The Valuation of Commodity-Contingent Claims

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This article describes a new approach to the valuation of commodity-contingent claims. The approach uses all the information contained in the term structure of commodity futures prices in addition to the historical volatilities of futures returns for different maturities. It is based on the principle that no arbitrage opportunities should exist when trading in futures contracts.

The framework is applied to price copper-contingent claims. We analyze the daily returns for all copper futures contracts traded at the Commodity Exchange of New York between 1978 and 1990. By applying principal components analysis to the data, we conclude that a three-factor model describes the stochastic movement of copper futures prices.

Finally, as an illustration of the approach, we use the factor loadings obtained in the principal components analysis to price the publicly traded copper interest-indexed notes issued by Magma Copper Company in 1988.

With the proliferation of financial instruments linked to the price of commodities, such as futures, options on futures, and commodity-linked bonds, the valuation of commodity-contingent claims is becoming an increasingly important problem in financial economics. It is particularly suited for the evaluation of natural resource investments.

The first commodity-contingent claims models, following the tradition of Black and Scholes [1973] for equity-contingent claims, assumed that all the uncertainty could be summarized in one factor, the spot price of the commodity. Models of this type include Schwartz [1982] and Gibson and Schwartz [1991] for pricing commodity-linked securities, and Brennan and Schwartz [1985], Paddock, Siegel, and Smith [1988], and Cortazar and Schwartz [1993] for valuing real assets.

Looking at the variability of the cost of carry for most commodities, it soon became apparent that a second stochastic variable is needed to value commodity-contingent claims properly. In their two-factor model, Gibson and Schwartz [1990] assume that the spot price of the commodity and the convenience yield, defined as the difference between the interest rate and the cost of carry, follows a joint stochastic process.

In this article we take a different approach to the valuation of commodity-contingent claims. Our starting point is the whole term structure of existing commodity futures prices, and we assume that movements in this term structure are determined by k factors. These movements are such that they preclude arbitrage opportunities when trading among futures contracts. We let the data tell us the number of relevant factors k. Our approach is similar to the arbitrage-free term structure of interest rate movement models developed by Ho and Lee [1986] and Heath, Jarrow, and Morton [1990, 1991, 1992], and applied by Reisman [1991] to commodities.

Our main contribution, however, is empirical. Very little work has been done in trying to implement this approach in the pricing of interest rate-contingent claims and none for commodities. Our aim here is to study the stochastic movement of futures prices of one particular commodity, copper, and to show how to implement the approach by pricing a bond whose coupon payments are linked to the price of copper.

We analyze the daily prices for all copper futures traded between 1978 and 1990 at the Commodity Exchange of New York (Comex). By applying principal components analysis to daily copper futures returns, we obtain a three-factor model that describes the stochastic movement of futures prices.

We find that the most important factor in explaining return variance represents shocks that have a constant impact on futures returns of all maturities. The other two factors impact differently on futures returns with short-term, medium-term, and long-term maturities, allowing for term structures of virtually any shape. Our results for the process of futures returns are found to be similar to those reported in the literature for bond returns.

Finally, as an illustration of the approach, we price the publicly traded copper interest-indexed notes issued by Magma Copper Company in 1988 using simulation techniques. Although we concentrate on copper-contingent claims, a similar approach can be used to value claims linked to other commodities such as gold, silver, or oil.

I. THE MODEL

The model is similar to the one proposed by Reisman [1991]. Let S(t) be the spot price of a commodity at time t, and F(t, T) the price of a futures contract at time t, written on the same commodity, for delivery at time T. At the maturity of the futures contract, its price is equal to the spot price, i.e., S(T) = F(T, T). We also assume that the futures contracts are traded in a frictionless continuous market, and that no arbitrage opportunities are available.[1]

As shown by Harrison and Kreps [1979] and Harrison and Pliska [1981], under very general conditions the absence of arbitrage opportunities in the economy implies the existence of a probability measure under which asset prices follow a martingale. This probability measure has also been called the risk-neutral probability measure because it would prevail in an economy populated by risk-neutral agents. Additional names given to it are equivalent-martingale measure and no-arbitrage-martingale measure.

The stochastic process followed by asset prices under the equivalent-martingale measure is called the risk-adjusted or risk-neutral stochastic process. A direct implication of the martingale property is that under this measure the instantaneous expected return on all financial assets is equal to the instantaneous riskless rate of interest, and the instantaneous return on all futures contracts (which require no investment) is equal to zero (see Cox, Ingersoll, and Ross [1981]).[2]

The starting point of our analysis is the risk-adjusted process for commodity futures prices, which, as mentioned above, has zero drift:

$$\frac{dF(t,\ T)}{F(t,\ T)} = \sum_{k=1}^{K} b_k(t,\ T)\, dW_k \qquad (1)$$

or equivalently, in stochastic integral form:

$$F(t,\ T) = F(0,\ T) + \int_0^t \sum_{k=1}^{K} b_k(s,\ T)\, F(s,\ T)\, dW_k(s) \qquad (2)$$

where W_1, W_2, ..., W_K are K independent Brownian motions under the equivalent-martingale measure, and $b_k(t, T)$ are volatility functions of futures prices.

