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A Two-Factor Model of the Term Structure: An Approximate Analytical Solution

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Abstract

This paper develops an approximate analytical solution to a two state-variable model of the term structure similar to the one proposed by Brennan and Schwartz. Unlike the BS model, which was based on the consol rate and the short rate of interest, our model is based on the consol rate and the *spread* (i.e., the difference) between the consol rate and the short rate. This change, merely a redefinition of variables, is made to exploit an assumption, for which there is substantial empirical evidence, that these two variables (the consol rate and the spread) are *orthogonal*. Employing orthogonal state variables provides the key simplification in providing an approximate solution to the fundamental valuation equation.

I. Introduction

The literature on the continuous-time approach to the term structure can be divided into three parts. First, Vasicek [12] Cox, Ingersoll, and Ross [7] and others have described models in which the short-term rate of interest is the single state variable. While these models generally admit closed-form solutions, their empirical promise for pricing nominal bonds is limited because they imply that all bond returns are locally perfectly correlated and that the long-term interest rate is a constant. Although there are, so far, no formal empirical tests of these models, casual empiricism and some indirect tests [9] suggest that their performance will not be good. The second group consists of models involving two state variables but where neither is an asset price. Two examples are the models given by Richard [10] and Cox, Ingersoll, and Ross [7]. The empirical implications of these models are more plausible but the state variables on which they are based (e.g., the inflation rate and the "real rate") often are not easily observed. Lastly, in the third group we have models based on two state variables both of which are readily observable and where at least one is an asset price. The main examples here are a series of papers by Brennan and Schwartz [3], [4], and [5]. Brennan

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and Schwartz (BS) have tested their model and the results are encouraging. The major drawback of the BS model is the absence of a closed form solution; instead the model must be solved numerically.

In this paper, we provide an approximate analytical solution to a model similar to the one proposed by Brennan and Schwartz. Unlike the BS model, which was based on the consol rate and the short rate of interest, our model is based on the consol rate and the *spread* (i.e., the difference) between the consol rate and the short rate. This change, merely a redefinition of variables, is made to exploit an assumption, for which there is substantial empirical evidence, that these two variables (the consol rate and the spread) are *orthogonal*. This idea was first proposed and tested by Ayres and Barry [1], [2]; further empirical support is to be found in Schaefer [11] and Nelson and Schaefer [9]. Employing orthogonal state variables provides the key simplification in providing an approximate solution to the fundamental valuation equation.

The plan of the paper is as follows. In Section II, we present our model and in Section III we derive the approximate solution. In Section IV, we compare the accuracy of this approximation with a full numerical solution. The results show that, for typical parameters, the approximate solution gives results that are surprisingly accurate. In Section V, we give our conclusions.

II. The Model

In the spirit of Brennan and Schwartz [3], [5], we assume that the prices of all default-free bonds can be expressed in terms of two state variables. Like Brennan and Schwartz, we take the consol rate l as one of the state variables. However, as mentioned above, the second state variable is taken to be the spread s between the instantaneously riskless rate r and the consol rate; i.e., $s \equiv r - l$. The consol rate is defined as the yield on a bond that has a constant continuous coupon and infinite maturity. In a different analytical framework, Ayres and Barry [1], [2] also use the spread and consol rate to model the term structure.

Under our assumptions, we can express the value of any default-free bond as $V(s, l, \tau; c)$ where τ is the time to maturity of the bond and c its coupon rate. The state variables, s and l, are assumed to follow the system of stochastic differential equations

$$
(1) \qquad \begin{aligned} ds &= \beta_1(s,l,t)dt + \eta_1(s,l,t)dz_1 , \\ dl &= \beta_2(s,l,t)dt + \eta_2(s,l,t)dz_2 , \end{aligned}
$$

where t denotes calendar time, and dz_1 and dz_2 are standard Gauss-Wiener processes with $\mathrm{E}[dz_1] = \mathrm{E}[dz_2] = 0$, $dz_1^2 = dz_2^2 = dt$, and $dz_1 dz_2 = \rho dt$; ρ is the instantaneous correlation between the processes. (In the general derivation of the model, we allow ρ to be nonzero. Later, when we specialize the model, we impose the restriction that ρ equals zero).

