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ABSTRACT

In this paper, we develop a model for valuing debt options that takes into account the changing characteristics of the underlying bond by assuming that the standard deviation of return is proportional to the bond's duration. The resulting model uses the bond price as the single state variable and thus preserves much of the simplicity and robustness of the Black-Scholes approach. The paper provides comparisons between option prices computed using this model and those using the Black-Scholes and Brennan and Schwartz models.

THE LAST DECADE HAS seen a dramatic increase in the volatility of bond markets, leading to a growing interest in financial instruments that can be used to hedge this risk. Thus, in the U.S., Canada, and the U.K., there has been a proliferation of new interest rate-dependent securities such as bond futures, options on bonds, options on bond futures, swap agreements, and bonds with option features such as callability, putability, retractability, and so forth.

With the development of these markets, the problems of valuation and the derivation of the hedging strategies have become subjects of increasing importance to both practitioners and academics. Several valuation methods have been suggested in the literature, but, so far, no consensus has emerged as to the best solution to these problems.

The simplest approach to the valuation of debt options, and probably the most widely used in practice, is the Black-Scholes model [3]. This method, originally developed for the valuation of stock options, assumes that the variance of the rate of return on the underlying security is constant. While this assumption is plausible for common stocks, it is clearly unreasonable for bonds with a finite maturity. The distinctive feature of the problem of debt-option valuation is that the characteristics of the underlying asset change over time because the price is constrained to converge to the face value of the bond at maturity. A second drawback of the Black-Scholes approach is that it assumes a constant short-term interest rate, an assumption that is clearly inconsistent with stochastic returns on bonds. In spite of these problems, the Black-Scholes model has been frequently used in practice to value debt options because of its overwhelming simplicity in comparison with other available methods.

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The second approach to the valuation of debt options derives from the equilibrium theories of the term structure. Cox, Ingersoll, and Ross [10], Vasicek [17], Brennan and Schwartz [4], and others have derived equilibrium models of the term structure assuming that one or more interest rates follow exogenously given stochastic processes.[1] Courtadon [7] has used a single-state-variable model based on the short rate to value debt options. Even though the single-state-variable approach allows for changes in the variance of bond returns over time and a stochastic short rate, it has the undesirable properties that the returns on bonds of all maturities are perfectly correlated and that the long-term zero-coupon yield is a constant. The Brennan and Schwartz two-state-variable model, based on the consol rate and the short rate, overcomes these difficulties while preserving the desirable properties of a stochastic short rate and time-dependent variance of bond returns. This is achieved at the cost of substantially increased computational complexity.

There are, however, three significant practical difficulties in applying the equilibrium approach. First, it requires the estimation of the stochastic process for either one or two interest rates. Second, it requires the estimation of a utility-dependent parameter: the market price of short-term interest rate risk. Third, because the underlying bond is not a state variable in the model, the bond price in the boundary conditions for an option must be computed from the interest rate state variables, thus adding both to complexity and to the possibility of error in the valuation.

Ball and Torous [1] have proposed a model for the valuation of European options on discount bonds based on the assumption that the rate of return on the underlying discount bond follows a Brownian bridge process. This ingenious idea allows the bond price to converge to its face value at maturity but does so in such a way that the variance of the rate of return on the bond is constant over time. This allows Ball and Torous to use Merton's [14] stochastic interest rate option model to derive a closed-form solution for the option value. A significant weakness of this approach, however, is that a constant variance of return implies that the variability of the yield to maturity increases without bound as the bond approaches maturity.

In this paper, we develop a model for valuing debt options that takes into account the changing characteristics of the underlying bond in a way that is both simple and realistic. We assume that the standard deviation of return on the underlying bond is proportional to the bond's duration. Duration, a present-value weighted time to payment of the bond's cash flows, is a measure of effective maturity, and, as maturity increases, a bond's duration converges to that of a consol bond. Despite its simplicity, recent empirical work on bond-hedging technique suggests that duration is a good measure of the relative variability of bond returns.[2]

In our option-valuation model, the single state variable is the price of the underlying bond with a standard deviation of return proportional to duration.

[1] In the Cox, Ingersoll, and Ross model, the interest rate process is derived in a general-equilibrium framework.

[2] See, e.g., Brennan and Schwartz [5] and Nelson and Schaefer [15].

This means that, like the equilibrium-based models and unlike Black-Scholes and Ball-Torous, our option values will reflect the fact that the characteristics of the underlying bond are changing over time. Moreover, it means that, like Black-Scholes and Ball-Torous and unlike the equilibrium-based models, our model does not require the estimation of stochastic processes for interest rates, the estimation of the market price of interest rate risk, or the computation of bond prices in the boundary conditions.

In common with Black-Scholes, our method has the theoretical weakness of assuming a constant short-term rate of interest. However, considering the advantages of our model described above, including its simplicity of application, we believe that, for practical purposes, this theoretical weakness is a price worth paying. This issue, however, can only be resolved empirically.