From this specification of the futures price process we can obtain the stochastic process for the spot price. Applying Itô's Lemma, Equation (2) can be rewritten as:

$$F(t, T) = F(0, T)\exp\left[-\frac{1}{2}\int_0^t \sum_{k=1}^{K} b_k^2(s, T)\,ds + \int_0^t \sum_{k=1}^{K} b_k(s, T)\,dW_k(s)\right] \tag{3}$$

By setting T = t we determine the process for the spot price S(t):

$$S(t) = F(0, t)\exp\left[-\frac{1}{2}\int_0^t \sum_{k=1}^{K} b_k^2(s, t)\,ds + \int_0^t \sum_{k=1}^{K} b_k(s, t)\,dW_k(s)\right] \tag{4}$$

Alternatively, we could start with the risk-adjusted stochastic process for the spot price S(t) written as:

$$\frac{dS(t)}{S(t)} = y(t)\,dt + \sum_{k=1}^{K} c_k(t)\,dW_k \tag{5}$$

or equivalently, in stochastic integral form, as:

$$S(t) = S(0) + \int_0^t y(s)S(s)\,ds + \int_0^t \sum_{k=1}^{K} c_k(s)S(s)\,dW_k(s) \tag{6}$$

where y(t) represents the instantaneous cost of carry for investing in the commodity and $c_k(t)$ its volatility parameters.

A simple interpretation of y(t) is that of the instantaneously riskless return obtained by buying one unit of the commodity spot and selling one futures contract maturing in the next instant of time. It is akin to defining the riskless interest rate as the return on a discount bond that matures in the next instant of time.

It is now standard in the commodity pricing literature to define the net convenience yield of a commodity, $\delta(t)$, as the flow of services that accrues to the holder of the physical commodity, but not to the owner of a contract for future delivery (Brennan [1991]). The no-arbitrage condition induces the instantaneous cost of carry y(t) to be equal to the riskless interest rate r(t), minus the net convenience yield, $\delta(t)$. In our model this instantaneous cost of carry is stochastic, reaching negative values when the net convenience yield is greater than the riskless interest rate.

To derive the process followed by y(t) we apply Itô's Lemma to Equation (4) and compare the corresponding drift and stochastic terms with those from Equation (5), obtaining:

$$c_k(t) = b_k(t, t) \tag{7}$$

$$y(t) = \frac{\partial \ln F(0, t)}{\partial t} - \int_0^t \sum_{k=1}^{K} b_k(s, t)\left(\frac{\partial b_k(s, t)}{\partial t}\right)ds + \int_0^t \sum_{k=1}^{K} \left(\frac{\partial b_k(s, t)}{\partial t}\right)dW_k(s) \tag{8}$$

Therefore, the specification of the process for the futures prices completely determines the process for the spot prices. The problem could also be formulated in terms of the forward cost of carry instead of futures prices.

Recall that in the interest-contingent claim literature both model specifications have been pursued. Ho and Lee [1986] initially developed the no-arbitrage approach to the movement of the term structure of interest rates by considering the process for the bond prices as the primitive, while Heath, Jarrow, and Morton [1992] started from the process for forward rates. In the case of commodities, specifying both the risk-adjusted process for the forward cost of carry and for the spot price is equivalent to specifying the risk-adjusted process for futures prices.

Let y(t, T) be the instantaneous forward cost of carry of investing in the commodity at time T, as perceived at time t. As in the definition of y(t), y(t, T) is the instantaneously riskless forward return obtained by buying one futures contract with maturity T and selling one with maturity T + dT. Then,

$$F(t, T) = S(t)\exp\left(\int_t^T y(t, s)\,ds\right) \tag{9}$$

We can write the stochastic process for y(t, T) as:

$$dy(t, T) = A(t, T)\,dt + \sum_{k=1}^{K} B_k(t, T)\,dW_k(s) \qquad (10)$$

or equivalently,

$$y(t, T) = y(0, T) + \int_0^t A(s, T)\,ds + \int_0^t \sum_{k=1}^{K} B_k(s, T)\,dW_k(s) \qquad (11)$$

Applying Itô's Lemma to (9), it can be shown that the relation between the parameters of Equations (10) and those in (1) and (5) are given by:

$$A(t, T) = -\sum_{k=1}^{K} [B_k(t, T)]\left(c_k(t) + \int_t^T B_k(t, s)\,ds\right) \qquad (12)$$

$$b_k(t, T) = c_k(t) + \int_t^T B_k(t, s)\,ds \qquad (13)$$

$$B_k(t, T) = \frac{\partial b_k(t, T)}{\partial T} \qquad (14)$$

Thus, given the process for the spot price and for the forward cost of carry, the process for the futures prices can be determined.

II. PRICING AND HEDGING A COMMODITY-CONTINGENT CLAIM

This model for arbitrage-free movements of commodity futures prices does not describe the actual movement of prices, but the one that would prevail in a risk-neutral world. Using risk-neutral pricing procedures (Cox and Ross [1976], Harrison and Pliska [1981], and Heath, Jarrow, and Morton [1992]), we can price contingent claims by computing the expectation of the discounted payoffs under these modified probabilities.

