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Time-Invariant Portfolio Insurance Strategies

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ABSTRACT

This paper characterizes the complete class of time-invariant portfolio insurance strategies and derives the corresponding value functions that relate the wealth accumulated under the strategy to the value of the underlying insured portfolio. Time-invariant strategies are shown to correspond to the long-run policies for a broad class of portfolio insurance payoff functions.

THE LINK BETWEEN PORTFOLIO insurance and investment strategy was first noted by Brennan and Schwartz [5], who pointed out that insurance companies that had guaranteed the minimum payments they would make under equity-linked life insurance policies could hedge the resulting liability by following an investment strategy derived from the Black-Scholes [4] option-pricing model. Pure portfolio insurance without any element of mortality insurance appears to have been offered first by the Harleysville Mutual Insurance Company in 1971; however, a lack of public interest in this product led to its withdrawal by 1979.[1] Despite this initial lack of success, there has been, in recent years, an explosive growth in the sale of portfolio insurance strategies to institutional portfolio managers.[2]

Under ideal conditions, a simple portfolio insurance strategy ensures that the value of the insured portfolio, at some specified date, will not fall below some specified level. This property may be of considerable significance to portfolio managers if their investment performance is monitored on a periodic basis and if poor performance is heavily penalized; it may also be of significance to the owner of an investment portfolio that is held to meet some known set of future liabilities. However, there are at least two difficulties with this simple type of portfolio insurance. First, under almost all circumstances, a simple portfolio insurance strategy is inconsistent with expected-utility maximization.[3] Second, in many circumstances, the specification of the precise date on which the insurance is to be effective is arbitrary because institutional investment portfolios typically have no predetermined final date. Moreover, the specification of the

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[1] See Gatto et al. [9].

[2] The pioneers in the sale of portfolio insurance strategies were Leland, O'Brien, Rubinstein Associates.

[3] See Brennan and Solanki [6] and Benninga and Blume [2].

effective date of the insurance induces an investment strategy that is strongly time dependent.

Leland [14] and Brennan and Solanki [6] have generalized the concept of portfolio insurance to payoffs that are arbitrary functions of the value of some reference portfolio and have analyzed the types of function that will be optimal for individuals with different tastes and expectations. More recently, Perold [19] and Black and Jones [3] have popularized one of these functions and the associated investment strategy; this is the function that is appropriate for an individual with constant proportional risk aversion if the investment opportunity set is stationary; as Merton [15] and Hakansson [10] had shown earlier, if the investment opportunity set is constant, such an investor keeps a constant proportion of his or her wealth in risky assets. Besides its obvious simplicity, this constant-proportion investment strategy has the advantage over the simple portfolio insurance strategy of being time independent.

It is known from the work of Mossin [18], Leland [13], Hakansson [11],[4] and others that, for broad classes of utility functions defined over terminal wealth, the optimal investment strategy becomes constant as the horizon recedes. Given the indefinite horizons of most institutional investment portfolios, it is therefore of interest to consider the class of investment strategies under which the fraction of wealth allocated to risky assets is independent of time. The constant-proportion strategy is, of course, one member of this class, while, for a finite horizon, the simple portfolio insurance strategy mentioned above is not.

In this paper, we offer a complete characterization of the class of time-invariant insurance investment policies and their associated payoff functions and present some particular examples. The setting is one in which there is a single risky portfolio and a riskless security. The return on the risky portfolio is assumed to follow an Itô process with a constant variance rate, and the return on the riskless security is assumed to be an intertemporal constant. Although there exists a broad class of time-invariant portfolio insurance strategies that are optimal for some risk-averse expected-utility maximizer as the investment horizon recedes, we consider the whole class of time-invariant strategies and ignore the issue of an appropriate objective function for an institutional portfolio manager.[5]

Section I provides a formal definition of time invariance and derives the main results. The value functions yielded by the time-invariant investment strategies are characterized in Section II. Section III concludes the paper.

I. Time-Invariant Strategies

We assume that P, the value of the underlying risky-asset portfolio (the insured or "reference" portfolio), follows a continuous stochastic process of the general

[4] Mossin [18] showed that, if the utility function is of the extended power class, the optimal investment strategy tends asymptotically to the constant-proportions policy. These results were extended by Leland [13], who examined the class of utility functions for which the measure of constant relative risk aversion converges as wealth approaches infinity. Hakansson [11] offers bounds on the utility function that are sufficient to yield the constant-proportions investment policy asymptotically.

[5] Indirect evidence that institutional portfolio managers are not concerned with expected utility maximization is to be found in the popularity of simple portfolio insurance.

type:

$$\frac{dP}{P} = \mu dt + \sigma dz, \tag{1}$$

where dz is the increment to a standard Gauss-Wiener process and μ is the (possibly stochastic) instantaneous expected rate of return on the portfolio.

Let $V(P, t)$ denote the value at time t of the funds accumulated under a particular investment strategy, the "value of the strategy". We shall be concerned with investment strategies under which

(i) the fraction of the reference portfolio held can be written as a right-continuous function of, at most, the current value of the portfolio and time, $z(P, t)$;
(ii) the balance of the funds invested under the strategy, $V(P, t) - z(P, t)P$, is held in the riskless security, which earns at the continuously compounded rate, r;
(iii) no funds are added or withdrawn so that the strategy is "self-financing" (we shall refer to such investment strategies as "(generalized) portfolio insurance (investment) strategies").[6]

The basis of the Black-Scholes [4], Merton [16] option-pricing model is that any terminal payoff function of the form $y(P)$ can be achieved by following an investment strategy of the type we describe and that the value of the funds held under the strategy, $V(P, t)$, $t \leq T$, where T is the payoff date, satisfies a certain partial-differential equation. The following lemma, which is related to a result of Merton [17],[7] states that the value function under any investment strategy of this type is a function only of P and t and that it satisfies the Black-Scholes partial-differential equation.