Section I of the paper describes in detail the stochastic process governing bond prices. Section II presents some empirical evidence on the relationship between the variability of bond returns in the U.K. and duration. In Section III, we develop the option-valuation model, and Section IV provides some numerical examples and compares our results with those from the Black-Scholes and Brennan-Schwartz models. Section V gives our conclusions.

I. A Stochastic Process for Bond Prices

As we mentioned earlier, the objective of this paper is to construct a model that (a) has the price of the underlying bond as the only state variable and (b) accommodates the fact that the dynamics of a bond's price change as the bond approaches maturity. We therefore assume that, at time t, the instantaneous rate of return on a default-free bond with price P is given by

$$\frac{dP}{P} = \mu dt + \sigma(P, t)dz, \tag{1}$$

where μ is the instantaneous rate of price appreciation on the bond, possibly stochastic, and $\sigma(P, t)$ is the instantaneous standard deviation of return. Note that, by writing the standard variation of return as a function of only P at t, our model becomes an arbitrage model in the spirit of Black and Scholes [3], rather than an equilibrium analysis such as those of Cox, Ingersoll, and Ross [10], Vasicek [17], or Brennan and Schwartz [4]. We take this route because the purpose of this paper is to develop a robust and simple procedure for valuing options on bonds rather than developing an equilibrium model of the term structure.

Despite this shortcoming, our approach has a number of advantages. First, by making the price of the bond the sole state variable, we are freed from the need to estimate any utility-dependent parameters as would be the case in all the equilibrium models mentioned above. Second, we are free to choose an empirically realistic characterization for the variance of the price process. This is to be contrasted with, for example, one-factor equilibrium models where the price variability of long-term zero-coupon yields tends to zero. Third, by having the price of the underlying asset as a state variable, the boundary conditions for option pricing are greatly simplified. In the case of Brennan and Schwartz [6], for example, the boundary conditions must be computed from the value of the underlying state variables. Fourth, by having only one state variable, the computational requirements are significantly less demanding than for two-state-variable models to the extent that implementation on a personal computer is straightforward. Finally, as will become apparent, the inputs for the model are easily obtained.

The key issue in equation (1) is the specification of the standard deviation $\sigma(P, t)$ and, in particular, the pattern of time dependent of $\sigma(P, t)$ as the bond approaches maturity. A significant literature exists that suggests that duration is a good measure of the standard deviation of bond returns.[3] Of the duration measures that depend only on P and t, the simplest is the Reddington [16] duration, which is defined as

$$D(P, t) = [\textstyle\sum_{i=1}^{n} (t_i - t)c_i\exp\{-y(t_i - t)\} + (t_n - t)F\exp\{-y(t_n - t)\}]/P, \quad (2)$$

where c_i, $i = 1, \cdots, n$, is the ith coupon on the bond paid at time t_i, F is face value, and y is the yield to maturity on the bond that is the solution to

$$P = [\textstyle\sum_{i=1}^{n} c_i\exp\{-y(t_i - t)\} + F\exp\{-y(t_n - t)\}]. \quad (3)$$

Macaulay's [12] duration, which is identical to (2) except that the discounting is peformed using the underlying zero-coupon yields rather than the yield to maturity on the bond, is an alternative and frequently used measure of duration. Indeed, it may appear to be a better measure of average maturity since the discounting is performed at market rates. However, Ingersoll [11] has shown that the differences between the two measures are almost always trivial[4] and, since Reddington's measure depends only on the own price of the bond rather than the entire term structure, we will use this measure.

In Section II, we provide some empirical evidence from the U.K. government bond data that suggests that duration is a good measue of the variability of bond returns.

In this paper, we assume that the standard deviation of a bond's return is proportional to duration, and we write (1) as

$$dP = \mu P dt + kP^{\alpha}D(P, t)dz \quad (4)$$

or, equivalently,

$$\sigma(P, t) = kP^{\alpha-1}D(P, t). \quad (5)$$

In (4) and (5), α and k are constants.

It is well known that equation (2) is equivalent to

$$D(P, t) = -\frac{1}{P}\frac{\partial P}{\partial y}, \quad (6)$$

[3] See, e.g., Nelson and Schaefer [15] and Brennan and Schwartz [5].

[4] See Ingersoll [11, p. 170, Table 4].

and it therefore follows that $kP^{\alpha-1}$ measures the local standard deviation of the change in yield to maturity.[5] Different values of α therefore characterize different patterns of a dependence between the variability of yield changes and the level of the bond price. This is most clearly seen in the case of a consol bond, for which

$$P = \frac{c}{y} \tag{7}$$

and

$$D = \frac{1}{y}. \tag{8}$$

Here, from equations (5), (7), and (8), the standard deviation of bond returns is given by

$$\sigma(P, t) = \frac{k}{c} P^{\alpha}, \tag{9}$$

which implies that the standard deviation of the consol yield is $kc^{\alpha-1}y^{1-\alpha}$. Notice that, when α equals zero, the standard deviation of consol returns is a constant, as in the Black-Scholes model, and the consol's yield, like its price, follows a geometric Brownian Motion. When α is unity, the standard deviation of consol returns is k/y, which is proportional to duration,[6] and the corresponding consol yield follows an arithmetic Brownian Motion.