The relevant discount factor for pricing purposes is the risk-free rate, r. More formally, let V_t be the value at time t of a commodity-contingent claim with payout X at time T. Then:

$$V_t = \hat{E}\left[X \exp\left(-\int_t^T r(s)ds\right)\right] \qquad (15)$$

where $\hat{E}$ represents the expectations operator under the risk-neutral probabilities.

Assuming the payout X is uncorrelated with the risk-free interest rate, Equation (15) becomes:[3]

$$V_t = \hat{E}[X]\,\hat{E}\left[\exp\left(-\int_t^T r(s)ds\right)\right] \qquad (16)$$

Therefore, we can price a commodity-contingent claim as follows. First, estimate the model for the movement of futures prices under the equivalent-martingale measure. Using these risk-neutral probabilities, simulate the stochastic paths followed by the futures prices, and determine all payoffs contingent on these futures prices. Given that we have used the equivalent-martingale measure in generating risk-adjusted futures returns, we discount these payoffs at the risk-free interest rate, obtaining the estimated value of the contingent claim.[4]

To hedge or replicate the payoffs of a commodity-contingent claim, we need the same number of futures contracts as the number of factors that explain the stochastic movement of the term structure of commodity futures prices, assuming that interest rates are not stochastic. The sensitivity of the return of the hedging portfolio of commodity futures contracts with respect to each of the factors should be the same as the sensitivity of the return of the contingent claim with respect to the same factors.

The sensitivity of the return of each futures contract with respect to the factors is given directly by the volatilities functions in Equation (1). The sensitivity of the return on the contingent claim with respect to the factors can be obtained numerically by perturbing each of the factors separately, and computing the change in price of the commodity-contingent claim divided by the perturbation. In practice, the futures contracts chosen for the hedging portfolio should have an exposure to the different factors that is as different as possible.

III. THE STOCHASTIC MOVEMENT OF COPPER FUTURES PRICES

The model we have developed can be applied to commodities such as oil, silver, or gold. We use it to estimate the stochastic movement of copper futures prices. We now describe the estimation methodology, the data used, and the results.

Methodology

The intuition behind the model is that the no-arbitrage condition imposes a restriction on the risk-adjusted drift of the whole term structure of the futures market. For the forward cost of carry process the risk-adjusted drift must be a function of the volatilities [see Equation (12)], and for the futures price process the risk-adjusted drift must be zero [see Equation (1)].

To estimate the model, we can use as the primitive stochastic process the one followed by either the futures prices, F(t, T), or the forward cost of carry, y(t, T). We have shown how to derive one from the other. For estimation purposes, however, it is more convenient to use the process for the futures prices, described in Equation (1), instead of the process for the forward cost of carry, described in Equation (11). (In our illustration of the use of the methodology to value a specific contingent claim, however, we use the forward cost of carry.)

Equation (1) describes the stochastic process for futures prices under an equivalent-martingale measure and not under the true probability measure. Although actual futures prices may exhibit a non-zero drift, volatility coefficients are not affected by the risk-neutrality assumption, and can therefore be estimated using historical data.[5]

To estimate Equation (1) we must determine the number of independent Brownian motions (K) and the associated volatility coefficients, $b_k(t, T)$. Assuming that volatility is only a function of time to maturity (T − t), we can use principal components analysis (PCA) to estimate jointly the number of orthogonal factors and the corresponding volatility coefficients.[6]

More concretely, given F(t, T), the price at time t of a futures contract for delivery of copper at time T, we can approximate the left-hand side of Equation (1) by:

$$\frac{dF(t,\ T)}{F(t,\ T)} \approx \frac{F(t,\ T) - F(t-1,\ T)}{F(t-1,\ T)} \tag{17}$$

Following this procedure we can fill in a matrix of N rows (each one representing a different trading date t = 1, 2, ..., N) and M columns (each one representing different times to maturity, $\tau = T - t$, with $\tau = 1, 2, ..., M$). Column τ of this matrix can be interpreted as the set of observations on the return on holding a futures with maturity τ. Equation (1) models these returns as a linear function of K independent factors.

In general we would require K = M factors to explain all sample variance. Principal components analysis, however, finds K < M independent factors that explain a high proportion of the total variance. Thus, each of the original M factors can be approximated by a linear combination of the computed principal components, in which the coefficients, properly scaled, correspond to the volatilities $b_k(t, T)$ in Equation (1).

Data Description

CHARACTERISTICS OF THE FUTURES MARKET FOR COPPER. The two most active copper futures markets are the Commodity Exchange of New York (Comex) and the London Metal Exchange (LME). These markets have different characteristics in terms of contract specifications and trading rules (Anthes [1984], Rivalland [1985], and *Statistical Yearbook* [1988]). We focus on the Comex copper futures market.

Two different copper futures contracts have been traded at the Comex. Until July 1988, the copper futures contract consisted of a commitment by the short position to deliver 25,000 pounds of Grade 2 electrolytic cathode copper at any Comex-approved warehouse during any day of the month when the future matures.[7] At the seller's option, other copper grades can be delivered at various prespecified discounts or premiums. Most futures contracts, however, are not settled by delivery of the underlying commodity, but rather by an offsetting futures contract.[8]

Trading of this copper futures contract was typically conducted in futures with maturities of the current month, the next two calendar months, and any January, March, May, July, September, and December falling within a twenty-three-month period beginning with the current calendar month. This contract traded at the Comex until January 1990, under the trading symbol CU.