By following Cox, Ingersoll and Ross [7] and Brennan and Schwartz [3], it is easy to show that, under the assumptions above, the value of all default-free bonds must satisfy the following partial differential equation

$$\frac{1}{2}\eta_1^2 V_{ss} + \rho\eta_1\eta_2 V_{sl} + \frac{1}{2}\eta_2^2 V_{ll} + V_s\left(\beta_1 - \lambda_1\eta_1\right) \tag{2}$$
$$+ V_l\left(\eta_2^2/l - sl\right) - V_\tau - V(s + l) + c = 0 .$$

Here λ_1 is the market price of "spread" risk and is at most a function of s, l and t, but independent of the maturity of the bond. To derive (2), we have used the fact that the consol rate is inversely proportional to the price of the consol bond that must also satisfy the diffential equation.[1] The bond must also satisfy an appropriate boundary condition determining its maturity value $V(s, l, 0; c)$.

In this paper, we are concerned with default-free discount bonds and the term structure of interest rates, so in what follows we will set $c = 0$. In a perfect market, any straight coupon bond can be valued as a portfolio of discount bonds, so there is no loss of generality in this analysis.

To solve equation (2), we must make specific assumptions about the nature of the stochastic processes given in (1) and about the functional form of the market price of spread risk, λ_1. We assume that the spread follows an Ornstein-Uhlenbeck process: a mean-reverting process with a constant variance. This process has been used previously by Vasicek [12] to model the short-term rate, r. However, it is probably more reasonable to assume that the spread rather than the short rate follows a process of this kind because it allows negative values. Also, the spread is more likely than the short rate to have a fixed mean, a point recognized by Brennan and Schwartz [5] in their specification of the joint process for r and l.

Like Brennan and Schwartz, we assume that the variance of changes in the consol rate depends on its level. However, the process we assume for the consol rate follows [7] in making the variance proportional to its level. The specific form of the stochastic process assumed is, therefore

$$ds = m(\mu - s)dt + \gamma dz_1 , \tag{3a}$$

$$dl = \beta_2(s, l, t)dt + \sigma\sqrt{l}\,dz_2 . \tag{3b}$$

In (3b), we have left the drift term in general form because, as we have already seen, any drift would be compatible with equation (2).

From this point, we shall also assume that λ_1, the market price of spread risk, is a constant, λ.[2] In addition, we now take into account the empirical regu-

[1] This has allowed us to substitute $\beta_2 - \lambda_2\eta_2 = \eta_2^2/l - sl$ in the coefficient of V_l in equation (2), where λ_2 is the market price of the consol rate risk.

[2] Without great difficulty, the model could be generalized to include the case where λ_1 is a linear function of s.

larity, mentioned earlier, that the spread is uncorrelated with the long rate and set $\rho = 0$. Under these assumptions, equation (2) for a discount bond becomes

$$\begin{aligned} \frac{1}{2}\gamma^2 V_{ss} &+ \frac{1}{2}\sigma^2 l V_{ll} + V_s m(\hat{\mu} - s) \\ &+ V_l\left(\sigma^2 - ls\right) - (l + s)V - V_\tau = 0 , \end{aligned} \tag{4}$$

subject to the terminal boundary condition

$$V(s,l,0) = 1.0 , \tag{5}$$

where

$$\hat{\mu} = \mu - \frac{\lambda\gamma}{m} .$$

Equation (4), subject to boundary condition (5), is the valuation equation of our model for discount bonds. Unfortunately, it has no known closed-form solution and so, in general, numerical procedures would be required for its solution. In the following section, we present an approximate analytical solution to this problem.