LEMMA: *Consider a self-financing investment strategy in which an amount $z(P, t)P$ is invested in a reference portfolio with a value that follows the stochastic process* (1) *and the balance is invested in riskless securities. Then the value of the wealth accumulated under the strategy may be written as $V(P, t)$, where*

(i) *the value function $V(P, t)$ satisfies the Black-Scholes equation:*

$$\tfrac{1}{2}\sigma^2 P^2 V_{pp} + rPV_p + V_t - rV = 0 \tag{2}$$

and

(ii) $$z(P, t) = V_p(P, t). \tag{3}$$

Proof: Let $H(\omega, t)$ denote the value function for the self-financing strategy $z(P, t)$ at time t, where $\omega \in \Omega$ is the state at time t. Then

$$H(\omega, t) = z(P, t)P + B(\omega, t), \tag{4}$$

[6] Cox and Leland [7] describe such investment strategies as "path independent" and show that these are the only efficient investment strategies for an expected-utility maximizer if the investment opportunity set is constant.

[7] See also Cox and Leland [7].

where $B(\omega, t)$ is the amount of funds held in the riskless security. Since the strategy is self-financing,

$$dH = zdP + rBdt. \tag{5}$$

Consider the value function $V(P, t)$. Itô's Lemma implies that

$$dV = V_p dP + [V_t + \tfrac{1}{2}\sigma^2 P^2 V_{pp}]dt. \tag{6}$$

Comparing coefficients in (5) and (6), it is apparent that $dV = dH$ if the value function $V(P, t)$ is such that

$$V_p(P, t) = z(P, t) \tag{3}$$

and

$$V_t + \tfrac{1}{2}\sigma^2 P^2 V_{pp} = rB. \tag{7}$$

Then, if $H(\omega, t_0) = V(P, t_0)$, $H(\omega, t) = V(P, t)$, so that the value of the wealth accumulated under the strategy depends only on the current value of P and t. Substitution for B from (4) into (7) yields (2). Q.E.D.

We are concerned with the characteristics of time-invariant investment strategies, which are defined as follows.

Definition: A portfolio insurance investment strategy is time invariant if the fraction of wealth under the strategy that is allocated to the reference portfolio is at most a function of the current value of the portfolio, P:

$$\frac{PV_p}{V} = f(P). \tag{8}$$

As shown in the Lemma, the wealth accumulated under a generalized portfolio insurance investment strategy at time T may be written as $V(P, T) = y(P)$, where $y(P)$ is the terminal payoff function. Given the long investment horizons of many institutional investors, it is of interest to characterize the class of payoff functions, $y(P)$, for which the associated investment strategy becomes time invariant as the horizon recedes. This class is described by the following turnpike theorem.

THEOREM 1: *A necessary and sufficient condition for the investment strategy that yields the payoff function $y(P)$ to be asymptotically time invariant is that*

$$\lim_{T\to\infty} \frac{E^*[y'(P(T))P(T)]}{E^*[y(P(T))]} = h(P(0)), \tag{9}$$

where $P(t)$ is the value of the reference portfolio at time t and $E^*[\cdot]$ denotes the expectation with respect to the "risk-adjusted" stochastic process[8] *or the "equivalent martingale measure".*[9]

Proof: $V(P, t)$ satisfies equation (2) subject to the boundary condition

[8] See Cox and Ross [8].

[9] See Harrison and Kreps [12].

$V(P, T) = y(P)$. The solution may be written, for $t = 0$, as

$$V(P, 0) = e^{-rT} \int_0^\infty y(x) \times \phi(x, T)\, dx, \tag{10}$$

where

$$\phi(x, T) = \frac{1}{x\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2T\sigma^2}\left[\ln\frac{x}{P} - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}$$

is the risk-adjusted lognormal density.

Differentiating (10) with respect to P yields, after some manipulation,

$$\frac{V_p \times P}{V} = -\frac{\ln P}{T\sigma^2} - \frac{r - \frac{1}{2}\sigma^2}{\sigma^2} + \frac{1}{T\sigma^2}\,\frac{\int_0^\infty y(x) \times \ln x \times \phi(x, T)\, dx}{\int_0^\infty y(x) \times \phi(x, T)\, dx}. \tag{11}$$

Using Stein's Lemma,[10] (11) may be written as

$$\frac{V_p \times P}{V} = \frac{\mathrm{E}^*[y'(P(T)) \times P(T)]}{\mathrm{E}^*[y(P(T))]}.$$

Hence,

$$\lim_{T\to\infty} \frac{V_p \times P}{V} = f(P)$$

if and only if condition (9) holds. Q.E.D.

We state without proof the following theorem, which characterizes a class of payoff functions that induce asymptotically time-invariant investment strategies.