A second copper futures contract began trading at the Comex during 1988, under the trading symbol HG. The main difference between this contract and the CU futures is the higher grade of the copper required for delivery, now defined to be Grade 1 electrolytic copper. Also, futures of additional maturities are traded, including the current and immediately following eleven calendar months, and every January, March, May, July, September, and December falling within a twenty-three-month period. During 1988, the combined trading volume for these two futures amounted to over two million contracts, or around $50 billion.

PRICES ON COPPER FUTURES. We obtained from the Center for Futures Research at Columbia University more than 70,000 daily prices corresponding to all copper futures traded at the Comex between January 3, 1966, and January 15, 1991. The data are organized in 287 files, 246 corresponding to prices of the CU futures and 41 corresponding to prices of the HG futures. Each file includes the prices of a copper futures contract with a specific maturity date. The 287 files correspond to futures that matured between January 1966 and January 1990 for the CU futures, and between January 1989 and September 1992 for the HG futures.

EXHIBIT 1
EXPLANATORY POWER OF PCA (THREE FACTORS)

Dates	Observations	Covariance	Correlation
Jan. 78-Dec. 90	3272	0.9875	0.9879
Jan. 78-Jun. 84	1636	0.9928	0.9930
Jul. 84-Dec. 90	1636	0.9909	0.9904
Jan. 78-Mar. 81	818	0.9963	0.9961
Apr. 81-Jun. 84	818	0.9974	0.9974
Jul. 84-Sep. 87	818	0.9984	0.9984
Oct. 87-Dec. 90	818	0.9888	0.9881

Notes: Fraction of total variance explained by the first three components of a principal components analysis on the daily returns on copper futures. Each PCA is performed on the covariance and the correlation matrixes, using data from different trading dates.

Thus, during the thirteen months between January 1989 and January 1990, the two different copper futures contracts matured. Also, each specific futures contract was traded at an average of almost 300 different dates.

DATA PREPARATION. First, we construct two matrixes of daily returns, one for each type of futures contract: CU or HG. Each element of the matrix is computed using Equation (17), with M equal to twenty-one months. The time to maturity is calculated assuming that futures expire on the twenty-eighth of the delivery month. Although we have some prices on futures that mature twenty-two and twenty-three months ahead, this information is discarded because of the thinness of this market.

Given that for any date futures with maturities on only some of the next twenty-one months will be traded, we aggregate the daily returns corresponding to several maturities into seven quarterly periods. For each trading date, we compute the average returns on copper futures (both CU and HG) corresponding to maturities that fall into a single period.[9] By aggregating returns into seven quarterly periods, we practically eliminate the missing observation problem.

We end up with a matrix of daily returns. For estimation purposes we use daily returns on futures traded only between January 1978 and December 1990, or thirteen years of daily returns.[10]

Empirical Results

Exhibit 1 shows the explanatory power of PCA for different time periods. The results are fairly similar for all the trading dates and for both the covariance and the correlation matrix.[11] With three factors, the model explains between 98.7% and 99.8% of total return variance.

Exhibit 2 shows the explanatory power of the one-factor, two-factor, and three-factor models across all maturities. For purposes of illustration, we use PCA on the covariance matrix of all daily returns from 1978 to 1990.

The principal components extracted from the data explain a large proportion of the return variance across all maturities. The first factor accounts for around 93% of the total variance, the second factor for around 4%, and the third factor for around 1%, for most maturities.[12]

To analyze the factor loadings obtained using PCA, we compute the loadings on each of the three main orthogonal factors that explain return volatility on futures of different maturities. Exhibit 3 plots these loadings.

The actual magnitude of each particular factor loading is not important because they have been normalized so that the total variance of each of the returns is equal to one. To obtain the actual percentage impact on the futures daily return, adequate scaling is required.

Examination of Exhibit 3 shows that the shocks represented by Factor 1 explain a great fraction of the futures return volatility. Notice in particular the high loading of Factor 1 relative to Factors 2 and 3. This is consistent with Exhibit 2. Also, it can be seen that the first-factor loading is fairly constant across all maturities, which implies that shocks have the same impact on futures returns of all maturities.[13]

Factor 2 is the next most important factor in explaining return volatility. It represents shocks that have opposite effects on the return of short- and long-term futures.[14] Finally, Factor 3 represents shocks that have the same effect on the return of both short and long maturities, but this effect is opposite to the effect on the returns of medium-term maturity futures.

The pattern of the factor loadings plotted in Exhibit 3 resembles those found by Litterman and Scheinkman [1991] and Ilmanen [1992] in factor analysis of bond returns. Litterman and Scheinkman [1991] call our first factor the *level* factor, the second the *steepness* factor, and the third the *curvature* factor, depending on the effect of shocks of each of these factors on the returns of instruments of different maturities. In an eight-period aggregation, representing maturities from six months to eighteen years, Litterman and Scheinkman find that the combined explanatory power of the first two factors is less than 91%.

Overall we find that the volatilities of copper futures and of bond returns behave rather similarly in terms of the number and the characteristics of the factors that explain them.

