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Portfolio Insurance and Financial Market Equilibrium

This article compares a capital market in which prices are set by a single expected utility maximizing investor with a market in which the expected utility maximizing investor owns only a part of the wealth, the balance being held by an investor who follows a portfolio insurance strategy. Comparative values for the market risk premium, the cost of insurance, the market volatility, and the level of interest rates are computed for different levels of portfolio insurance.

I. Introduction

The explosive growth in the popularity of portfolio insurance investment programs that has taken place over the last few years and the events of October 1987 have created concern that the simultaneous use of such strategies by a large number of market participants may have the effect of substantially increasing the volatility of stock market prices,[1] with adverse consequences for both the stability of the financial system and the cost of funds raised by private sector corporations.[2]

Portfolio insurance is most conveniently defined as an investment strategy whose object is to ensure that the value of the funds under management is a convex function of the value of

1. Industry sources suggest that between 50 and 100 billion dollars of funds are currently managed under portfolio insurance programs. This is about 2%–4% of the total equity market capitalization of approximately 2,250 billion dollars.

2. See "Is Prudence to Blame for a More Volatile Market?" *New York Times* (February 1, 1987). "The market's volatility makes portfolio insurance more attractive, and as money managers flock to that approach, their activities in the futures markets, provide more room for program trading—thus laying the groundwork for bigger swings in stock prices." The possible negative effects of portfolio insurance have received widespread attention in the aftermath of the 1987 stock market crash. As Hart and Kreps (1986) have pointed out, increased variability in speculative prices is not necessarily welfare decreasing.

(*Journal of Business*, 1989, vol. 62, no. 4)

0021-9398/89/6204-0001$01.50

some underlying insured or reference portfolio.[3] It has been shown that, under standard assumptions of stationarity, the optimal reference portfolio is the mean variance efficient portfolio which, according to the capital asset pricing model (CAPM), is the market portfolio of all risky assets;[4] and, in fact, a large proportion of existing portfolio insurance programs are based on Standard and Poor's 500 (S&P 500) or some other such proxy for the market portfolio. A portfolio strategy that is designed to give a convex payoff function will require that units of the reference portfolio be sold after its price has declined and be bought after its price has risen. It is this aspect of portfolio insurance that has met with criticism, for, it is argued, such selling on market weakness will give rise to further price declines, and purchases on market strengths will accentuate the price increases. Thus, it is argued, portfolio insurance strategies increase market volatility.[5]

Portfolio insurance and related dynamic investment strategies have become commercially feasible only as a result of the dramatic reduction in the costs of trading portfolios brought about by the development of stock index futures contracts. As Duffie and Huang (1985) emphasize, the ability to engage in continuous dynamic trading strategies may serve to complete an otherwise incomplete securities market if a Radner equilibrium of plans, prices, and price expectations is achieved. Neglecting endowment effects, this will represent a welfare improvement. However, if investor plans are not coordinated, then it is possible that the trading induced by portfolio insurance may cause liquidity problems of the type described by Grossman (1987) and Leland (1987). Even if such coordination problems do not arise, portfolio insurance will affect the properties of the financial market equilibrium such as the level and volatility of security prices. In this article we analyze the potential effects of portfolio insurance on financial markets abstracting from the possible liquidity problems that may be caused by a lack of coordination between investors or by institutional frictions.

Since dynamic investment strategies are known to be optimal for a wide class of investment strategies,[6] it seems that the case against portfolio insurance must be made on grounds other than that it gives rise to an increase in the amount of trading in response to price changes or that portfolio insurance increases volatility. One possibility is that portfolio insurance strategies are employed, not to maximize the welfare of the investor, but to protect the interests of an agent who is

3. See Leland (1980). Also see Brennan and Schwartz (1976).
4. See Brennan and Solanki (1981).
5. Grossman (1987) makes the somewhat different point that the creation of synthetic securities by dynamic investment strategies may reduce the information available to investors from market prices, thereby creating coordination problems in the implementation of the strategies.
6. See, e.g., Merton (1971).

delegated the task of managing a portfolio; such an agent may have no interest in the return on the portfolio, per se, but only insofar as it affects his wage. If agents' incentive schemes are inappropriately defined, then dynamic investment strategies may enable them to game the reward scheme and perhaps to have adverse consequences for market volatility in the process.

Thus the strongest case against the likely consequences of portfolio insurance can be made if it is assumed that the investors who follow such strategies are not expected utility maximizing individuals but automata who blindly follow the investment rule required by the strategy whatever its consequences for the distribution of their final portfolio payoffs. In this article we adopt this extreme viewpoint and compare a capital market in which prices are set by a single expected utility maximizing investor[7] with a market in which the expected utility maximizing investor owns only a part of the wealth, the balance being held by an investor who follows a simple portfolio insurance strategy.