III. An Analytical Approximation

The procedure we used to obtain our approximation is to derive the *exact* solution to an equation that is closely related to (4). We are able to solve this related equation by separating it into two parts: one part depends only on s and the other depends only on l. The related equation is identical to (4) except that the term s in the coefficient of V_l is a constant, $\hat{s}$

$$\frac{1}{2}\gamma^2 \overline{V}_{ss} + \frac{1}{2}\sigma^2 l \overline{V}_{ll} + \overline{V}_s m(\hat{\mu} - s) + \overline{V}_l\left(\sigma^2 - l\hat{s}\right) - (l + s)\overline{V} - \overline{V}_\tau = 0, \tag{6}$$

with boundary condition

$$\overline{V}(s,l,0) = 1.0 . \tag{7}$$

The solution to (6) subject to (7) can be written[3] as

$$\overline{V}(s,l,\tau) = X(s,\tau)Y(l,\tau) , \tag{8}$$

where $X(s, \tau)$ is itself the solution to

$$\frac{1}{2}\gamma^2 X_{ss} + X_s m(\hat{\mu} - s) - sX - X_\tau = 0 , \tag{9}$$

with terminal boundary condition

$$X(s,0) = 1.0 , \tag{10}$$

and $Y(l, \tau)$ is the solution to

$$\frac{1}{2}\sigma^2 l Y_{ll} + \left(\sigma^2 - l\hat{s}\right)Y_l - lY - Y_\tau = 0 , \tag{11}$$

[3] See [6], p. 33.

with boundary condition

$$Y(l,0) = 1.0 . \tag{12}$$

Equation (9), in s, is essentially the same as the valuation equation, in r, derived by Vasicek (12). The solution is

$$\begin{aligned} X(s,\tau) = \exp\Big[&\frac{1}{m}(1 - \exp(-m\tau))\big(s_\infty - s\big) - \tau s_\infty \\ &- \frac{\gamma^2}{4m^3}(1 - \exp(-m\tau))^2\Big] , \end{aligned} \tag{13}$$

where

$$s_\infty = \hat{\mu} - \frac{1}{2}\frac{\gamma^2}{m^2} .$$

Similarly, equation (11), in l, is isomorphic to an equation in r given by Cox, Ingersoll, and Ross [7]. The solution to (11) with boundary condition (12) is

$$Y(l,\tau) = A(\tau)\exp[-B(\tau)l] , \tag{14}$$

where

$$A(\tau) = \left[\frac{2\alpha\exp\{(\hat{s} + \alpha)\tau/2\}}{(\hat{s} + \alpha)(\exp(\alpha\tau) - 1) + 2\alpha}\right]^2,$$

$$B(\tau) = \frac{2(\exp(\alpha\tau) - 1)}{(\hat{s} + \alpha)(\exp(\alpha\tau) - 1) + 2\alpha} ,$$

and

$$\alpha = \sqrt{\hat{s}^2 + 2\sigma^2} .$$

The product of equations (13) and (14) is the solution to equation (6), subject to boundary condition (7). Thus, the yield on a pure discount with maturity τ is given by

$$\begin{aligned} R(s,l,\tau) &\equiv -\frac{1}{\tau}\operatorname{Ln}\overline{V}, \\ &= s_\infty - F(\tau)\big(s_\infty - s\big) + G(\tau) + \frac{1}{\tau}\{B(\tau)l - \operatorname{Ln}(A(\tau))\} , \end{aligned} \tag{15}$$

where s_∞, $A(\tau)$ and $B(\tau)$ are as defined above,

$$F(\tau) = \frac{(1 - \exp(-m\tau))}{m\tau} ,$$

and

$$G(\tau) = \frac{\gamma^2}{4m^3\tau}(1 - \exp(-m\tau))^2.$$

Equation (15) is used in Section IV, where we compare yields from our approximation with those from a full numerical solution to equation (4).