EXHIBIT 2
EXPLANATORY POWER OF PCA (ONE, TWO, AND THREE FACTORS) ACROSS MATURITIES

τ	1 Factor	1, 2 Factors	1, 2, 3 Factors
1	0.8963	0.9753	0.9872
2	0.9197	0.9735	0.9735
3	0.9292	0.9310	0.9973
4	0.9732	0.9847	0.9847
5	0.9628	0.9928	0.9938
6	0.9321	0.9830	0.9872
7	0.9301	0.9881	0.9911

Note: Proportion of the variance in the returns of futures of different maturities explained by the one-factor, the two-factor, and the three-factor PCA models, on the covariance matrix generated by all daily returns on copper futures corresponding to 1978-1990.

IV. PRICING A COMMODITY-CONTINGENT CLAIM

There are many bonds whose final payment or coupons are linked to the price of one, and sometimes more, commodities (see, for example, Smithson and Chew [1992]). And there are many options on commodity futures currently traded in the U.S. market. It will be useful therefore to apply our results on the

EXHIBIT 3
PRINCIPAL COMPONENTS ANALYSIS FACTOR LOADINGS

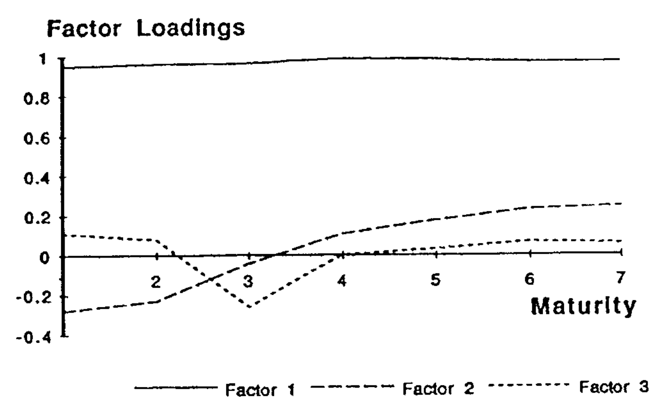


EXHIBIT 4
INDEXED INTEREST RATE ON MAGMA NOTES

Copper Price (U.S.$ per pound)	Indexed Annual Interest Rate (annual percent)
2.00 or above	21.0
1.80	20.0
1.60	19.0
1.40	18.0
1.30	17.0
1.20	16.0
1.10	15.0
1.00	14.0
0.90	13.0
0.80 or below	12.0

Note: Annual interest rate corresponding to the quarterly coupon of Magma's copper interest-indexed notes, contingent on the price of the closest-to-maturity copper futures.

stochastic movement of copper futures prices to price an actual copper-contingent security.

We choose a rather innovative hybrid debt security issued by Magma Copper Company in 1988 (see Smithson and Chew [1992], *Moody's Bond Survey*, and *Moody's Annual Bond Record*). What is interesting about this ten-year debenture is that it has embedded within it forty option positions on the price of copper futures, each one determining the coupon payment according to the maturity of an option.

Description of Magma's Copper Notes Offering

Magma Copper Company is one of the four largest U.S. producers of refined copper, with mines located in Arizona. Until 1987, Magma was a wholly owned subsidiary of Newmont Mining Corporation. In 1988 Magma's debt was rated for the first time. Magma's Copper Notes received a B1 Moody's rating.

The copper notes, with face value of $200 million, were issued on November 23, 1988, and are due November 15, 1998. Quarterly coupons are contingent on the prevailing copper price, according to Exhibit 4. For copper prices in between those reported in the table, the indexed interest rate is determined by linear interpolation. Thus, the higher the price of copper, the higher the interest payment received by the holder of the security.

Interest payments are due February 15, May 15, August 15, and November 15 of each year, with the principal due November 15, 1998. The copper notes have other features that we disregard in our illustration of the methodology.

Pricing Methodology

To price the simplified version of Magma's copper notes, we start by simulating the stochastic process of futures prices. For each of the remaining coupon payment dates until maturity, we generate random shocks on the futures prices, according to the factor structure estimated for copper as above. Using these futures prices, we determine the coupon payments, according to Exhibit 4. Once each coupon is determined, we discount it at an appropriate rate.

We value the security on November 28, 1991, just after the November coupon has been paid. The risk-free term structure of interest rates on that date was obtained from the prices of U.S. Treasury Strips. The relatively low Moody's credit rating implies a substantial default risk embedded in the security. Since our model does not consider default risk, we must add to the risk-free rate an appropriate risk premium. We estimate this risk premium by looking at the default premium of similar securities at the estimation time and later price Magma's copper notes using a range of possible risk premiums.

Our model for the stochastic process of futures prices assumes that the current futures prices for all maturities relevant for computing payoffs are known. Thus, we need the current price of futures that mature during each of the twenty-eight quarters between November 1991 and November 1998. This information is not available because the copper futures that are publicly traded have a much shorter maturity.

To estimate the initial futures prices for all twenty-eight quarters we extrapolate existing prices as follows. For the first six quarters, we use actual prices of traded futures that mature in December 1991, March, June, September, and December of 1992, and March 1993. Given that during these first six quarters we have actual market prices for both the bonds and the futures, we are able to compute the implicit quarterly convenience yield during this period, finding that it ranges from 2% to 2.9%.