In Section II we introduce the basic valuation framework and describe the information structure that determines the stochastic evolution of market prices. In Section III we specialize this framework to an economy with isoelastic utility; this is used in Section IV to provide quantitative estimates of the effects of portfolio insurance on financial markets. Section V concludes the article.

II. The Valuation Framework

Consider a pure exchange economy that lasts for a single period. Consumption takes place at the beginning and end of, but not during, the period. However, trading is continuous. Prices are set by a single representative risk-averse investor who takes prices as given and whose utility function may be written as

$$U(C_{0r}) + \rho\, U(W_r), \tag{1}$$

where C_{0r} denotes his consumption at the beginning of the period, and W_r is his wealth at the end of the period, wealth which is available for consumption. Market clearing implies that the initial consumption and terminal wealth of the representative investor be equal to the aggregate supplies C_0 and W.

The information structure of the economy known to all agents is represented by a geometric Brownian motion y:

$$\frac{dy}{y} = \eta dz, \tag{2}$$

7. It is well known that if markets are Pareto efficient, prices are set as though there exists a single representative investor; see, e.g., Constantinides (1982).

where dz is the increment to a Wiener process. The end-of-period aggregate wealth is determined by $y(1)$, the value of y at the end of the period, and we adopt the normalization $W = y(1)$, so that for any time t, $0 > t > 1$,

$$E[W \mid y(t)] = y(t). \tag{3}$$

Conditional on $y(t)$, the information available at time t, aggregate terminal wealth, W, is lognormally distributed with parameters $[y(t) - \frac{1}{2}\eta^2(1 - t), \eta^2(1 - t)]$ so that the uncertainty about terminal wealth is resolved at a constant rate over the period. Moreover, the information structure implies that asset prices follow continuous sample paths, so that any contingent claim on the market portfolio can be constructed by an appropriate dynamic strategy involving the market portfolio and the riskless asset.[8]

The first-order condition for the portfolio problem of the representative investor implies that $P_j(y(0),0)$, the price at the beginning of the period of asset j whose risky terminal payoff is X_j, is given by

$$P_j(y(0),0) = E[U'(W) \cdot X_j \mid y(0)]/(E[U'(W) \mid y(0)] \cdot R_F), \tag{4}$$

where

$$R_F \equiv U'(C_0)/\rho E[U'(W) \mid y(0)], \tag{5}$$

and R_F is the gross, riskless interest rate for the period.

Since the interest rate is undefined for $t > 0$, it is convenient to use the riskless asset as a numeraire and define the normalized prices:

$$p_j(y(t),t) = E[U'(W) \cdot X_j \mid y(t)]/E[U'(W) \mid y(t)] \quad \text{for } 0 \leq t \leq 1. \tag{6}$$

This completes the description of the base economy into which we now introduce portfolio insurance. The representative portfolio insurer is assumed to be a pure automaton who follows a portfolio strategy that yields an insured position in the market portfolio.[9] His terminal wealth, W_I, is given by

$$W_I = g(W), \tag{7}$$

where $g(W)$ is a convex terminal payoff function. The strategy of the portfolio insurer and his resulting payoff function are known to all market participants.[10] However, the representative investor is now representative of the expected utility maximizing investors, but not of the portfolio insurers. Thus, introduction of the representative insurer corresponds to an assumed change in the behavior of some of the investors, from expected utility maximization to automaton-like port-

8. See Huang (1985) for a formal analysis.

9. By portfolio insurer we mean the individual or institution that follows a portfolio insurance strategy. This usage, while standard in this context, contrasts with that in the traditional insurance literature.

10. The implications of relaxing this assumption are the focus of Grossman's (1987) analysis.

folio insurance. When we move from the base economy to the corresponding economy with portfolio insurance we do not change the risk aversion of the representative investor. This is tantamount to assuming that the investors who become portfolio insurers are also representative. The effect of introducing portfolio insurance estimated in this way is an upper bound on the effects likely to be observed in practice since in practice portfolio insurers are likely to be drawn from the more risk-averse participants in the market.

The terminal wealth of the representative investor is equal to the difference between aggregate wealth and the wealth of the representative portfolio insurer:

$$W_r = W - g(W). \tag{8}$$

Prices of risky claims are now determined by the optimizing decisions of the representative investor so that, corresponding to (6), the normalized price of claim j is given by

$$p_j^I(y(t),t) = E[U'(W - g(W)) \cdot X_j | y(t)]/E[U'(W - g(W))|y(t)], \tag{9}$$

where the superscript I distinguishes the prices that prevail in the market with portfolio insurance. In what follows we shall be interested in the behavior of $p_M(y(t),t)$ and $p_M^I(y(t),t)$, the values of the market portfolio in the base economy and the economy with portfolio insurance, respectively. These are obtained by substituting W for X_j in (6) and (9).