Some limited empirical evidence on the value of α is given in Section II. It should be pointed out, however, that even though with α equal to zero the process for bond returns converges to the Black-Scholes process as time to maturity increases, the same will not be true for bonds with short maturities. In the case of the latter, duration will be relatively less sensitive to price, and, therefore, the standard deviation of return will be inversely related to the level of price. This is to be contrasted with the Black-Scholes case, where the standard deviation of returns is independent of price.

On the other hand, when α equals unity, the standard deviation of returns on short bonds will be insensitive to the price level, whereas the standard deviation of consol returns will be proportional to price.

The above discussion suggests that, if (5) provides a good description of the standard deviation of bond returns, then the Black-Scholes model will not correctly price options on both short- and long-term bonds simultaneously.

Finally, it is interesting to consider the case of discount bonds. Here, duration

[5] To see this, consider the yield to maturity, y, as the primitive process with standard deviation σ_y. Using Itô's Lemma on $P = P(y, t)$, the local standard deviation of dP/P is $(1/P)(\partial P/\partial y)\sigma_y$, which, from (5) and (6), implies that $\sigma_y = kP^{\alpha-1}$. The process described by equation (4) has the drawback that the bond price does not necessarily tend to the face value as the bond approaches maturity. For options with maturities close to the maturity of the bond, this may produce inaccuracies. A number of solutions to this problem appear possible, but perhaps the simplest would be to reformulate the model in terms of the bond's yield to maturity, y, rather than the price. Since $y = y(P, t)$, this is no more than a change of variable, but it will nonetheless overcome the difficulty described above.

[6] Note that, when $\alpha = 1$, $\sigma(P, t)$ is also proportional to duration for nonconsol bonds.

is equal to time to maturity and hence insensitive to bond price or yield. In this case, (5) shows that the standard deviation of return is insensitive to price when α equals one rather than zero.

In this section, we have proposed a specification of the stochastic process for bond prices. While other specifications are possible, the process we have chosen seems particularly attractive since it not only incorporates the well-documented relationship between duration and the standard deviation of bond returns but also accommodates a variety of patterns of dependence between the standard deviation of yield changes and the level of price.

II. Duration and the Variability of Bond Returns: Some Empirical Evidence

A fundamental feature of our specification of the stochastic process is the proportional relationship between duration and the standard deviation of bond returns. While it is not our intention in this paper to provide an exhaustive analysis of this topic, we have carried a limited investigation and report our results in this section.

The data were daily prices and returns on all the U.K. government bonds (excluding index linked to convertible issues) over the period from October 1, 1985 through December 1, 1985. The total number of bonds in the sample was eighty-nine, with maturities ranging from three months to twenty-eight years.

Figure 1 shows the annualized standard deviations of return for the total sample plotted against their average daily durations over the period. The strong positive association between these variables is apparent, and a regression of standard deviation ($\hat{\sigma}$) against average duration ($\bar{D}$) gave the following results:

$$\begin{aligned} \hat{\sigma} = &\underset{(0.0011)}{0.0055} + \underset{(0.0002)}{0.0055}\ \bar{D} + \varepsilon, \\ \bar{R}^2 = &\ 0.91. \end{aligned}$$

Even though duration explains ninety-one percent of the variability of $\hat{\sigma}$, the following two points should be noted. First, the intercept is greater than zero; this may well be due to a small amount of noise in the price data (which, in turn, probably derives from the bid-ask spread). Second, there are a number of outliers in the data, particularly in the mid- to high-duration range, where a number of observations are below the main body of the data. More detailed investigation revealed that the majority of the outliers were low-coupon bonds, and, in Figure 2, bonds with coupons below ten percent have been excluded. As can be seen, this step (which eliminates thirty-two bonds) substantially reduces the number of outliers, and the corresponding regression becomes

$$\begin{aligned} \hat{\sigma} = &\underset{(0.0011)}{0.0045} + \underset{(0.0002)}{0.0059}\ \bar{D} + \varepsilon, \\ \bar{R}^2 = &\ 0.94. \end{aligned}$$