Using the actual term structure of interest rates and an estimated constant convenience yield for the seventh to twenty-eighth quarter, we compute our estimate for the initial futures prices. Later we price Magma's copper notes using a range of initial convenience yields. Details of the procedure are described in the appendix.

Exhibit 5 shows the result of this extrapolation process. It plots the risk-free discount function, the initial six futures prices, and the final twenty-two extrapolated futures prices, assuming a constant quarterly convenience yield of 2.5% from Period 7 to Period 28.

Finally, for simulating the twenty-eight remaining coupons of the security, we need volatility estimates for each of the three factors, corresponding to the twenty-eight maturities. Through principal components analysis on futures prices we estimate the factor loadings for the first seven maturities. These loadings are scaled so that the resulting return volatility matches the historical volatility, for each maturity. For the remaining twenty-one maturities we assume the loadings for each of the factors are the same as those estimated for Period 7.

Exhibit 6 plots the estimated (Periods 1 to 7) and extrapolated (Periods 8 to 28) factor loadings, properly scaled.

The simulation program to price Magma's copper interest-indexed notes starts by filling in futures prices for Quarters 7 to 28, according to the extrapolation procedure described above. Then, it computes risky discount factors for each of the twenty-eight quarters by adding the default risk premium to the riskless rate implicit in the inputted riskless discount bond prices. The program then generates a random path for the futures prices, according to the three-factor model loadings.

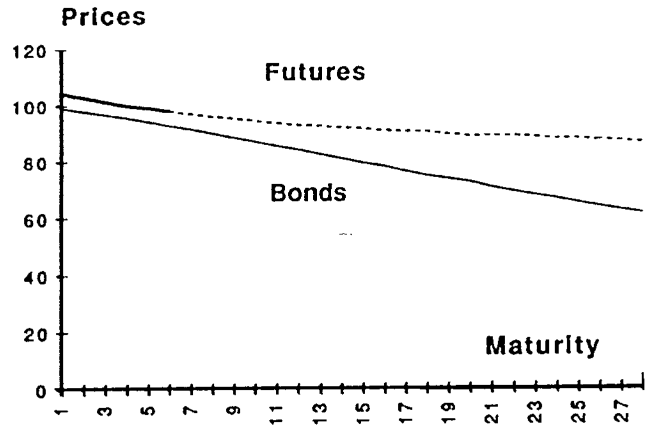
For each coupon payment date, it determines the price of the closest-to-maturity futures and computes the coupon amount using Exhibit 4. This coupon payment is then discounted at the risky discount factor. In Period 28, the principal is added to the corresponding coupon payment and discounted at the risky discount factor.

The sum obtained from the addition of all discounted payments represents one simulated value of Magma's copper interest-indexed notes. This step is repeated until 10,000 simulated prices of the security are obtained. Finally, the average price across all simulations, and the standard deviation of this estimate, is computed.

Pricing Results

Exhibit 7 plots the value of Magma's copper interest-indexed notes given a quarterly convenience

EXHIBIT 5
U.S. TREASURY STRIPS AND FUTURES PRICES



Prices for all U.S. Treasury Strips and for the first six futures are market values as of November 28, 1991. Prices for futures with maturity seven to twenty-eight periods ahead are extrapolations as shown in the appendix.

EXHIBIT 6
FACTOR VOLATILITIES ESTIMATED AND EXTRAPOLATED

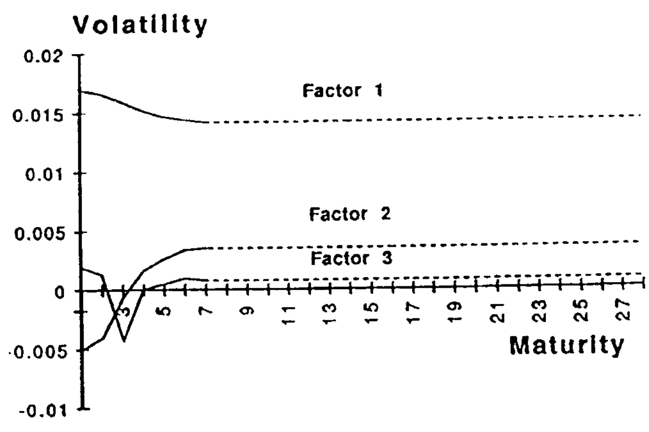
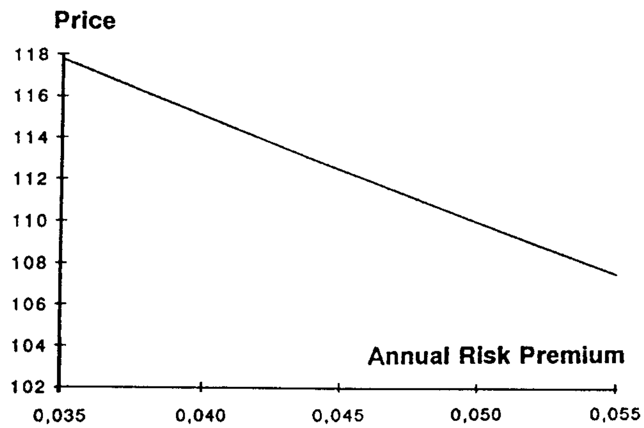