It follows from Ito's lemma and the assumed information structure (2) that the instantaneous standard deviation of return on the market portfolio is given by[11]

$$\sigma(y(t),t) = \frac{y}{p_M^I} \cdot \frac{\partial p_M^I}{\partial y} \cdot \eta. \tag{10}$$

Thus, to determine the effect of portfolio insurance on market volatility it is necessary only to specify the insurance payoff function, $g(W)$, the utility function of the representative investor, and the underlying risk of the economy that is represented by η. Then the market volatility, $\sigma(y,t)$, may be found from equations (9) and (10) and the stochastic process for y, equation (2).

III. An Economy with Isoelastic Utility

Suppose now that the utility function of the representative investor can be written as

$$U(C_{0r},W_r) = \frac{1}{1-\gamma}[C_{0r}{}^{1-\gamma} + \rho\, W_r{}^{1-\gamma}], \tag{11}$$

with $\gamma \geq 0$.

11. Returns are defined using the riskless asset as a numeraire.

Setting X_j equal to W in equation (6) and evaluating the expectations using (11) and the properties of the log-normal density, we obtain the following expression for the value of the market portfolio in the base economy for $0 \leq t \leq 1$:

$$p_M(y,t) = y \cdot e^{-\gamma\eta^2(1-t)}. \tag{12}$$

Similarly from (5) the interest rate at the beginning of the period rate is given by

$$R_F = y^\gamma \cdot e^{-1/2\gamma(\gamma+1)\eta^2} \cdot \frac{U'(C_0)}{\rho}. \tag{13}$$

Substituting for p_M and $\partial p_M/\partial y$ from (12) in (10), it is immediate that

$$\sigma(y,t) = \eta, \tag{14}$$

so that, in the absence of portfolio insurance, the market volatility is a constant equal to the underlying risk parameter η. Note that, while the degree of risk aversion affects the value of the market portfolio and the interest rate, it does not affect the volatility of the market portfolio.

IV. The Effects of Portfolio Insurance

We consider a simple portfolio insurance payoff function:

$$g(W) = \max\,[\alpha W, \beta], \tag{15}$$

where β is the minimum guaranteed return and α is the fraction of the market portfolio that is subject to portfolio insurance.

We define the units for wealth so that $\beta = \alpha$; then $W = 1$ corresponds to the level of aggregate wealth (return on the market portfolio) at which the guarantee becomes effective. Then, for an economy in which a fraction α of the market portfolio is insured, the normalized value of the insured portfolio is written $p_I^\alpha(y,t)$, the normalized value of the market portfolio is written as $p_M^\alpha(y,t)$, and the interest rate is $R_F^\alpha(y,t)$.

The guarantee that is offered by portfolio insurance cannot, even under idealized conditions, be unconditional, for the existence of limited liability implies that the payoff to portfolio insurers cannot exceed aggregate wealth W. Therefore, the payoff function after normalization must be modified to

$$g_\alpha(W) = \min\,[W, \max\,(\alpha W, \alpha)]. \tag{16}$$

The critical parameters in the analysis are the relative risk aversion of the representative investor and the underlying risk parameter η. For the latter we use 0.2 per year, which roughly corresponds to the average volatility of common stocks over the period 1926–81 as reported by Ibbotson and Sinquefield (1982).

Estimates of aggregate, relative risk aversion have been obtained both from cross-sectional data on asset holdings and from time-series data on asset returns. Using the former approach, Friend and Blume (1975) obtain an estimate of approximately 2. Estimates obtained from time-series data on asset returns include Grossman and Shiller (1981) with an estimate of about 4, Hansen and Singleton (1983) with a range of 0.07–0.62, Ferson (1982) with a range of −1.4–5.4, and Brown and Gibbons (1985) with an estimate of about 2. To provide an indication of the sensitivity of our results to the precise measure of risk aversion, we report results for risk-aversion parameters of 2 and 4.

Portfolio insurance changes the allocation of risk-bearing across market participants and, in principle, will affect the prices of all financial assets as well as the interest rate. In what follows we shall consider first the effect of the amount of portfolio insurance on the risk premium on the market portfolio and the cost of insurance, using the riskless bond as a numeraire. Since portfolio insurance is not, in general, an optimal policy for an expected utility maximizer investor (see Brennan and Solanki 1981) we shall also consider the opportunity cost of following the insurance strategy relative to the expected utility maximizing strategy. Second, we consider the effects of the amount of portfolio insurance on market volatility, as measured both by the instantaneous standard deviation of return on the market portfolio and the volatility implied by the cost of insurance using the Black-Scholes approach to valuation. Finally, we shall assess the effect of the amount of portfolio insurance on the level of interest rates in our simple economy.