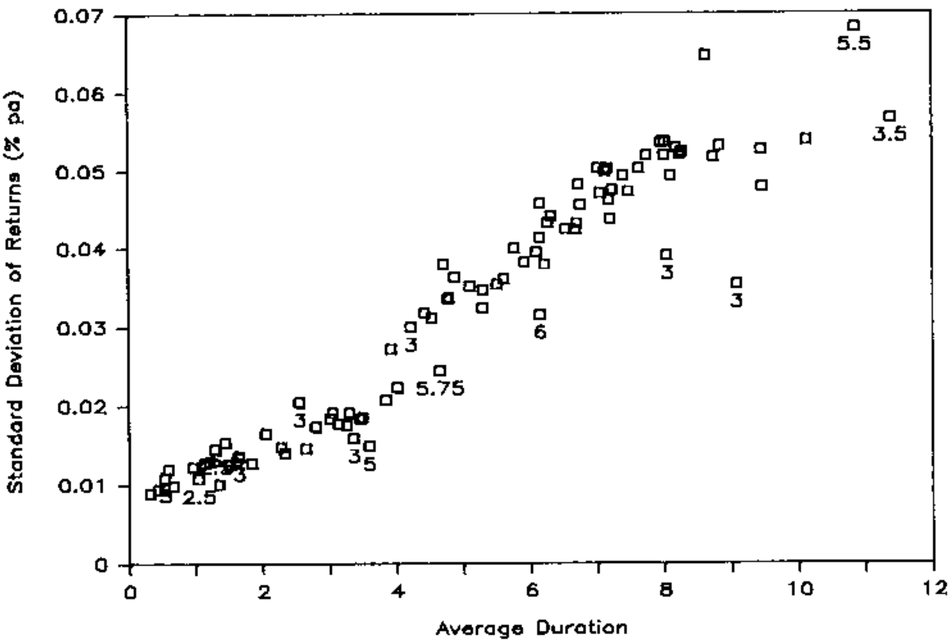

Figure 1. The relationship between the volatility of bond returns and average duration. The figure plots the annualized time-series volatility of daily returns on all British Government Securities ("gilts") against their average duration. The data cover the period from October 1 through December 1, 1985. The diagram also indicates those bonds that have coupons of less than six percent.

While these results suggest many interesting avenues for research, it is clear that duration alone explains a very significant fraction of the cross-sectional variation in bond-return variability.

Finally, we have made some very preliminary attempts to estimate α. It should be remembered that the role of the term $P^{\alpha-1}$ in equation (5) is to model the effect of changes in the level of yields on the variability of yield changes. Consequently, the estimation of α should be carried out within a time-series analysis rather than in a cross-section. To simplify the analysis, we used data on a very long-term bond (twelve percent Exchequer 2013–2017) and estimated equation (9), which related to a consol.[7]

Taking logs of equation (9), we obtain

$$\log \sigma = \log(k/c) + \alpha \log(P).$$

We estimated this equation using values for the standard deviation of returns derived from forty-one nonoverlapping two-month periods of daily returns between January 1979 and November 1985. The price, P, in the regression was taken as the last price in each two-month period (though the results were little

[7] We used data on the twelve percent Exchequer 2013–2017 rather than on one of the outstanding "consols" as the price data on the former were felt to be more reliable.

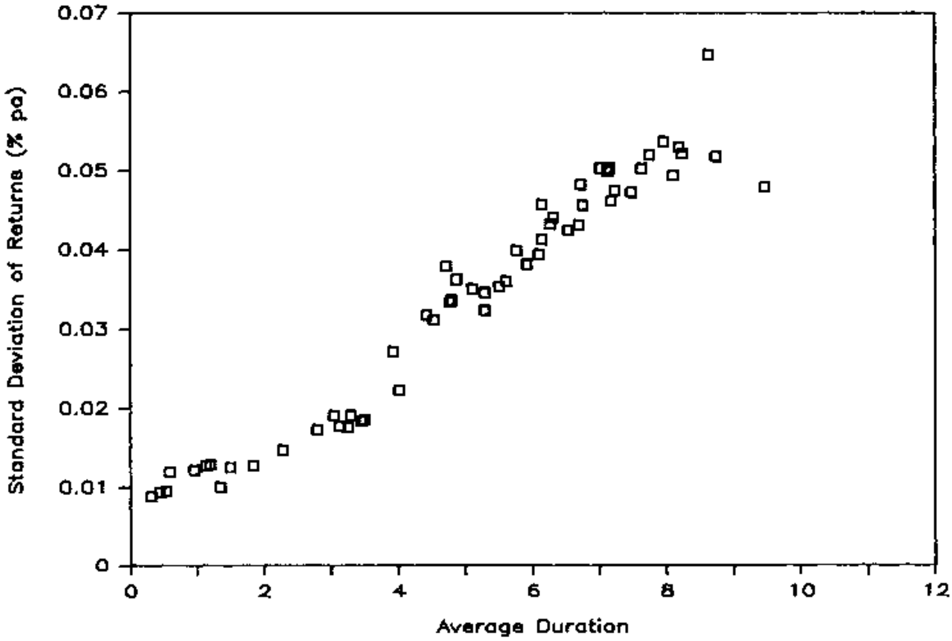

Figure 2. The relationship between the volatility of bond returns and average duration. The data are as described for Figure 1 except that bonds with coupons less than ten percent are excluded.

affected by using the first price). The results of this regression were

$$\log \hat{\sigma} = \underset{(1.77)}{3.16} - \underset{(0.38)}{1.19} \log P + \varepsilon,$$

$$\bar{R}^2 = 0.18; \quad \text{Durbin Watson} = 1.39.$$

This negative estimated value of α suggests that, for the period considered, the variability of U.K. bond returns *increased* with the level of yields. We point out, however, that the limited period covered by the data does not allow any general conclusions on the value of α to be drawn.