The remaining task is to specify the value of $\hat{s}$ to be used in equation (14). For this purpose—but not, of course, in deriving the solution to (6)—we ignore uncertainty in both the original equation (4) and the related equation (6).[4] These equations then become

$$(4') \qquad m(\hat{\mu} - s)V_s + \left(\sigma^2 - ls\right)V_l - (l + s)V - V_\tau = 0$$

$$(6') \qquad m(\hat{\mu} - s)\overline{V}_s + \left(\sigma^2 - l\hat{s}\right)\overline{V}_l - (l + s)\overline{V} - \overline{V}_\tau = 0 .$$

Both of these equations, with boundary conditions (5) and (7), respectively, can be solved analytically. The value of $\hat{s}$ used in our approximation is the one that makes the solutions to (4′) and (6′) equal. The details of the solution are in the Appendix.

In general, $\hat{s}$ will depend on the current values of s and l and the time to maturity of the discount bond, in addition to the parameters of the stochastic processes. This means that the yield on a pure discount bond is not, in general, a linear function of s and l as equation (15) might suggest.

This completes the description of the approximate analytical solution. Its usefulness will depend on its accuracy in pricing bonds for realistic values of the parameters of the equations. In the following section, we compare the analytical approximation ($\overline{V}$) with the numerical solution to the true equation (V) for reasonable values of the parameters of the equation.

IV. Accuracy of the Approximation

In this section, we compare yields obtained using the analytical approximation, represented by equations (8), (13), and (14) with those obtained by the numerical solution of the exact equation (4). The numerical procedure employed was the Alternative Direction Implicit (ADI) method as described by McKee and Mitchell [8].

In our calculations, we used the two sets of parameters shown in Table 1. The first set, the Base Case, reflects values of the same order of magnitude as those reported in [5] and [9]. Because in computing $\hat{s}$ we have ignored the stochastic part of the processes for s and l (see equations (4′) and (6′)), we wished to investigate the sensitivity of the approximation to an increase in variance. Accordingly, in the High Variance Case in Table 1, the standard deviation of the s and l processes have doubled. In all our calculations, we have assumed a zero value for λ. This is unlikely to influence our results to the extent that, for every nonzero value of λ, there is a corresponding value of μ that would leave the value of $\hat{\mu}$ unchanged (see the definition of $\hat{\mu}$ after equation (4)). In practice, the value of λ (or alternatively $\hat{\mu}$) must be estimated from bond prices (see [5]).

[4] It is to be assumed that because of this the approximation will be less accurate for higher variances. For our purposes, however, the relevant question is the effect on bond prices and yields for reasonable values of the parameters. The effect of this simplification on bond prices and yields is examined in Section IV.

TABLE 1

Parameters of Stochastic Process and Market Price of Spread Risk

Base Case	High Variance Case
m = .72	m = 72
μ = −.01	μ = − 01
γ = .007	γ = 014
σ^2 = .0003	σ^2 = 0012
λ = 0	λ = 0

Stochastic Process

$ds = m(\mu - s)\,dt + \gamma\, dz_1$

$dl = \sigma\sqrt{l}\,dz_2$

We have carried out two tests of the accuracy of the approximation. First, we compared the yields on discount bonds estimated with the approximation with yields computed from a numerical solution of the exact equation. In evaluating these comparisons, we must remember that the numerical solution itself is subject to error. The second test was to use the approximation to calculate the price, and from this the yield, on a consol. This computed yield can be compared with the value of the consol yield, which is an input (state variable) in the model. This latter test investigates the internal consistency of the model without relying on comparison with a numerical solution.

Table 2 reports summary statistics on the difference between yields on discount bonds computed using the approximation and yields derived from the numerical solution. The most relevant results are given in panels *a* and *c*. Panel *a* uses the base-case parameters and gives summary statistics for a range of values of the spread between −5 percent and +5 percent and values of the consol yield between 0 percent and 20 percent. The results show that for maturities of between one and twenty years the maximum absolute error was 3.25 basis points. The mean errors are less than one basis point for all maturities and the root mean square reached a maximum of 1.24 basis points at 20 years. Errors of this magnitude can safely be ignored for empirical purposes.