THEOREM 2: *A sufficient condition for the portfolio insurance investment strategy to become time invariant as the horizon recedes is that the payoff function $y(P)$ be given by*

(i) $$y(P) = \max(P, k) + \eta(P),$$

(ii) $$y(P) = \textstyle\sum_i a_i P^{\beta_i} + \eta(P),$$

where $\eta(P)$ is the payoff on any contingent claim with present value that approaches zero as the date of the contingent payment recedes (i.e., $|\eta(P)| < a + bP^\beta$ *for some* $\beta < 1$).

Since any monotone increasing payoff function is optimal for some risk-averse utility function,[11] the above theorem extends the portfolio turnpike results of Hakansson [11] and Ross [20] to the class of utility functions for which the above payoff functions are optimal.

The following two theorems characterize the whole class of value functions and investment strategies that are permissible if the investment strategy is time invariant.

[10] See Rubinstein [21].

[11] See Brennan and Solanki [6].

THEOREM 3: *Under a time-invariant investment strategy, the value function, $V(P, t)$, is of the form*

$$V(P, t) = e^{\gamma t}[C_1 P^{\alpha_1} + C_2 P^{\alpha_2}], \tag{12}$$

where C_1, C_2, and γ are constants that are chosen to satisfy the initial budget constraint, $\gamma \leq (r + \frac{1}{2}\sigma^2)/2\sigma^2$, and

$$\alpha_1 = \frac{-(r - \frac{1}{2}\sigma^2) + \sqrt{(r + \frac{1}{2}\sigma^2)^2 - 2\gamma\sigma^2}}{\sigma^2},$$

$$\alpha_2 = \frac{-(r - \frac{1}{2}\sigma^2) - \sqrt{(r + \frac{1}{2}\sigma^2)^2 - 2\gamma\sigma^2}}{\sigma^2}.$$

Proof: Integrating (8), the condition for time invariance, we obtain

$$V(P, t) = k(t)g(P), \tag{13}$$

where $k(t)$ is a constant of integration and

$$g(P) = \exp\left[\int_0^P f(x)/x \, dx\right]. \tag{14}$$

Since $V(P, t)$ satisfies the partial-differential equation (2), (13) implies for $g'(\cdot) \neq 0$ and $k(\cdot) \neq 0$ that

$$\tfrac{1}{2}\sigma^2 P^2 g'' + rPg' - (r - k'/k)g = 0. \tag{15}$$

Since $g(\cdot)$ is a function only of P, $k'/k = \gamma$, a constant, so that $k(t) = k_0 e^{\gamma t}$. Then, substituting the solution to (15) in (13) and using the definition of $k(t)$ yield (12). Q.E.D.

THEOREM 4: *Under a time-invariant investment strategy $f(P)$, the fraction of wealth allocated to the reference portfolio is given by*

$$f(P) = w(P)\alpha_1 + (1 - w(P))\alpha_2, \tag{16}$$

where

$$w(P) = C_1 P^{\alpha_1}/(C_1 P^{\alpha_1} + C_2 P^{\alpha_2}). \tag{17}$$

Proof: This follows immediately from (8) and (12). Q.E.D.

The value function (12) is multiplicatively separable in a function of time and a function of the value of the reference portfolio. The parameter γ is a "growth parameter"; it is the rate at which the value function grows for a given value of the reference portfolio. As we shall see below, it is possible to construct value functions with a minimum, in which case γ is the guaranteed minimum long-run rate of return. For a given value of the parameter γ, the constants of integration C_1 and C_2 must be chosen to satisfy the constraint that the initial value of the strategy be equal to the funds available for investment; therefore, for any value of γ, there exists a one-parameter family of value functions and investment strategies.

Two simple cases correspond to one hundred percent investment in the reference portfolio, $f(P) = 1$, and one hundred percent investment in riskless securities, $f(P) = 0$. It may be verified that these strategies correspond to the value functions $V(P, t) = C_1 \times P$ and $V(P, t) = C_1 e^{rt}$, respectively.

While the investment strategy characterized above is time invariant, the joint distribution of the rate of return earned under this strategy and the rate of return on the reference portfolio in general will be nonconstant, even if the distribution of the return on the reference portfolio is constant. An institutional portfolio manager is most often evaluated on the basis of the realized distribution of returns on his or her portfolio. If the evaluation rule is constant, he or she is likely to choose a strategy with a *return distribution* that is constant. The class of such strategies is characterized by the following theorem.

THEOREM 5: *A necessary and sufficient condition for the rate of return under a portfolio insurance investment strategy to follow a constant stochastic process (independent of P and t), if the return on the reference portfolio follows a constant stochastic process, is that C_1 or C_2 in equation* (12) *be equal to zero or, equivalently, that $w(P)$ in equation* (16) *be equal to zero or one.*

Then

$$V(P, t) = C_1 e^{\gamma t} P^{\alpha_1} \tag{18}$$

or

$$V(P, t) = C_2 e^{\gamma t} P^{\alpha_2}. \tag{19}$$

Proof: (i) Sufficiency: Under this strategy, the fraction of wealth allocated to the reference portfolio is constant, which implies that the instantaneous return on the strategy is a fixed linear function of the return on the reference portfolio. (ii) Necessity: Since the rate of return under the policy is a linear combination of the riskless interest rate and the risky return on the reference portfolio, its stochastic process can be constant only if the portfolio weight $f(P)$ is a constant. However, this implies that $w(P)$ is constant, so that, from (17), $C_1 = 0$ or $C_2 = 0$. Q.E.D.