III. A Pricing Model for Bond Options

In our pricing model, we shall assume that the short-term rate of interest is constant. This is clearly a weakness since a deterministic short rate is logically inconsistent with stochastic variation in the prices of long bonds and is also inconsistent with the empirical evidence. We make this bold assumption based on the success of the Black-Scholes model in valuing stock options since, in this case, even though the application of their model is not subject to the first criticism, it is certainly subject to the second.[8] In theory, it would be easy to include a stochastic short rate of interest in our model, but doing this would

[8] Clearly, when interest rates are both high and variable, the assumption of a deterministic short rate becomes less satisfactory.

involve considerable complication in its practical application. In the first place, it would be necessary to estimate the stochastic process governing the short rate; second, it would require the estimation of a preference-dependent function describing the market price of the short-term interest rate; third, the numerical solution of the resulting partial-differential equation would be substantially more involved. Given the empirical evidence on the performance of the Black-Scholes model, it appears that the influence of random variation in the short-term rate on equity option prices is a second-order effect. Therefore, in spite of the theoretical considerations, the inclusion of a stochastic short rate may not be justified for valuing debt options in practice. The main contribution of the model presented here is to adapt the highly practical features of the Black-Scholes model to the complexities introduced by the changing nature of the variability of bond returns.

Making the standard market assumptions of the continuous-time framework and assuming bond price dynamics described by (4), we can write the value at time t of an option on a bond with price P as $V(P, t)$. Using the familiar Black-Scholes hedging argument, it is easy to show that the value of the option must conform to the following partial-differential equation:

$$\tfrac{1}{2}P^2\sigma^2(P, t)V_{PP} + (rP - c)V_P + V_t - rV = 0, \tag{10}$$

where c is the rate of the continuously paid coupon on the bond,[9] r is the short-term interest rate, and $\sigma(P, t)$ is given by (5). If T is the maturity of the option, then the terminal boundary condition for a call is

$$V(P, T) = \max[0, P - E], \tag{11}$$

where E is the exercise price of the option. Similarly, the terminal boundary condition for a put is

$$V(P, T) = \max[0, E - P]. \tag{12}$$

For American options, the following boundary conditions must also hold for any time $t \leq T$:

$$V(P, t) \geq \max[0, P - E] \quad \text{for a call and} \tag{13}$$

$$V(P, t) \geq \max[0, E - P] \quad \text{for a put.} \tag{14}$$

The partial-differential equation (10), subject to the boundary conditions for either a put or a call, has no analytical solution. However, numerical solutions are easily obtained using, for example, finite-difference methods. In the following section, we provide some solutions to (10) for call options.

IV. Application and Examples

In this section, we discuss some aspects of the application of the model and also provide some examples of bond-option prices.

Apart from the variance function, the remaining parameters in our model are identical to those in the Black-Scholes model and are easily observable (the

[9] For simplicity, we assume continuous coupons; discrete coupons could be easily accommodated.

current bond price, exercise price of the option, time to maturity of the option, and the short-term interest rate). The problem of estimating the parameters k and α in the function for standard deviation given by (5) is precisely the same as that of estimating the corresponding parameters in the Cox [8] constant-elasticity-of-variance model. Section II described one possible approach to this problem, and several others have been suggested in the literature. (See, e.g., Beckers [2] and MacBeth and Merville [13]). The best method to use in the context of bond returns is a subject for further research. In our examples, we assume a value of 0.5 for α and use equation (5) to derive k from a given value of the instantaneous standard deviation of returns, $\hat{\sigma}(P, t)$:

$$k = \hat{\sigma}(P, t)/P^{-1/2}D(P, t)). \tag{15}$$

Table I contains values for American call options on bonds with maturities of two, five, ten, and twenty years. The exercise price in each case is \$100, and values are given for underlying bond prices of \$95, \$100, and \$105 and for options with initial maturities of three months, six months, one year and three years. All these values have been computed using a short-term interest rate of ten percent and an instantaneous standard deviation of return of ten percent.[10] The underlying bond in each case has a face value of \$100 and a coupon rate of ten percent.