The Market Risk Premium

Since y is the expected terminal value of the market portfolio and $P_M^\alpha(y,0)$ is the beginning of period value,

$$y/P_M^\alpha(y,0) = 1 + \bar{r}_M,$$

where $\bar{r}_M$ is the expected rate of return on the market portfolio.

Then, using relations (4), (5), and (6):

$$\begin{aligned} \frac{y}{p_M^\alpha(y,0)} - 1 &\equiv \frac{1}{R_F^\alpha(y,0)} \cdot \frac{y}{P_M^\alpha(y,0)} - 1 \\ &\equiv \frac{1 + \bar{r}_M}{1 + r_f} - 1 \qquad (17) \\ &\approx \bar{r}_M - r_f, \end{aligned}$$

where r_f is the 1-year riskless interest rate and $\bar{r}_m - r_f$ is the market risk premium.

In the absence of portfolio insurance, $p_M^\alpha(y,0)$ is obtained from ex-

TABLE 1 Market Risk Premium for Alernative Amounts of Portfolio Insurance

y	$\alpha = 0\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
$\gamma = 2$:					
.80	8.32	8.45	8.95	9.71	11.84
.90	8.32	8.42	8.83	9.43	11.05
1.00	8.32	8.40	8.70	9.15	10.32
1.10	8.32	8.38	8.59	8.89	9.68
1.20	8.32	8.36	8.49	8.68	9.17
$\gamma = 4$:					
.80	17.35	17.63	18.88	20.81	27.10
.90	17.35	17.59	18.64	20.23	25.00
1.00	17.35	17.54	18.39	19.65	23.30
1.10	17.35	17.50	18.13	19.08	21.76
1.20	17.35	17.45	17.90	18.56	20.40

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured. Nos. in table indicate percentages.

pression (6) with $X_j = W$. When there is portfolio insurance, the normalized value of the market portfolio, $p_M^\alpha(y,0)$ is given by expression (9) with $X_j = W$.

For the economy with isoelastic utility the market risk premium in the absence of portfolio insurance is $[\exp(\gamma\eta^2) - 1]$, as seen from equation (12). In order to estimate the effect of portfolio insurance on the market-risk premium, expression (9) with $X_j = W$ was integrated numerically using the lognormal density for W and the power utility assumption.[12]

The results are reported in table 1. In interpreting this and the subsequent tables, it is useful to remember that the guarantee becomes effective for $W \leq 1$ and that y is the expected value of W. The central row of the table ($y = 1.0$) corresponds to the situation in which the expected terminal value of the market portfolio is precisely "at the money" as illustrated in figure 1. The columns of the tables correspond to different assumptions about the fraction of the market portfolio which is subject to portfolio insurance.

When the coefficient of relative risk aversion is 2, the market-risk premium in the absence of portfolio insurance is 8.32%, which corresponds closely with the historical average risk premium. The effect of portfolio insurance is to increase the market risk premium. When y is at the money and the proportion insured is 5% the effect is to increase the

12. As the referee has pointed out, the expectations in (9) are not defined for power utility if there is positive probability that the representative investor has wealth. To resolve this technicality the payoff function (16) was modified to

$$\min\,[W - \epsilon, \max\,(\alpha W, \alpha)] \quad \text{for } \epsilon > 0.$$

The value of ϵ was set at 1% of the level of aggregate wealth at which the guarantee becomes effective. The results were insensitive to the choice of ϵ since, even for $\alpha = .20$, a return eight standard deviations below the mean is required before ϵ affects the payoff.

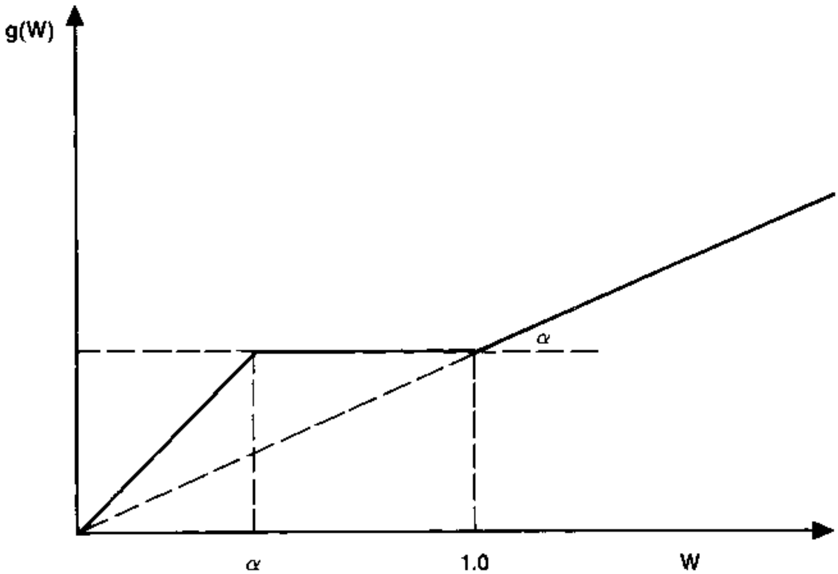

FIG. 1.—The aggregate payoff on insured portfolios when a fraction of the market is insured.