EXHIBIT 7
PRICE OF MAGMA'S COPPER INTEREST-INDEXED NOTES VERSUS DEFAULT RISK PREMIUM

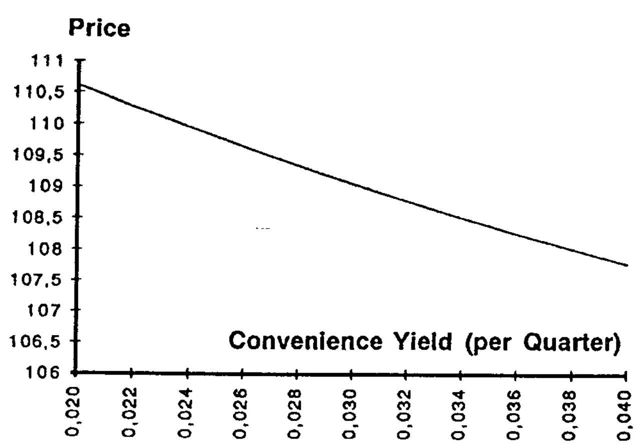


Constructed by running 10,000 simulations and assuming a quarterly convenience yield of 2.5%.

yield of 2.5%, and annual risk premiums that vary from 3.5% to 5.5%. For each set of parameters we ran 10,000 simulations. Notice that as the default risk premium increases, the value of the security decreases.[15]

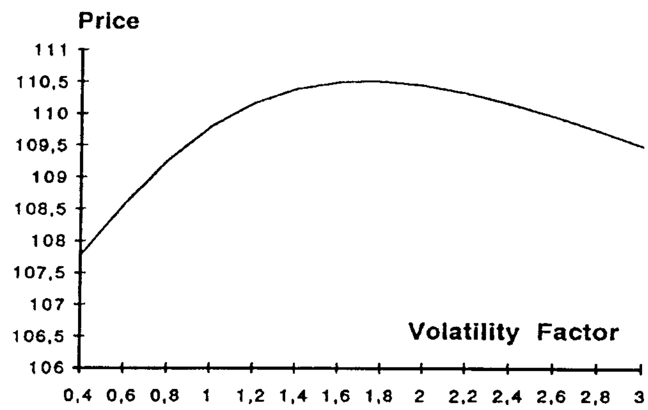
Exhibit 8 shows the value of Magma's copper

EXHIBIT 8
PRICE OF MAGMA'S COPPER INTEREST-INDEXED NOTES VERSUS CONVENIENCE YIELDS



Constructed by running 10,000 simulations and assuming an annual default risk premium of 5%.

EXHIBIT 9
PRICE OF MAGMA'S COPPER INTEREST-INDEXED NOTES VERSUS VOLATILITY



Constructed by running 10,000 simulations, assuming an annual default risk premium of 5%, a quarterly convenience yield of 2.5%, and volatility shocks that range from 0.4 to 3 times historical averages.

interest-indexed notes considering an annual default risk premium of 5% and quarterly convenience yield of 2% to 4%. As expected, as the convenience yield increases, the value of the security decreases.

Finally, Exhibit 9 analyzes the effect of the volatility of futures prices on the value of Magma's copper interest-indexed notes. To perform this experiment we multiply all three factors by the same volatility coefficient. We plot the price estimate assuming an annual default risk premium of 5% and a quarterly convenience yield of 2.5%, for volatility factors that range from 0.4 to 3.[16]

We find no monotonic relationship between volatility and security price. This is not surprising, given the shape of the payoff function of each coupon: it is convex for low copper prices, but concave for high copper prices. These payoffs can be mimicked by a portfolio of short and long positions on bonds, calls, and puts, with volatility increasing the value of some of these positions, while decreasing the value of others and rendering an ambiguous net effect.

V. CONCLUSIONS

Our approach to the valuation of commodity-contingent claims uses all the information embodied in the current term structure of futures prices in addition to the volatilities of historical returns on futures contracts of different maturities. Our illustration shows its application to value copper-contingent claims.

We analyze a data set composed of thirteen years of daily copper futures traded at the Comex. Using principal components analysis, we determined that three orthogonal factors are able to explain over 98% of daily return variance.

Finally, as an illustration of the methodology, we price by simulation an actual copper-linked security issued by Magma Copper Company, using the factor loadings for copper futures returns.

This approach offers a promising alternative for the pricing of commodity-contingent claims. Further research is needed on the relative merits of modeling the movement of the whole term structure of futures prices compared to earlier approaches that estimate the stochastic process of prices corresponding to one or two maturities only. This is especially critical when we are interested in the behavior of futures prices for maturities for which we do not have current prices.

Several extensions and refinements are possible. In modeling the factor structure for copper futures prices we are, in effect, combining shocks to interest rates and convenience yields. Some exploration into separating these two effects by jointly analyzing returns on copper futures and discount bonds could prove fruitful.

It would also be of interest to apply our framework to other commodities such as gold or oil. Our results could also be used to price commodity-contingent claims with early exercise features (American options), which require numerical solutions because they cannot be valued using our simulation method.

Finally, and perhaps most importantly, this approach would be useful to value real assets with payoffs contingent on commodity prices.