It is interesting to compare our results with those derived from the Black-Scholes model. There are, of course, a number of different ways in which the Black-Scholes model could be applied to the problem of valuing bond options, and, in Table I, we provide two alternatives. For each alternative, we have made an adjustment to the variance used in the Black-Scholes model so as to reflect the known fact that the variance of the rate of return on the bond decreases with time to maturity. We assume that the variance declines linearly with time from its current level (0.10 p.a. in the example) to zero at maturity. The first case, BS1, assumes a constant (continuous) coupon and accommodates the American option characteristics. Note that, for the parameters of the example, Merton's [14, p. 156, equation (13)] sufficient condition for no premature exercise holds. The second case, BS2, then treats the option as European and assumes a constant proportional coupon to enable us to compute a closed-form solution.

A primary purpose of our model is to capture the attenuation of a bond's standard deviation of return as it approaches maturity. This effect is most clearly seen in those cases where the maturity of the option is a significant fraction of the maturity of the underlying bond. For example, in Panel C of Table I, the value of an at-the-money one-year call on a two-year bond is \$2.99, whereas the value of a similar option on a twenty-year bond is some twenty-five percent higher at \$3.75. Notice also that the differences in option values become less as the maturity of the underlying bond increases. This is because duration, and therefore the variability of returns, decreases more quickly for shorter maturities.

In comparing our values with those from the various versions of Black-Scholes described above, several points emerge. First, the BS1 values for three- and six-

[10] Notice that it is neither realistic nor consistent with our model for the instantaneous standard deviation of return to be the same for bonds of all maturities. We use the same value of $\hat{\sigma}$ in each case simply to facilitate comparison between the option values we obtain and the corresponding Black-Scholes values.

Table I

Values for Call Options[a]

Panel A: Values for Three-Month Call Options

Bond Maturity (Years)	95: Duration Model	95: BS1	95: BS2	100: Duration Model	100: BS1	100: BS2	105: Duration Model	105: BS1	105: BS2
2	0.31	0.34	0.32	1.85	1.89	1.88	5.38	5.39	5.38
5	0.34	0.36	0.34	1.92	1.92	1.92	5.42	5.41	5.40
10	0.36	0.37	0.35	1.94	1.94	1.93	5.43	5.42	5.41
20	0.37	0.37	0.35	1.95	1.94	1.94	5.43	5.43	5.41

Panel B: Values for Six-Month Call Options

Bond Maturity (Years)	95: Duration Model	95: BS1	95: BS2	100: Duration Model	100: BS1	100: BS2	105: Duration Model	105: BS1	105: BS2
2	0.66	0.76	0.72	2.45	2.56	2.51	5.79	5.85	5.81
5	0.80	0.84	0.79	2.64	2.67	2.62	5.93	5.94	5.89
10	0.85	0.86	0.82	2.70	2.70	2.65	5.97	5.96	5.91
20	0.88	0.87	0.83	2.73	2.72	2.67	5.98	5.98	5.93

Panel C: Values for One-Year Call Options

Bond Maturity (Years)	95: Duration Model	95: BS1	95: BS2	100: Duration Model	100: BS1	100: BS2	105: Duration Model	105: BS1	105: BS2
2	1.05	1.30	1.18	2.99	3.28	3.13	6.22	6.44	6.30
5	1.46	1.56	1.42	3.51	3.59	3.42	6.66	6.71	6.55
10	1.61	1.64	1.49	3.68	3.69	3.52	6.83	6.80	6.63
20	1.70	1.67	1.53	3.75	3.74	3.56	6.85	6.84	6.68

Panel D: Values for Three-Year Call Options

Bond Maturity (Years)	95: Duration Model	95: BS1	95: BS2	100: Duration Model	100: BS1	100: BS2	105: Duration Model	105: BS1	105: BS2
5	2.48	2.77	2.17	4.71	5.01	4.28	7.77	8.02	7.24
10	3.18	3.29	2.55	5.47	5.51	4.71	8.48	8.51	7.65
20	3.53	3.44	2.73	5.83	5.75	4.92	8.79	8.73	7.85

[a] Parameter values used: exercise price = \$100, short-term interest rate = ten percent p.a., instantaneous standard deviation of bond return = ten percent p.a., α = 0.5, coupon rate on bonds = ten percent, and face value of bond = \$100.

month call options on bonds with ten and twenty years to maturity are always within a penny from our values. It seems that the variance-adjusted Black-Scholes method for valuing short-term options on long-term bonds is a very good approximation to our duration procedure. Second, the BS1 procedure overvalues all options on short-term bonds and undervalues long-term options on long-term bonds, relative to our method. The BS2 results are rather mixed. For options on long-maturity bonds, our values tend to be larger than the corresponding BS2 values; for options on shorter maturity bonds, the reverse is true. Clearly, there can be several possible explanations for these results, including differences in treatment of the American feature and of the coupon. However, the results may also reflect the different rates of variance attenuation implicit in the two models. All this serves to illustrate the difficulties in devising simple modifications to Black-Scholes that produce results similar to our model.