With the higher variance (panel *c*) and the same range of values for s and l, the maximum absolute error rises to 8.59 basis points. However, the root mean square error is below.4 basis points at 20 years and this level of accuracy would be acceptable for most purposes. These results give some support to the conjecture in footnote 4 that the approximation would be less accurate for higher variances. We emphasize that the parameters used in panel *c* are high compared with those typically observed in post-war U.S. data.

Panels *b* and *d* show that the errors are larger for extreme values and s and l, but here again the maximum absolute error is only 9 basis points for the base-case parameters and 21 basis points for the high variance case.

For Table 3, we used the approximation to calculate the price of a consol.[5] The table shows the yields implied by these prices and these yields should be

[5] We actually computed the price of a 200-year annuity.

TABLE 2

Summary Statistics on Differences between Approximate Analytical Solution and Numerical Solution

Time to Maturity (Years)	Differences (Basis Points)				
	Mean	Root Mean Square	Mean Absolute	Maximum	Minimum
Panel *a:* Base Case Parameters with Spread Range ± 5% and Consol Range 0, 20%					
1	0.84	1.11	1.01	2.08	−1.97
5	0.65	0.80	0.74	1.46	−0.97
10	0.37	0.58	0.49	1.24	−0.73
15	0.12	0.75	0.61	2.46	−1.99
20	−0.03	1.24	0.98	2.79	−3.25
Panel *b:* Base Case Parameters with Spread Range ± 10% and Consol Range 0, 25%					
1	0.80	1.30	1.11	3.29	−1.97
5	0.40	0.85	0.75	1.46	−3.00
10	0.15	0.89	0.72	1.28	−3.56
15	−0.09	1.60	1.21	3.10	−5.51
20	−0.22	2.88	2.17	6.67	−9.08
Panel *c:* High Variance Case with Spread Range ± 5% and Consol Range 0, 20%					
1	0.59	0.81	0.71	1.57	−1.56
5	0.13	0.49	0.41	1.06	−1.16
10	−0.74	1.35	1.05	1.40	−3.79
15	−1.21	2.36	1.74	3.07	−6.33
20	−1.20	3.51	2.69	6.88	−8.59
Panel *d:* High Variance with Spread Range ± 10% and Consol Range 0, 25%					
1	0.55	1.08	0.90	2.90	−1.93
5	−0.25	1.01	0.68	1.06	−4.67
10	−1.17	2.33	1.59	1.43	−8.31
15	−1.57	4.34	3.15	7.61	−13.80
20	−1.30	7.30	5.52	17.77	−20.81

Notes: 1. Base case parameters are those given in Table 1.
2. High variance case parameters are those give in Table 1 but with standard deviations doubled; i.e., $\gamma = 0.014$ and $\sigma^2 = 0.0012$.

compared with the consol yield used in the computation. In Panel *a*, which gives the results for the base-case parameters, the maximum error is 4 basis points and this occurs for a consol yield of 5 percent and a +5 percent spread (i.e., a 10 percent short rate).

It is interesting to point out that in these results, the implied yields for high values of the consol yield are extremely accurate whereas, in the comparison with the numerical solution, it is precisely in these areas, for long maturities, that the error is greatest. The reason for this, of course, is that when interest rates are high, even relatively large errors in spot rates for long maturities have little effect on the prices of consols and of long-term bonds in general. In other words, it seems that the approximation is least accurate where it matters least.

In Panel *b* of Table 3, we give the corresponding results for the high vari-

TABLE 3

Internal Consistency of the Analytical Approximation
(Entries are computed consol yields (annual) in percent.)