APPENDIX
EXTRAPOLATION OF FUTURES PRICES

In this appendix we explain the procedure to estimate initial futures prices for all twenty-eight maturities, given that only the first six are observed in the market.

Recall the relation between spot and futures described in Equation (9):

$$F(t, T) = S(t)\exp\left(\int_t^T y(t, s)\,ds\right)$$

in which $y(t) = r(t) - \delta(t)$.

A discretized version of the above is:

$$F_t = S_0 \exp\left(\sum_{\tau=1}^{t}(r_\tau - \delta_\tau)\Delta\tau\right) \tag{A-1}$$

from which it follows that:

$$\frac{F_t}{F_{t-1}} = e^{(r_t - \delta_t)\Delta\tau} \tag{A-2}$$

By a similar argument, we have the relation involving the price of a bond at time t, B_t:

$$\frac{B_t}{B_{t-1}} = e^{-r_t\Delta\tau} \tag{A-3}$$

Solving for δ_t, using Equations (A-2) and (A-3), we have:

$$\delta_t = \frac{1}{\Delta\tau}\ln\frac{B_{t-1}F_{t-1}}{B_t F_t} \tag{A-4}$$

During the first six periods we have actual market prices for both the bonds and the futures. For each month for which we have futures prices, we compute the implicit quarterly convenience yields using Equation (A-4), and report them in Exhibit A. Notice that convenience yields range from 2.0% to 2.9%.

Assuming that the convenience yield is constant for Periods 7 to 28, but using the term structure for the period obtained from actual bond prices, we obtain extrapolated prices for the initial futures prices for all twenty-eight maturities, using the relation:

$$F_{t+1} = F_t \times \frac{B_t}{B_{t+1}}\exp(-\delta\Delta\tau) \tag{A-5}$$

ENDNOTES

The authors thank the Center for Futures Research at Columbia University for providing the data for

EXHIBIT A
IMPLICIT CONVENIENCE YIELDS — DECEMBER 1991-MARCH 1993

Maturity	Futures	Bonds	δ
Dec. 91	104.25	99.69	0.024
Jan. 92	103.75	99.37	0.027
Feb. 92	103.15	99.06	0.020
Mar. 92	102.85	98.69	0.026
Apr. 92	102.35	98.31	0.029
May. 92	101.75	97.94	0.027
Jun. 92	101.25	97.53	0.027
Jul. 92	100.75	97.13	0.025
Sep. 92	99.90	96.34	0.022
Dec. 92	99.05	95.04	0.029
Mar. 93	97.90	93.44	

this study, and participants at seminars in British Columbia, Michigan, Osaka, Sydney, and the Western Finance Association meetings in Whistler for helpful comments. Special thanks to FONDECYT and DIUC-*Pontificia Universidad Catolica de Chile* for partial financial support.

[1]Frequently, futures contracts allow the short position to make delivery of any of several grades of a single commodity, at any exchange-approved location. In our model we define the *spot* price as the futures contract price at maturity. Prices for immediate delivery of a commodity of a particular grade at a particular location are sometimes referred to as the *cash* price. The model does not require actual *cash* prices to equal *spot* prices, or even to exist. In developing the model, we require only that no arbitrage exist in trading among futures.

[2]If interest rates are non-stochastic, forward prices are also a martingale under the risk-adjusted process. For stochastic interest rates, however, in general forward prices are not a martingale under the risk-adjusted process.

[3]For example, if interest rates are non-stochastic. In this case, all the uncertainty in the model would be related to spot prices and convenience yields.

[4]Using the risk-free interest rate assumes there is no default risk.

[5]This is strictly correct only for instantaneous returns. Because in the empirical tests we use daily returns, it will be approximately correct.

[6]Heath, Jarrow, and Morton [1991] use a similar procedure, but applied to percentage changes in forward interest rates. For good reviews of principal components analysis, see Morrison [1990] and Harman [1967].

[7]Actually the delivery date rule is somewhat more complex, including an obligation that the seller submit to the clearing house a *notice of intention to deliver* within some prescribed dates, prior to the actual delivery.

[8]Roughly 2% of all commodity futures contracts are settled by deliveries (Horn [1984]).

[9]Even though prices of the CU and HG contracts are significantly different, daily returns should be similar. Averaging all daily returns on futures with maturities in neighboring months is equivalent to considering that investors hold portfolios of these futures.

[10]We discard returns on futures traded during the first fifteen days of 1991 and before 1978 because of severe missing observation problems.

[11]Recall that if principal components analysis is performed on the covariance matrix, observations are weighted by their variance, while if it is performed on the correlation matrix, all observations have the same weight.

[12]Note that the model assumes that the relevant factors totally explain the movements in futures prices.

[13]To have the *same impact on returns* in our context means to explain the same fraction of the total return variance.

[14]Notice that the magnitude and the sign of each shock is random, with mean zero. Thus, a positive or negative loading does not provide any information by itself, but only in relation to the sign of the other factor loadings.

[15]For a reasonable annual default risk premium of 5% we obtain an estimated price of $109.8, which is 1.67% higher than the $108 recorded as the closing price for that day at the American Stock Exchange.

[16]A volatility factor equal to 2.0 induces a return volatility that is twice the historical figure.

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