The drawbacks of applying Black-Scholes to valuing debt options are well understood, and it is more interesting to compare our model with those that take into account the changing nature of bond returns. The most sophisticated of these is the Brennan and Schwartz model, and, in Brennan and Schwartz [6], values are provided that allow such a comparison.

In Table II, we compare our results with those provided by Brennan and Schwartz and also examine the sensitivity of our results to different values of α. Panel A reports values for call options, with initial maturities of six months, one year, and two years, on a twenty-year bond for values of the bond price between \$90 and \$110. Panel B gives the corresponding values when the underlying bond has initially five years to maturity.

One input to the Brennan and Schwartz model is the standard deviation of returns on a consol bond. To compare our results with theirs, we use their value of 10.32 percent p.a. to estimate the value of k in equation (5). For a consol bond, we have, from equation (9),

$$k = \frac{\sigma(\text{consol}) \times \text{coupon}}{\text{price}(\text{consol})^{\alpha}}. \tag{16}$$

The other parameters are unchanged from our earlier examples.

The first three rows in both panels of Table II report our option values for α of zero, 0.5, and unity. In neither case is there any change in the value of the at-the-money options. The values for in-the-money and out-of-the-money options are influenced by α, but the differences are generally not substantial. For options on the five-year bond, the maximum difference between values for different α's is two cents. For options on the twenty-year bond, the differences are larger, and the maximum difference of fifteen cents occurs for the deep-out-of-the-money two-year option. These results are consistent with those for the Cox [8] constant-elasticity-of-variance model reported in Cox and Rubinstein [9, p. 364, Table 7.1]. It is reassuring that the value of α, which is inherently difficult to estimate, does not appear to be critical for option valuation.

The fourth row in both panels gives the Brennan and Schwartz values. For at-the-money options on a twenty-year bond (Panel A), our values are very close to those of Brennan and Schwartz. For both in-the-money and out-of-the-money options, our values are always somewhat lower than theirs and, in the case of a six-month deep-out-of-the-money option on a twenty-year bond, substantially so. This is somewhat surprising as the differences actually decrease for longer maturity options.

Table II

Values of Call Options on Bonds[a]

Panel A: Values of Call Options ($) on Twenty-Year Bond

Option Maturity:	6 months					1 year					2 years				
Bond Price ($):	90	95	100	105	110	90	95	100	105	110	90	95	100	105	110
$\alpha = 0$	0.12	0.69	2.43	5.74	10.16	0.43	1.39	3.35	6.47	10.54	1.06	2.36	4.49	7.49	11.25
$\alpha = 0.5$	0.13	0.71	2.43	5.72	10.14	0.47	1.42	3.35	6.44	10.50	1.13	2.41	4.49	7.44	11.18
$\alpha = 1.0$	0.15	0.73	2.43	5.70	10.13	0.52	1.46	3.35	6.41	10.46	1.21	2.46	4.49	7.39	11.11
Brennan-Schwartz	0.28	1.07	2.31	6.16	10.41	0.63	1.72	3.30	6.84	10.81	1.31	2.69	4.54	7.84	11.41
Black-Scholes	0.21	0.95	2.82	6.06	10.31	0.67	1.82	3.90	6.99	10.90	1.50	3.01	5.27	8.26	11.91

Panel B: Values of Call Options ($) on Five-Year Bond

Option Maturity:	6 months					1 year					2 years				
Bond Price ($):	90	95	100	105	110	90	95	100	105	110	90	95	100	105	110
$\alpha = 0$	0.00	0.04	1.07	5.04	10.00*	0.00	0.13	1.42	5.13	10.00*	0.02	0.29	1.77	5.29	10.02
$\alpha = 0.5$	0.00	0.04	1.07	5.03	10.00*	0.00	0.14	1.42	5.13	10.00*	0.02	0.30	1.77	5.28	10.02
$\alpha = 1.0$	0.00	0.04	1.07	5.03	10.00*	0.01	0.14	1.42	5.12	10.00*	0.03	0.31	1.77	5.27	10.01
Brennan-Schwartz	0.00	0.04	1.12	5.20	10.08	0.01	0.18	1.55	5.40	10.15	0.10	0.59	2.17	5.70	10.27
Black-Scholes	0.21	0.95	2.82	6.06	10.31	0.67	1.82	3.90	6.99	10.90	1.50	3.01	5.27	8.26	11.91

[a] Parameter values used: exercise price = $100, short-term interest rate = ten percent p.a., instantaneous standard deviation of return on consol bond = 10.32 percent p.a., coupon rate on bond = ten percent and face value of bond = $100.

* Exercise value.

For options on five-year bonds (Panel B), we observe that at-the-money values are similar for six-month and one-year options but that our method gives substantially lower values for two-year options. As in Panel A, we find that the values for away-from-the-month options are lower in the case of our model, but, contrary to Panel A, these differences now increase with option maturity.

