs of debt and no countervailing tax benefits, then the optimal ex-ante capital structure will contain no debt. This is not to say however that a firm with debt outstanding should immediately retire it. While retirement of debt reduces agency costs and thereby increases the value of the firm, it also reduces the risk of the outstanding debt, and this confers gains on bondholders. Consequently, the net effect of debt retirement on the value of the equity can be either positive or negative. The arrows in Figure 2a indicate the direction of the change in the debt ratio under the optimal policy. It is apparent that the currently *optimal* capital structure depends not only on the current profitability of the firm, but also on the current *actual* capital structure—and under the optimal policy the current actual capital structure will depend upon the past history of profitability of the firm. It seems therefore that there may be little reason to expect firms which are apparently similar but have different histories to have similar capital structures.

Figure 3 relates to the same example as Figure 2 except that the firm is now assumed to be subject to a 25% corporate tax. Naturally this reduces the firm value and growth rate below the no tax case. Note that the value schedules are perfectly flat over considerable capital structure ranges, contrary to the usual textbook figures. The reason for this is the combination of optimal behaviour by firms and rational expectations by investors. If firm value were an increasing function of leverage and it were feasible to increase the leverage, firms would find it optimal to do so immediately: since under rational expectations the firm value reflects this, the value schedule cannot slope upwards. This observation casts doubt on the value of attempts to detect the influence of the tax deductibility of corporate interest payments on firm value by relating firm values to their capital structures.



Firm Value and Leverage (a) Optimal Growth Rate and Leverage (b)

Figure 3. Case (2): 25% Corporate Tax, Loose Covenants



Firm Value and Leverage (a) Optimal Growth Rate and Leverage (b)

Figure 4. Case (3): 25% Corporate Tax, Restrictive Covenants

Our final example, Case (3), relates to a much stricter bond indenture as shown in column (2) of Table 1. The effect of this is to reduce both the potential tax savings and the potential agency costs created by debt. It is to be expected that the tax effects will predominate when the firm is highly profitable, while the agency costs will be more important at lower levels of profitability. This is indeed the case. For $x = 18\%$ the restrictive indenture reduces firm value by about 1%. For $x = 9\%$ and 2% the restrictive indenture increases the value of the firm by 6% and 7.4% respectively. It can also be seen by comparing Figures 3b and 4b that for low levels of profitability and leverage the restrictive indenture leads to a higher level of investment.

The preceding examples suggest that the problem of optimal financial policy has at least three distinct facets: the design of an optimal indenture; the choice of an optimal initial capital structure; and the choice of an optimal financing decision given the existing capital structure. This last reveals that financial policy has a dynamic aspect which has hitherto been largely neglected. The model developed in this paper represents a first step towards the analysis of a dynamic financing policy.

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DISCUSSION

DAVID EMANUEL*: In this paper, Professors Brennan and Schwartz have tackled a problem of considerable merit and horrendous difficulty, namely that of applying the continuous time valuation framework to firms that are subject to realistic constraints. In particular, they distinguish between the book and market values of the assets and liabilities of a firm and financial constraints are posed in terms of plausible accounting ratios. I like the basic approach that is employed and Brennan and Schwartz have used it quite successfully and productively in this paper. It is clear that this is one of the first steps in a fertile and challenging new area of corporate financial theory. My remaining comments are critical of relatively minor details and should not be viewed as indicating dissatisfaction with the overall paper.

The paper starts with the development of an equilibrium valuation model for contingent claims in the manner of Cox, Ingersoll and Ross (1978). The general model is then simplified considerably to make it analytically tractable. I do not particularly like having to assume constant proportional risk aversion, but there is not any choice as the need to simplify will be with us forever. I do not think

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