risk premium by 0.38%.[13] The effect becomes less pronounced when y is in the money, for then the dynamic strategy of portfolio insurers requires them to hold a higher proportion of equities, leaving less risk to be borne by the price-setting representative investor. The reverse is true when y is out of the money. As y decreases, the portfolio insurers invest almost entirely in bonds supplied by the representative investors so that the risk premium rises. As one would expect, the effect on the risk premium is increasing in the proportion of the market portfolio under portfolio insurance.

The lower part of the table reports analogous results when the coefficient of risk aversion is equal to 4. In this case the risk premium, even without portfolio insurance, is 17.35%, which is implausibly high. However, we include this result to emphasize the importance of the risk-aversion assumption.

The Cost of Insurance

It is to be expected that, since portfolio insurance is supplied willingly by the optimizing representative investors, its cost will increase as the number of investors demanding insurance increases. We consider two measures of cost. The first is the value of the insured portfolio payoff

13. This fraction is of the same order as the fraction of aggregate equity values subject to portfolio insurance prior to the events of October 1987.

TABLE 2 **Value of the Insurance Payoff as a Fraction of the Value of the Market Portfolio for Alternative Amounts of Portfolio Insurance: $P_I^{\alpha}(y,0)/P_M^{\alpha}(y,0)$**

y	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
$\gamma = 2$:				
.80	1.36	6.84	13.77	28.07
.90	1.23	6.15	12.36	25.05
1.00	1.13	5.67	11.38	22.95
1.10	1.07	5.37	10.75	21.61
1.20	1.04	5.19	10.39	20.82
$\gamma = 4$:				
.80	1.47	7.44	15.12	31.80
.90	1.32	6.64	13.44	27.91
1.00	1.20	6.04	12.20	25.07
1.10	1.12	5.63	11.33	23.06
1.20	1.07	5.36	10.75	21.73

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured. Nos. in table are percentages.

relative to the value of the corresponding uninsured (market) portfolio. The (normalized) value of the insured portfolio is obtained from equation (9) with $X_j = g_\alpha(W)$. The (normalized) cost of the uninsured market portfolio is also obtained from (9) with $X_j = W$. Both expressions are evaluated numerically using the log-normal density and the power-utility assumption.

Table 2 reports the value of the fraction of current aggregate wealth accounted for by the insured portfolio payoff, given the fraction of terminal wealth that is insured. Thus, in the upper half of the table, when α equals 5% and y equals 1.0, the value of the insured portfolio payoff is 5.67% of aggregate wealth, so that the cost of insuring a 5% share of terminal wealth is 0.67% of the value of current wealth, or 13.4% of the value of the insured portfolio.

The cost of insurance rises more than proportionately to the amount insured, so that, when α equals 20%, the cost of insurance is 2.95%, which is 4.40 times the cost when α equals 5%. The nonlinearity becomes more pronounced in the extreme case when the coefficient of risk aversion is 4.

A different measure of the cost of portfolio insurance is the opportunity cost to a representative investor of switching from the expected utility maximizing investment strategy to the portfolio insurance strategy. The certainty equivalent per normalized dollar of initial wealth for a representative investor who follows the optimal strategy, given that a fraction α of aggregate wealth is insured, may be written as $\omega^r(y,\alpha)$, where y is the initial value of the expectations index. This certainty

TABLE 3 **Difference between Certainty Equivalents for Optimizing Investors and for Portfolio Insurers: Cents per Dollar of Investment**

y	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
$\gamma = 2$:				
.80	3.67	4.16	4.94	7.49
.90	2.91	3.27	3.84	5.61
1.00	1.98	2.22	2.59	3.70
1.10	1.17	1.31	1.52	2.13
1.20	.61	.68	.78	1.09
$\gamma = 4$:				
.80	7.89	9.08	11.07	18.78
.90	6.72	7.68	9.24	14.69
1.00	5.10	5.81	6.96	10.75
1.10	3.42	3.90	4.65	7.08
1.20	2.05	2.33	2.77	4.17

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured.