In comparing our values for options on five-year bonds with those of Brennan and Schwartz, it should be pointed out that, in the case of our method, we have imputed the standard deviation of returns on the five-year bond from the consol standard deviation. This step, which undoubtedly introduces error into our values, would, of course, be unnecessary in practice as the standard deviation of returns would be estimated directly from the returns on the five-year bond.

Finally, and for completeness, the last row in both panels of Table II gives the values obtained using the Black-Scholes model. Following Brennan and Schwartz, we use the consol standard deviation in all Black-Scholes calculations, and this results in identical values in both panels.[11] Not surprisingly, this method grossly overvalues options on the shorter bond relative to the other methods since it implicitly assumes that the standard deviation of return on five-year bonds equals that of a consol.

Options models are used, not only for valuation but also for hedging. The key statistic here is the "hedge ratio," defined by

$$HR = \partial V/\partial P, \tag{17}$$

which gives the number of units of the underlying asset that must be sold short (purchased) in order to hedge one call (put). Table III gives the hedge ratios corresponding to the call option values given in Table I. It is interesting to note that the differences between the hedge ratios for our model and those for Black-Scholes are much smaller than the differences in option values. This is most marked in the case of at-the-money options where the greatest difference is only 0.01. The largest difference occurs in Panel C of Table III for an out-of-the-money one-year call on a two-year bond, where the difference is 0.03.

V. Conclusions

In this paper, we have developed a new approach to valuing debt options. The key feature of the model is that it incorporates in a simple yet realistic manner the well-documented fact that the variance of returns on a bond tends to decline as it approaches maturity. The inputs to the model are fairly readily obtained, and the computational complexity is of the same order of magnitude as that for the Black-Scholes model applied to American options.

The option values that are presented in the paper indicate that there can be substantial differences between the values produced by our model and those produced by Black-Scholes, in particular when the maturity of the option is a significant fraction of the maturity of the underlying bond. These differences, however, are trivial for short-term options on long-term bonds. Comparison with

[11] It should be noted that there seems to be an error in the Black-Scholes values reported in Brennan and Schwartz [6].

Table III

Hedge Ratios for American Call Options[a]

Panel A: Hedge Ratios for Three-Month American Call Options

	Bond Price ($)					
	95		100		105	
Bond Maturity (Years)	Duration Model	BS1	Duration Model	BS1	Duration Model	BS1
2	0.13	0.14	0.51	0.51	0.86	0.86
5	0.14	0.15	0.51	0.51	0.85	0.85
10	0.15	0.15	0.51	0.51	0.85	0.85
20	0.15	0.15	0.51	0.51	0.85	0.85

Panel B: Hedge Ratios for Six-Month American Call Options

	Bond Price ($)					
	95		100		105	
Bond Maturity (Years)	Duration Model	BS1	Duration Model	BS1	Duration Model	BS1
2	0.20	0.22	0.51	0.51	0.79	0.79
5	0.22	0.23	0.51	0.51	0.78	0.78
10	0.23	0.23	0.51	0.51	0.77	0.77
20	0.24	0.24	0.52	0.51	0.78	0.78

Panel C: Hedge Ratios for One-Year American Call Options

	Bond Price ($)					
	95		100		105	
Bond Maturity (Years)	Duration Model	BS1	Duration Model	BS1	Duration Model	BS1
2	0.25	0.28	0.51	0.52	0.75	0.74
5	0.29	0.30	0.51	0.52	0.72	0.72
10	0.30	0.31	0.52	0.52	0.71	0.72
20	0.31	0.31	0.52	0.52	0.72	0.72

Panel D: Hedge Ratios for Three-Year American Call Options

	Bond Price ($)					
	95		100		105	
Bond Maturity (Years)	Duration Model	BS1	Duration Model	BS1	Duration Model	BS1
5	0.35	0.37	0.52	0.53	0.67	0.68
10	0.38	0.38	0.52	0.53	0.66	0.67
20	0.40	0.39	0.53	0.53	0.66	0.66

[a] Parameter values used: exercise price = \$100, short-term interest rate = ten percent p.a., instantaneous standard deviation of bond return = ten percent p.a., $\alpha = 0.5$, coupon rate on bonds = ten percent, and face value of bond = \$100.

the Brennan and Schwartz model shows that our method produces broadly similar results. Moreover, the differences in practical application may be smaller than those reported here since now the variability of the underlying bond would be estimated directly rather than being imputed from consol variance.

In deriving our model, we have made a number of simplifying assumptions including a constant short-term rate of interest. The only way to establish whether this and the other assumptions we have made are reasonable is through empirical tests of their ability to price debt options and, in particular, by comparing the performance of our model with those of the alternatives.

Finally, the approach we have developed here can also be applied to the valuation of options on bond futures.

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