This investment policy has been previously recognized as the optimal portfolio insurance strategy for an investor with constant proportional risk aversion when the opportunity set is time invariant by Brennan and Solanki [6]; a related strategy has been recently popularized by Black and Jones [3], who limit the fraction of wealth allocated to the reference portfolio to unity.

An alternative notion of time invariance is that the *value function* $V(P, t)$ is time invariant. If this condition is satisfied, then the relative performance of the strategy and the reference portfolio will also be time invariant. Investment strategies yielding time-invariant value functions are characterized by the following theorem.

THEOREM 6: *The necessary and sufficient condition for the value function $V(P, t)$ to be time invariant (independent of t) is that the fraction of wealth allocated to the reference portfolio be given by*

$$f(P) = w(P) - (1 - w(P))2r/\sigma^2, \tag{20}$$

where

$$w(P) = C_1P/(C_1P + C_2P^{(-2r/\sigma^2)}). \tag{21}$$

Then the value function $V(P)$ *is*

$$V(P) = C_1P + C_2P^{(-2r/\sigma^2)}. \tag{22}$$

Proof: For a time-invariant value function, $V_t = 0$. Then (22) is the complete solution to the ordinary differential equation obtained from (2). The investment strategy (20)–(21) follows immediately from (8) and (22).

Comparing (12) with (22), it is apparent that the time-invariant value function (22) follows from the time-invariant investment strategy in which $\gamma = 0$. Q.E.D.

II. Characterizing the Value Functions

To illustrate the possible types of value function that are attainable under time-invariant portfolio insurance investment strategies, Figures 1 through 6 depict some representative value functions for different assumptions about γ and for a fixed pair of environmental parameters ($r = 0.07$ and $\sigma^2 = 0.04$). In constructing

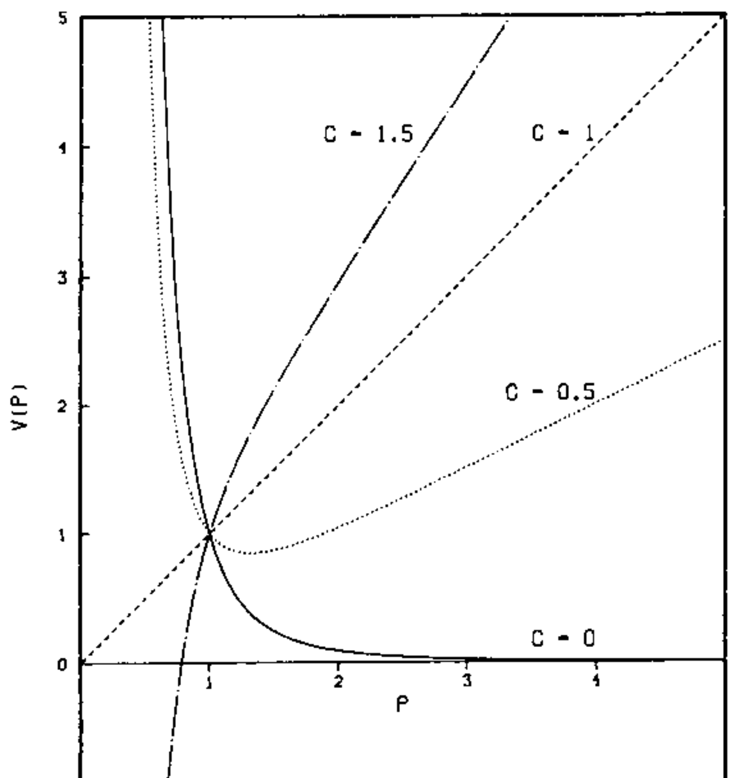

Figure 1. Value Functions for Time-Invariant Portfolio Insurance Policies When $\gamma = 0.0$.

($r = 0.07$; $\sigma^2 = 0.04$)

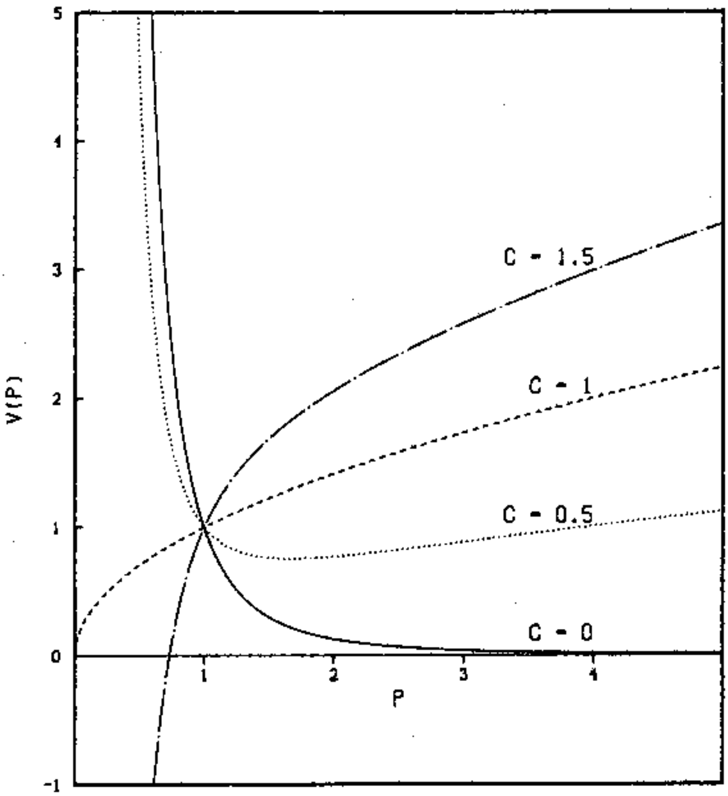

Figure 2. Value Functions for Time-Invariant Portfolio Insurance Policies at $t = 0$ When $0 < \gamma < r$.