equivalent is defined by[14]

$$\omega^r(y,\alpha) = \frac{\{E[(W - g_\alpha(W))^{(1-\gamma)}|\,y]\}^{\frac{1}{1-\gamma}}}{p_M^\alpha(y,0) - p_I^\alpha(y,0)}. \tag{18}$$

Similarly, the certainty equivalent per normalized dollar invested for a representative investor following the insurance strategy, $\omega^i(y,\alpha)$, is

$$\omega^i(y,\alpha) = \frac{\{E[g_\alpha(W)^{1-\gamma}|\,y]\}^{\frac{1}{1-\gamma}}}{p_I^\alpha(y,0)}. \tag{19}$$

Then $\delta(y,\alpha) = \omega^r(y,\alpha) - \omega^i(y,\alpha)$ is the opportunity cost to a representative investor of switching to the insurance strategy. Table 3 reports these opportunity costs.

It is seen that when $y = 1.0$ and $\alpha = 5\%$ the opportunity cost for a representative investor of switching to the insurance strategy is 2.22 cents per dollar, or 2.22%. This means that an investor with representative tastes who follows the insurance strategy is effectively throwing away 2.22% of his wealth each year. Of course, this cost is relevant only for investors with representative tastes. Nevertheless, the fact that this opportunity cost rises as the fraction of wealth insured increases suggests that the higher is the proportion of investors following insurance strategies, the greater is the disincentive for new investors to join them. The increase in the opportunity costs is the result of both an

14. This certainty equivalent is calculated by noting that the representative investor's payoff is given by expression (8), while his initial investment is given by the denominator of expression (18).

TABLE 4 **Instantaneous Market Volatility Relative to Market Volatility in the Absence of Portfolio Insurance**

y	α = 1%	α = 5%	α = 10%	α = 20%
γ = 2:				
.80	1.00	1.01	1.02	1.06
.90	1.00	1.01	1.02	1.06
1.00	1.00	1.01	1.03	1.06
1.10	1.00	1.01	1.02	1.06
1.20	1.00	1.01	1.02	1.05
γ = 4:				
.80	1.00	1.02	1.04	1.16
.90	1.00	1.02	1.04	1.13
1.00	1.00	1.02	1.05	1.13
1.10	1.00	1.02	1.05	1.13
1.20	1.00	1.02	1.05	1.12

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured.

increase in the certainty equivalent for the optimizing investor and a decrease of the certainty equivalent of the insurer.

Market Volatility

Market volatility is measured in two ways. First, the instantaneous standard deviation of the return on the market portfolio at the beginning of the period is computed using equation (10). The ratio of the market volatility to its value when there is no portfolio insurance is reported in table 4.

The effect of portfolio insurance on instantaneous market volatility increases more than proportionately with the proportion of the market subject to insurance. However, for a coefficient of risk aversion of 2 the effects are modest when $\alpha \leq 20\%$.

Our second measure of market volatility is based on the Black-Scholes approach to the valuation of contingent claims. The basic assumption of the Black-Scholes model is that the price of the underlying asset follows a diffusion process with a nonstochastic variance rate. Under the stronger assumption that the variance rate is a constant, it is straightforward to invert the Black-Scholes formula for the variance rate implied by observed prices. In the current context the underlying asset is the market portfolio, and in the presence of portfolio insurance the instaneous variance rate will depend upon the state variable y, as seen in table 4.

Nevertheless, the risk-neutral pricing principle that underlies the Black-Scholes model will continue to hold since both the asset price (the value of the market portfolio) and its instantaneous variance rate depend on the single-state variable y. Therefore, we use the risk-neutral valuation principle to infer the implied volatility corresponding to the value of the portfolio insurance contract and the value of the

TABLE 5 **Implied Market Volatility Relative to Market Volatility in the Absence of Portfolio Insurance**

y	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
$\gamma = 2$:				
.80	1.00	1.01	1.01	1.03
.90	1.00	1.01	1.01	1.03
1.00	1.00	1.00	1.01	1.03
1.10	1.00	1.01	1.01	1.03
1.20	1.00	1.01	1.01	1.03
$\gamma = 4$:				
.80	1.00	1.01	1.02	1.07
.90	1.00	1.01	1.03	1.07
1.00	1.00	1.01	1.03	1.07
1.10	1.00	1.01	1.03	1.07
1.20	1.00	1.01	1.03	1.07

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured.

market portfolio. This implied volatility provides a measure of the average volatility over the remaining life of the contract.[15]

The implied volatility is defined as the instantaneous standard deviation of return on the market portfolio that would account for the observed relation between the normalized prices of the market portfolio and the insurance contract on the assumption that the return on the market portfolio is lognormally distributed and its expected value is equal to zero.[16]

Specifically, the implied volatility $\hat{\sigma}$ is given by the solution to

$$p_I^{\alpha}(y,0) = \int g_{\alpha}(W) \cdot \frac{1}{W\hat{\sigma}\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\ln \frac{W}{p_M^{\alpha}(y,0)} + \tfrac{1}{2}\hat{\sigma}^2\right)\Big/\hat{\sigma}^2\right] dW. \tag{20}$$

In this equation, $g_{\alpha}(W)$ is the payoff on the insured portfolio given by equation (15). Values $p_I^{\alpha}(y,0)$ and $p_M^{\alpha}(y,0)$ are the normalized values of the insured portfolio and the market portfolio, respectively. These are computed as before, using equation (9).