Consol Yield	Spread −5%	Spread 0%	Spread +5%
Panel a: Base Case Parameters			
5	4.99	5.01	5.04
10	9.99	10.00	10.01
15	14.99	15.00	15.01
20	20.00	20.00	20.00
25	25.00	25.00	25.00
Panel b: High Variance Case Parameters			
5	5.35	5.45	5.57
10	10.01	10.07	10.14
15	14.97	15.00	15.04
20	19.98	20.00	20.02
25	24.98	24.99	25.01

ance case. As we would expect, the results are less accurate, particularly for low values of the consol yield.

IV. Conclusion

A significant barrier in the implementation of continuous time models of the term structure is that realistic models generally have no closed-form solutions. The purpose of this paper has been to present a model that is empirically relevant and to develop a method for obtaining solutions of adequate accuracy without resorting to a full-scale numerical solution. The conclusive test of the model and the approximation is, of course, in testing it using actual data. This will be the subject of further research by the authors.

Appendix

In this appendix, we describe our procedure for calculating $\hat{s}$. This is equivalent to solving equations (4′) and (6′) and then deriving the value of $\hat{s}$ that makes the solution to (6′), which depends on $\hat{s}$, equal to the solution to (4′), which does not. To solve (4′) and (6′), it is useful to view the solution to (4)—the "true" valuation equation—as an expectation.

Cox, Ingersoll, and Ross [7] have shown that the solution to (4) can be written as

$$\text{(A1)} \qquad V(s,l,\tau) = \hat{E}_{s,l}\left[\exp\left\{-\int_0^\tau (s(t) + l(t))\,dt\right\}\right],$$

where $\hat{E}(\cdot)$ denotes expectation with respect to the risk-adjusted processes for s and l. For our problem, the risk-adjusted versions of the original processes (3a) and (3b) are

$$ds = m(\hat{\mu} - s)dt + \gamma dz_1 , \tag{A2a}$$

and

$$dl = \left(\sigma^2 - sl\right)dt + \sigma\sqrt{l}\,dz_2 . \tag{A2b}$$

For the purposes of computing $\hat{s}$, we ignore uncertainty in equation (A1). This allows us to write (A1) as

$$\overline{\overline{V}}(s,l,\tau) = \exp\left\{-\int_0^\tau s(t)dt\right\}\exp\left\{-\int_0^\tau l(t)dt\right\} , \tag{A3}$$

where $\overline{\overline{V}}(\cdot)$ is the solution to (A1) under certainty, and where the paths of s and l are described by the following pair of deterministic differential equations

$$\frac{ds}{dt} = m(\hat{\mu} - s) , \tag{A4a}$$

and

$$\frac{dl}{dt} = \sigma^2 - sl . \tag{A4b}$$

To compute $\overline{\overline{V}}$, we must solve the system (A4) with initial conditions $s = s_0$ and $l = l_0$ for $t = 0$ and then compute the two integrals in (A3). This is trivial for s but nontrivial for l. This latter calculation is equivalent to computing the mean of l from time zero to time τ, that is

$$\bar{l} = \frac{1}{\tau}\int_0^\tau l dt . \tag{A5}$$

The solution to the system (A4), and, therefore, to (A5), depends on whether the initial value of s (s_0) is greater than, equal to, or smaller than $\hat{\mu}$. Thus we have

a) For $s_0 > \hat{\mu}$,

$$\begin{aligned}\bar{l} = {} & \left[\frac{\exp\left(-V_0\right)V_0^{\alpha}l_0}{m\tau} + \frac{\sigma^2\gamma\left(\alpha, V_0\right)}{m^2\tau}\right]\sum_{m=0}^{\infty}\frac{V_0^{n-\alpha} - V_\tau^{n-\alpha}}{(n-\alpha)n!} \\ & - \frac{\sigma^2\Gamma(\alpha)}{m^2\tau}\sum_{n=1}^{\infty}\frac{V_0^n - V_\tau^n}{n\Gamma(\alpha+n+1)} - \frac{\sigma^2}{\alpha m} .\end{aligned} \tag{A6a}$$