($r = 0.07$; $\sigma^2 = 0.04$; $\gamma = 0.04$)

these figures, the reference portfolio is standardized so that $V(1, 0) = 1$ and the value functions are shown for $t = 0$. The standardization implies that the value function (9) may be written as

$$V(P, t) = e^{\gamma t}[cP^{\alpha_1} + (1 - c)P^{\alpha_2}] \tag{12'}$$

or, for $\gamma = 0$, as

$$V(P) = cP + (1 - c)P^{-2r/\sigma^2}. \tag{22'}$$

We shall refer to the value functions corresponding to $c = 0$ or $c = 1$ as the "basic" value functions since all value functions may be constructed as weighted combinations of these.

A. Zero Growth Rate: $\gamma = 0$

Figure 1 shows the time-invariant value functions that are obtained when $\gamma = 0$. Since $\alpha_1 = 1$, the basic value function for $c = 1$ is linear and corresponds to a policy of investing one hundred percent of wealth in the reference portfolio. The negatively sloped basic value function $c = 0$ follows from a policy of shorting

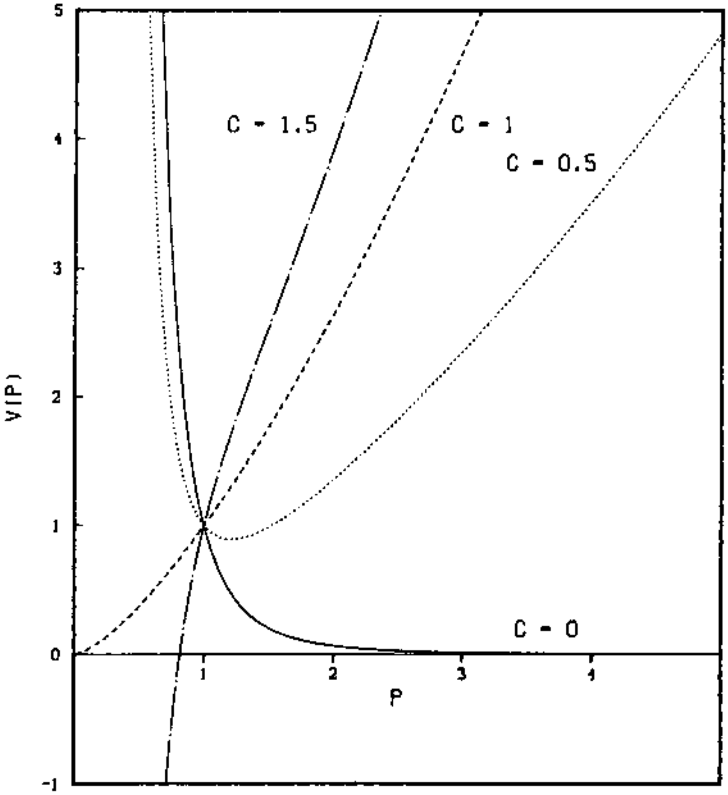

Figure 3. Value Functions for Time-Invariant Portfolio Insurance Policies at $t = 0$ When $\gamma < 0$.

($r = 0.07$; $\sigma^2 = 0.04$; $\gamma = -0.04$)

the reference portfolio by a constant fraction of wealth α_2. Monotonically increasing and strictly concave value functions are obtained by choosing values of the mixing parameter $c > 1$. However, by choosing convex combinations of the two basic value functions (e.g., $c = 1/2$), it is possible to obtain a value function with a minimum or "guaranteed" value. Moreover, in contrast to traditional portfolio insurance strategies,[12] the minimum value is realized on a set of measure zero. However, like traditional portfolio insurance strategies, nonmonotone value functions are unlikely to be optimal for expected utility maximizers.[13]

The value function is completely determined by the mixing parameter c, and, for $0 < c < 1$, it has a natural interpretation in terms of the "cost" of insurance. Thus, define $\rho(P; c)$, the *relative* value function, as the ratio of the wealth realized under a particular portfolio insurance strategy to the value of the reference portfolio:

$$\rho(P; c) \equiv \frac{V(P)}{P} = c + (1 - c)P^{-1-2r/\sigma^2}. \tag{23}$$

[12] Cf. Brennan and Schwartz [5] and Rubinstein [22].

[13] See Arrow [1].