Relative values of the implied volatility are reported in table 5. As with the instantaneous market volatility, the effect of portfolio insurance on the implied market volatility is modest. For example, if the risk-aversion coefficient is 2, the implied volatility increases by only 3% when the fraction of the market insured is 20%, and even when the risk-aversion coefficient is 4, the effect is still only 7%.

15. Although use of the implied volatility cannot be rigorously justified in this context, the implied volatility has the advantage of being widely used in practice.

16. The zero expected return arises in this context because both the price of the underlying asset and the claim price are expressed in terms of the price of end-of-period units.

TABLE 6 **Proportion of the Market Portfolio Held by Portfolio Insurers for Alternative Amounts of Portfolio Insurance**

y	α = 1%	α = 5%	α = 10%	α = 20%
γ = 2:				
.80	.08	.38	.73	1.29
.90	.20	1.00	1.94	3.62
1.00	.38	1.88	3.71	7.09
1.10	.57	2.83	5.59	10.89
1.20	.73	3.63	7.21	14.20
γ = 4:				
.80	.03	.16	.24	.39
.90	.11	.52	.96	1.52
1.00	.24	1.16	2.21	3.84
1.10	.41	2.00	3.88	7.11
1.20	.58	2.87	5.62	10.63

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α = fraction of market portfolio insured. Nos. in table are percentages.

These findings with respect to market volatility suggest that portfolio insurance is likely to have small effects on the variability of stock prices, at least in perfect markets in which the activities of portfolio insurers are fully anticipated.

Trading Volume

Portfolio insurance is implemented by a dynamic strategy of moving funds between equities and a riskless asset. The effect of this is to reallocate the burden of risk bearing between portfolio insurers and the optimizing representative investors. Thus, after a decline in stock prices, portfolio insurers will sell stock, increasing the share of risk borne by the rest of the market. Since, in less-than-perfect markets, there may be some limits to the ability of the rest of the market to absorb sudden increases in risk, it is important to gain some idea of the magnitude of the risk transfers that portfolio insurance is likely to involve in practice. This can be measured by the volume of trading induced by portfolio insurance strategies. To estimate portfolio-insurance-induced trading volume, the fraction of the market portfolio held by portfolio insurers was computed for different values of the expectational variable y. It is well known that this fraction is equal to the "hedge ratio" or partial derivative of the value of the insured portfolio with respect to the value of the market portfolio:

$$\partial p_I^\alpha / \partial p_M^\alpha = \frac{\partial p_I^\alpha(y,0)/\partial y}{\partial p_M^\alpha(y,0)/\partial y}. \tag{21}$$

The hedge ratios, which were calculated numerically using equation (9), are reported in table 6. For the range of parameters values considered, the value of the market portfolio $P_M^{\alpha}(y,0)$ is approximately proportional to the information variable y. Therefore, it is possible to gauge approximately the amount of portfolio-insurance-associated trading that will be induced by a given change in the level of stock prices. If the fraction of the market portfolio subject to portfolio insurance is 5% and the risk aversion parameter is 2, then a change in y from 1.00 to 1.10 induces portfolio insurers to purchase an additional 2.83% − 1.88% = 0.95% of the market portfolio. Thus each 1% change in stock prices induces portfolio-insurance-related trading equal in value to approximately 0.1% of the value of the market portfolio. Trading effects are roughly twice as large when the fraction of the market subject to insurance is 10%. By way of comparison, average daily turnover in the New York Stock Exchange in 1986 was 0.3% and an additional 0.5% was traded in the S&P 500 futures contracts. On October 19, 1987, when the market dropped by more than 20%, the combined trading in these two markets was of the order of 2% of the value of the underlying stocks.

It seems, therefore, that under current institutional arrangements even modest levels of portfolio insurance may impose major strains on the liquidity of the markets in the event of large changes in expectations.