(b) For $s_0 = \hat{\mu}$,

$$\bar{l} = \frac{l_0\hat{\mu} - \sigma^2}{\hat{\mu}^2\tau}(1 - \exp(-\hat{\mu}\tau)) + \frac{\sigma^2}{\hat{\mu}} . \tag{A6b}$$

(c) For $s_0 < \hat{\mu}$,

$$\begin{aligned}\bar{l} = \frac{\sigma^2}{m^2\tau}\Bigg\{&\left[\frac{\exp\left(V_0\right)V_0^{\alpha}l_0 m}{\sigma^2} + \frac{V_0^{\alpha}}{\alpha} + \sum_{n=1}^{\infty}\frac{V_0^{\alpha+n}}{(\alpha+n)n!}\right]\\ &\times\left[\frac{V_\tau^{-\alpha} - V_0^{-\alpha}}{\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^n\left(V_0^{n-\alpha} - V_\tau^{n-\alpha}\right)}{(n-\alpha)n!}\right]\\ &-\frac{1}{\alpha}\left[m\tau + \sum_{n=1}^{\infty}\frac{(-1)^n\left(V_0^{n} - V_\tau^{n}\right)}{(n)n!}\right]\\ &-\left[\frac{\exp\left(-V_\tau\right) - \exp\left(-V_0\right)}{\alpha+1} + \sum_{n=2}^{\infty}\frac{\gamma\left(n,V_0\right) - \gamma\left(n,V_\tau\right)}{(\alpha+n)n!}\right]\Bigg\} ,\end{aligned} \tag{A6c}$$

where

$$V_0 = \frac{|s_0 - \mu|}{m} ,$$

$$V_\tau = V_0\exp(-m\tau)$$

$$\alpha = -\frac{\mu}{m} > 0,^6$$

$\Gamma(x)$ is the gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\exp(-t)dt ,$$

and $\gamma(p,x)$ is the incomplete gamma function

$$\gamma(p,x) = \int_0^{x} t^{p-1}\exp(-t)dt .$$

Under equation (6), the risk-adjusted process, corresponding to (A2 a-b), gives rise to the following pair of deterministic differential equations (which correspond to A4 a-b)

$$\frac{ds}{dt} = m(\hat{\mu} - s) , \tag{A7a}$$

$$\frac{dl}{dt} = \sigma^2 - \hat{s}l . \tag{A7b}$$

Equations (A7a) and (A4a) are identical, but in (A7b) the coefficient of l is a fixed parameter, $\hat{s}$, rather than the variable s that appears in (A4b). We now choose $\hat{s}$, so that the values of $\overline{V}$, under (A7a-b) and under (A4a-b), are the same. Because the process for s is identical in the two systems, the value of $\overline{V}$ is preserved if the mean value of l under (A7b) is the same as the mean value of l under (A4a-b). The latter is given in equations (A6a-c); the former is easily obtained as[7]

$$\text{(A8)} \qquad \bar{l} = \frac{l_0\hat{s} - \sigma^2}{\hat{s}^2\tau}(1 - \exp(-\hat{s}\tau)) + \frac{\sigma^2}{\hat{s}} .$$

Equating (A8) and (A6) we obtain a nonlinear equation for $\hat{s}$ that can be easily solved numerically. Comparing (A8) with (A6b) we see that when s_0 is equal to $\hat{\mu}$, then $\hat{s}$ is also equal to $\hat{\mu}$.

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[6] The solution (A6) applies only in the case $\alpha > 0$, i.e., $\mu < 0$ if $m > 0$. This would appear to be the empirically relevant case but similar solutions are obtained for $\alpha < 0$.

[7] Note that when $\hat{s} \rightarrow 0$ in (A8) $\bar{l} \rightarrow l_0 + \frac{1}{2}\sigma^2\tau$.