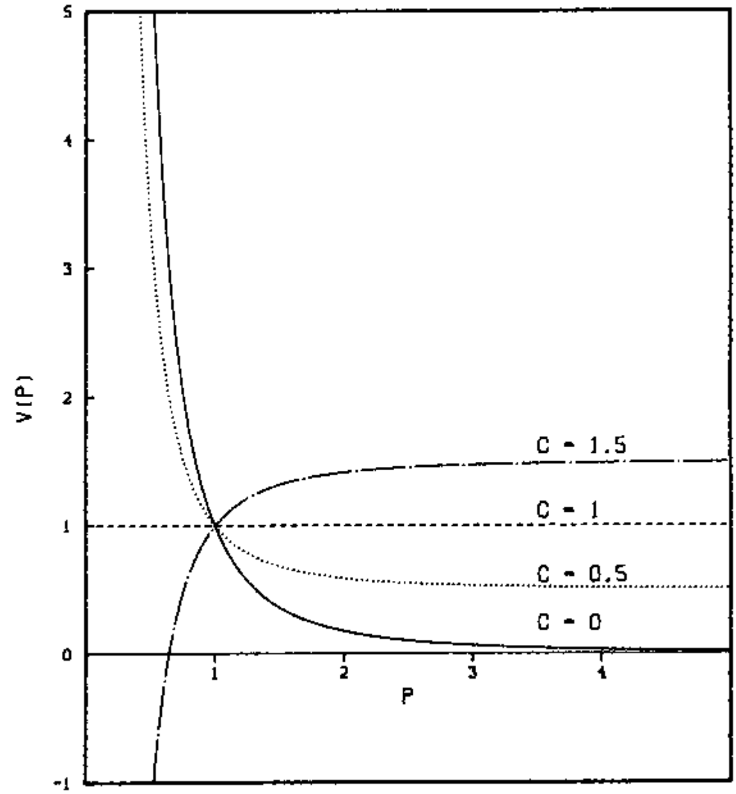

Figure 4. Value Functions for Time-Invariant Portfolio Insurance Policies at $t = 0$ When $\gamma = r = 0.07$.

$(\sigma^2 = 0.04)$

Then

$$\lim_{P\to\infty} \rho(P; c) = c. \tag{24}$$

Therefore, $(1 - c)$ may be interpreted as the proportional cost of the insurance strategy under favorable outcomes. The benefit of the insurance strategy depends on the minimum of the value function. Differentiating the value function (22′), it can be seen that the value function is minimized[14] at

$$P^* = \left[\frac{2r(1 - c)}{\sigma^2 c}\right]^{(\sigma^2/\sigma^2+2r)} \tag{25}$$

The minimum value is obtained by substituting from (25) in (22′). Table I gives the minimum values for different values of the mixing parameter c. For example, $c = 0.95$ guarantees that the value function will never fall below 0.84 and that this minimum is not attained until the reference portfolio value falls to 0.69. The implied "insurance cost" as measured by the value of the strategy relative to the

[14] The second-order condition is satisfied if $0 < c < 1$.

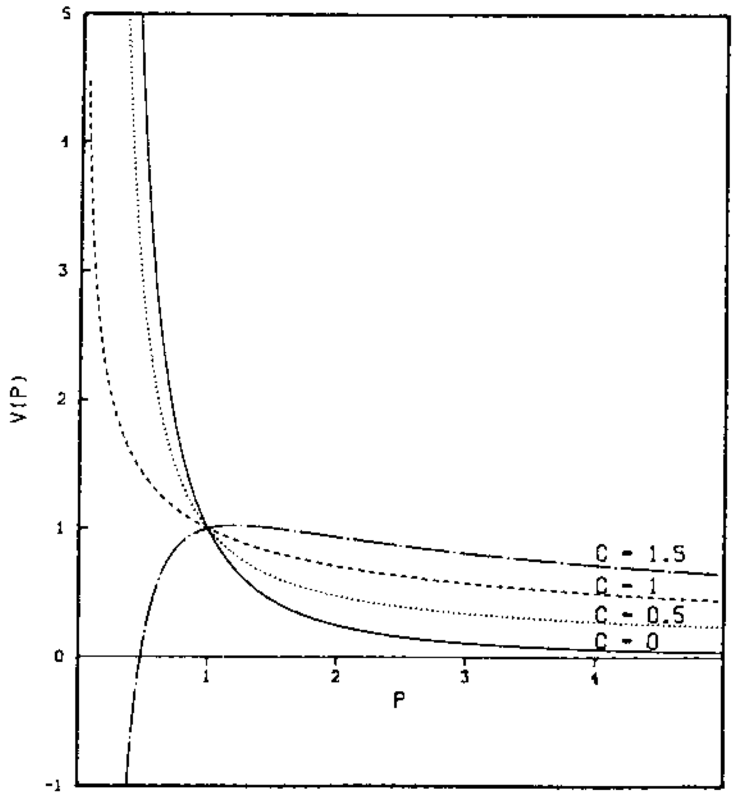

Figure 5. Value Functions for Time-Invariant Portfolio Insurance Policies at $t = 0$ When $\gamma > r$ and $\sigma^2 < 2r$.

$(r = 0.07; \quad \sigma^2 = 0.04; \quad \gamma = 0.09)$

value of the reference portfolio for high values of the reference portfolio is $1 - 0.95$ = five percent. Similarly, an "insurance cost" of about twenty percent guarantees that the value function never falls below its initial value of unity. Under this strategy, the guaranteed minimum rate of return for any horizon is zero.

B. Growth Rate below the Interest Rate: $\gamma < r$ and $\gamma \neq 0$

Figures 2 and 3 depict the value functions obtained when $0 < \gamma < r$ and $\gamma < 0$, respectively. The basic value function $c = 1$, which is linear when $\gamma = 0$, becomes increasing concave for $\gamma > 0$ and increasing convex for $\gamma < 0$. The basic value function $c = 0$ remains decreasing convex for all values of γ, the convenxity decreasing in γ.