Interest Rates

Thus far we have not considered the possible effects of portfolio insurance on the level of interest rates. In the absence of portfolio insurance, the gross interest rate is given by equation (5), which, under the joint assumptions of lognormality and power utility, implies that

$$R_F = \frac{C_0^{-\gamma}}{\rho} y^{\gamma} \exp\left[-\tfrac{1}{2}\gamma\eta^2(1+\gamma)\right]. \tag{22}$$

In order to estimate the effect of portfolio insurance on the interest rate we assume that portfolio insurers represent a fraction f of all investors, that all investors have the same initial wealth, and that initial consumption is the same for portfolio insurers and the other (representative) investors.

The fraction of investors who are designated as portfolio insurers is $P_I^{\alpha}(y,0)/P_M^{\alpha}(y,0)$. This is the fraction of the original identical investors who have exchanged their claims to the market portfolio for the insured claim. Then, the initial consumption of the (price-setting) representative investor is $(1-f)\,C_0$ so that the gross interest rate is

$$R_F^{\alpha}(y,0) = \frac{(1-f)^{-\gamma}\,(C_0)^{-\gamma}}{\rho E[[W - g_{\alpha}(W)]^{-\gamma}|y]}. \tag{23}$$

The results reported in table 7 assume that the expected growth rate in aggregate consumption is a constant m so that $C_0 = y/m$. In particu-

TABLE 7 Riskless Interest Rate for Alternative Amounts of Portfolio Insurance

y	$\alpha = 0\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 20\%$
$\gamma = 2$:					
.80	2.92	2.82	2.36	1.74	.31
.90	2.92	2.84	2.47	1.96	.79
1.00	2.92	2.86	2.58	2.20	1.29
1.10	2.92	2.88	2.69	2.42	1.78
1.20	2.92	2.90	2.78	2.61	2.20
$\gamma = 4$:					
.80	−14.23	−14.43	−15.21	−16.05	−16.18
.90	−14.23	−14.40	−15.07	−15.83	−16.52
1.00	−14.23	−14.37	−14.92	−15.58	−16.48
1.10	−14.23	−14.34	−14.76	−15.29	−16.20
1.20	−14.23	−14.31	−14.61	−15.00	−15.77

NOTE.—γ = coefficient of relative risk aversion; y = expected terminal value of market portfolio; α fraction of market portfolio insured. Nos. in table are percentages.

lar, we assume that $m = 1.05$ and that the impatience parameter ρ is equal to 0.95.

As seen in the table, the effect of portfolio insurance is to reduce the net interest rate, and the effect is more pronounced the lower the expectational variable y. Thus when y is at the money the effect of 5% portfolio insurance is to reduce the real interest rate from 2.92% to 2.58%, when γ equals 2. When γ equals 4 the effect is roughly twice as large. The reason for the reduction is that the portfolio insurers increase demand for the riskless asset while reducing their demand for shares in the market portfolio.

Time Horizon

We have arbitrarily set the length of the single period at 1 year. The effect on the market risk premium and volatility of changing the length of the period is identical to that of changing the exogenous variance η^2. A doubling of the variance corresponds to a doubling of the time horizon. Simulations in which the variance rate was increased by a factor of 4 had no significant effect on the market-risk premium and volatility reported above when interpreted on a yearly basis.

V. Conclusion

In this article we have presented estimates of the effects of portfolio insurance in a frictionless economy characterized by a single representative agent with power utility and rational expectations. Portfolio insurance was introduced by assuming that a fraction of agents were able to purchase claims on the end-of-period market portfolio with characteristics similar to those promised by portfolio insurance strategies.

These claims and the market portfolio itself were assumed to be priced by the remaining expected utility maximizing representative investors.

In this context, the effect of portfolio insurance on market volatility was found to be slight for reasonable parameter values. Moreover, the more widely followed the portfolio insurance strategy is, the more costly it becomes, both when the cost is measured by the value of the implicit put option and when it is measured by the difference in certainty equivalents achieved by optimizing portfolio insurance investors with the same utility function. The increasing cost of portfolio insurance suggests that the popularity of such strategies will be self-limiting. On the other hand, even modest levels of portfolio insurance potentially involve trading volume that is large relative to current turnover rates. This suggests that there may exist additional liquidity-related costs to following portfolio insurance strategies under current institutional arrangements.

An important assumption of the analysis is that a Radner equilibrium of plans, prices, and price expectations is achieved, that is to say, that individual investors are able to take into account the strategies of other investors in formulating their own investment strategies. In this context the assumption implies that optimizing investors are aware of the extent of portfolio insurance strategies. This seems to be a reasonable assumption for analyzing the effects of portfolio insurance in the long run. However, to the extent that the assumption is violated, the effect of portfolio insurance on market volatility may be greater than our calculations suggest because of liquidity problems of the type discussed by Grossman (1987) or possible misinterpretation of the information content of portfolio-insurance-induced transactions.

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