Note that, when $\gamma \neq 0$, the value functions are no longer time invariant but shift up at the (possibly negative) rate γ.

The investment strategies discussed by Black and Jones [3] and by Perold [19] correspond to the basic value function $c = 1$, which results from investing a constant fraction of wealth in the reference portfolio. The richer class of time-

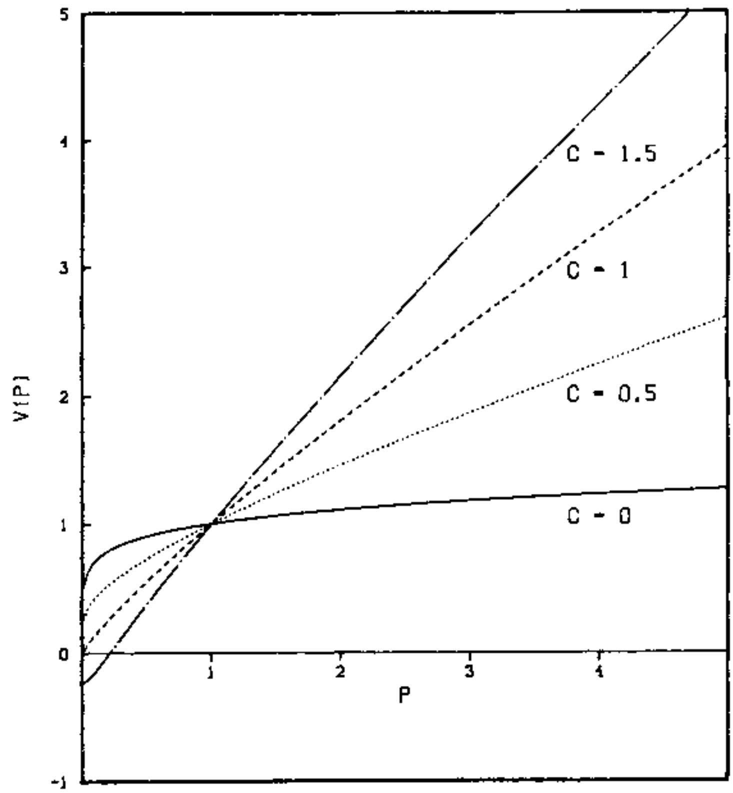

Figure 6. Value Functions for Time-Invariant Portfolio Insurance Policies at $t = 0$ When $\gamma > r$ and $\sigma^2 > 2r$.

Table I

Minima of the Value Function When $\gamma = 0$ for Alternative Values of the Mixing Parameter c[a]

c	P^*	$V(P^*)$
0.80	0.97	1.00
0.90	0.81	0.94
0.95	0.69	0.84
0.97	0.61	0.76
0.99	0.48	0.61

[a] $r = 0.07$; $\sigma^2 = 0.04$.

invariant investment strategies described in Theorem 4 yields arbitrary linear combinations of the two basic value functions as shown in equation (12′). Moreover, for convex combinations ($0 < c < 1$), the value function attains a minimum that rises over time at the rate γ. These value functions yield guaranteed minimum or "insured" long-run rates of return equal to γ. Differentiating

Table II

Minima of the Value Function at $t = 0$ for Alternative Values of γ and the Mixing Parameter c[a]

$\gamma = -0.04$		
c	P^*	$V(P^*, 0)$
0.80	0.93	0.99
0.90	0.80	0.90
0.95	0.70	0.78
0.97	0.63	0.69
0.99	0.51	0.52
$\gamma = 0.04$		
c	P^*	$V(P^*, 0)$
0.80	1.12	0.99
0.90	0.89	0.99
0.95	0.72	0.94
0.97	0.62	0.89
0.99	0.45	0.77

[a] $r = 0.07$; $\sigma^2 = 0.04$.

(12′) with respect to P, it is seen that this minimum is attained at

$$P^* = \left[-\frac{(1-c)\alpha_2}{c\alpha_1}\right]^{(1/\alpha_1-\alpha_2)}, \tag{26}$$

and the guaranteed minimum value for any time t is obtained by substituting (26) into (12′).

Table II gives values of P^* and the initial guaranteed minimum value, $V(P^*, 0)$, for different values of the guaranteed growth rate γ and the mixing parameter c.

C. Growth at the Interest Rate: $\gamma = r$

Figure 4 shows that, when $\gamma = r$, the basic value function $c = 1$ becomes a horizontal line rising at the interest rate as the investment strategy consists of investing everything in the riskless asset. Values of $c > 1$ yield monotonically increasing, strictly concave value functions, but there exist no value functions with a minimum.

D. Growth above the Interest Rate: $r < \gamma \leq (r + \frac{1}{2}\sigma^2)^2/2\sigma^2$[15]

In this case, both basic value functions slope down and approach zero for large values of P as long as $\sigma^2 < 2r$. Then all value functions either are monotonically decreasing or contain an interior maximum. Figure 5 illustrates this case.

When $\sigma^2 > 2r$, both basic value functions are positive, monotonically increas-

[15] The upper bound is the maximum value of γ for which the value function is defined. At this bound, the basic value functions are coincident.

ing, and strictly concave so that all convex-combination value functions share this property. Figure 6 depicts an example of this case.

E. Switching Strategies

As we have already mentioned, the time-invariant portfolio insurance strategies we have described that offer a minimum guaranteed rate of return when $\gamma < r$ possess nonmonotone value functions that imply short positions in the reference portfolio over the negatively sloped range. A feasible policy that avoids this problem while retaining most of the properties we have discussed[16] involves switching between two time-invariant strategies according to $P \gtrless P^*$.

Consider the policy of switching between the two classes of stationary strategies described in Theorem 2:

$$\begin{aligned} f(P) &= w(P)\alpha_1 + (1 - w(P))\alpha_2 \quad \text{if} \quad P > P^* \\ &= 0 \quad \text{if} \quad P \leq P^*, \end{aligned}$$

where

$$P^* = \left[\frac{-\alpha_2(1 - c)}{\alpha_1 c}\right]^{(1/\alpha_1 - \alpha_2)}. \tag{27}$$

Under this hybrid policy, which is depicted in Figure 7, wealth increases at the riskless interest rate (and the whole value function shifts up at the same rate) if $P \leq P^*$. If $P > P^*$, the whole value function shifts up at the rate γ.

Thus, the value function may be written as

$$\begin{aligned} \tilde{V}(P, t) &= e^{\tilde{\delta}t}[cP^{\alpha_1} + (1 - c)P^{\alpha_2}] \quad \text{for} \quad P > P^* \\ &= e^{\tilde{\delta}t} \quad \text{for} \quad P \leq P^*, \end{aligned} \tag{28}$$

where

$$\tilde{\delta} = \tilde{q}\gamma + (1 - \tilde{q})r$$

and $\tilde{q}$ is the fraction of the time since $t = 0$ for which $P > P^*$.

It is apparent that this policy guarantees a minimum rate of return of γ.

III. Conclusion

We have analyzed a class of time-independent portfolio insurance strategies in a simplified setting in which the interest rate and the risk of the underlying portfolio are known constants. To each strategy there corresponds a value function relating the value of the funds accumulated under the strategy to the value of the underlying reference portfolio. These value functions shift up at the rate γ. Indeed, they are the only class of constant-growth-rate value functions. A natural generalization is to consider value functions that shift up at the rate γ and shift to the right at a different rate π.[17] This class of value functions is attainable by allocating to the reference portfolio at time t a fraction of wealth

[16] The proposed policy, however, is not path independent.

[17] π may be thought of as the long-run rate of inflation.

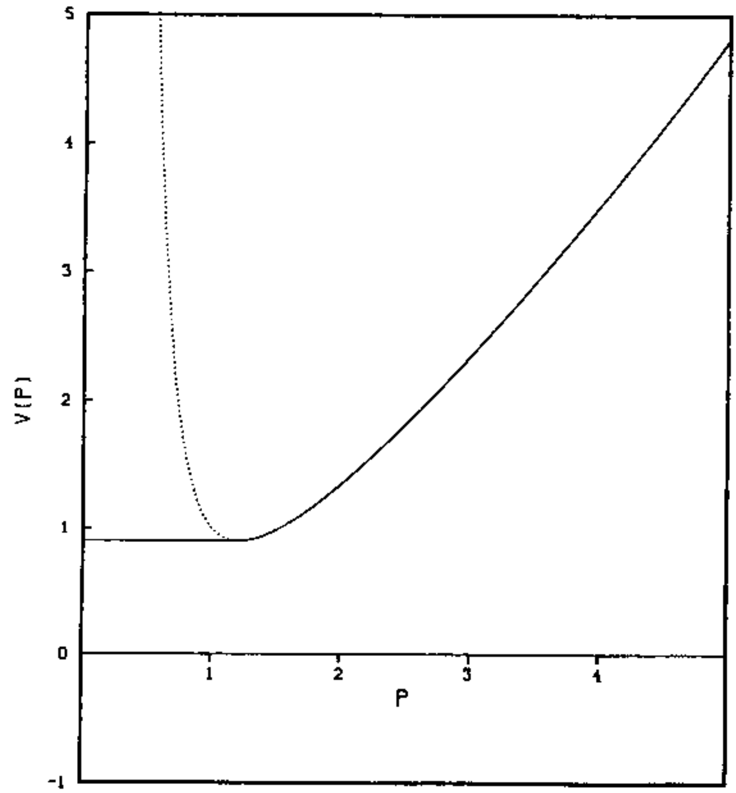

Figure 7. Value Functions for a Strategy of Switching between Two Time-Invariant Policies as $P \gtreqless P^*$. (The functions shift up at a rate γ when $P > P^*$ and at a rate r when $P < P^*$.)

$$(r = 0.07; \quad \sigma^2 = 0.04; \quad \gamma = -0.04; \quad c = 0.5)$$

equal to some function of the "discounted" value of the reference portfolio, $f(Pe^{-\pi t})$. While such policies are not time independent, they may be of interest in their own right and correspond to time-invariant strategies if the reference portfolio is defined in real terms and the rate of inflation is a constant π.

Time-independent investment strategies were shown to be the appropriate long-run portfolio insurance strategies for a broad class of insurance payoff functions. Since all monotone increasing payoff functions may be supported by some risk-averse utility function, our results extend earlier turnpike theorems.

We have not attempted to deal with the issue of an appropriate objective function for institutionally managed portfolios. A complete theory must deal with the problems created by the agency relationship between the portfolio sponsor and the portfolio manager, as well as taking account of transaction costs and uncertainty regarding the interest rate and risk parameters.

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