Optimal Investment and Production Decisions and the Value of the Firm

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Abstract. We explore the effects of uncertainty on a firm that can respond by modifying its investment or production schedule (or both simultaneously) to variations in output price. Investment may increase capacity and/or reduce costs. We consider a firm with finite resources.

Our model uses option theory instead of the more traditional net present value framework. One of the early papers using this approach is Brennan and Schwartz (1985) in which an investment project to extract a finite natural resource is valued. In that paper, the value of the firm is a function of two state variables, the finite resource to be extracted (output to be produced in the future) and the commodity spot price. In order to maximize firm value, the manager can respond by modifying one control variable, the production level. In our model we handle instead three state variables (spot price, resources, accumulated investment) and two control variables (production rate and investment rate), and solve numerically.

We obtain both the value and the optimal policy of a firm that has investment projects that increase capacity and/or reduce costs and illustrate optimal policies as resources and available investments decrease over the life of the firm. Firms may start by only investing, then invest and produce, to end only producing.

Keywords: real options, investment and production under uncertainty

1. Introduction

During their operation firms face multiple uncertainties. Some may be labeled as *economic*, while others as *technical*, depending on whether they are or are not related to market conditions. Examples of the former are input and output prices, and of the latter are the copper reserves in a given mine. Traditional finance theory states that only economic uncertainty plays a role in the value of a given expected cash flow, because technical uncertainties are diversifiable. Under this framework the Net Present Value approach for valuing cash flows amounts to measuring the economic risk of a given expected cash flow and using the market price of risk obtained through some equilibrium model such as the Capital Asset Pricing Model to determine the risk premium in the discount rate.

Despite the attractiveness of the above approach, based primarily on its simplicity, academics and practitioners have increasingly become aware of its limitations for valuing some investment projects. The major flaws of the Net Present Value approach lie in the difficulty of determining the expected value of cash flows (which may depend on economic as well as on technical variables) and the risk premiums when firms have the option to respond as uncertainty unfolds.

An alternative approach to the Net Present Value approach which improves on its weaknesses is to use Option Theory. The last decade has seen expanding work in this area, primarily analyzing optimal responses from the firm to different uncertainties and their impact on firm value giving rise to what has been called Real Options Literature.

One of the early papers in the real options literature is Brennan and Schwartz (1985) in which an investment project to extract a finite natural resource is valued using arbitrage theory. The source of uncertainty is the output price and firms can respond by delaying production temporally or permanently depending on price levels and volatility. Given that futures prices are used, the project value is determined without requiring predictions on spot prices, one of the main sources of error on natural resource investment valuations. The optimal response by the firm (when to delay or resume production) is obtained jointly with the firm value. In this setting the value of the firm is a function of two state variables, the finite resource to be extracted (output to be produced in the future) and the commodity spot price. In order to maximize its value, the firm manager can respond by modifying one control variable, the production level.

Several other models have been developed to take into account the particular characteristics of investment projects. For example, Majd and Pindyck (1989) include the effect of the learning curve by considering that accumulated production reduce unit costs. Others like McDonald and Siegel (1986) and Majd and Pindyck (1987) consider that the control variable is the investment rate, instead of the production level. Some include two production level controls like He and Pindyck (1992) with output for two different products and Cortazar and Schwartz (1993) with two-stage production system. The source of uncertainty can be commodity prices (Brennan and Schwartz (1985) or Paddock et al. (1988)), exchange rates (Dixit (1989) or Cortazar (1992)), or costs (Pindyck (1993)). Finally, models are tailored to different kinds of investment projects like copper mines (Brennan and Schwartz (1985)), oil reserves (Ekern (1985), Paddock et al. (1988)), research and development (Schwartz and Moon (1997)), environmental technologies (Cortazar et al. (1997)), or flexible production (He and Pindyck (1992)).[2]

Regardless of the variety and robustness of the real option models found in the literature, they have remained rather simple and naive in their specification of possible firm responses to uncertainty by limiting the number of specified state and control variables. Most of the models found in the literature handle only one control variable and two state variables simultaneously. The reason for this limitation lies in the increasing complexity of the numerical methods required for solving models with more variables. In many cases this complexity restriction is not binding in that the main intuition of the optimal response by the firm is adequately captured with these simplified models. In others, however, it can seriously hinder the understanding of the operational behavior of a firm subject to high uncertainty and high operational flexibility.

In this paper we explore the effects of uncertainty on a firm that can respond to variations in output price by modifying its investment or production schedule (or both simultaneously). Our model has three state variables (spot price, resources, accumulated investment) and two control variables (production rate and investment rate). Investment may increase capacity and/or reduce costs. We consider a firm with finite resources (or finite accumulated output) which allows the model to be used for natural resource investments. We can view this

model as an extension to Brennan and Schwartz (1985) in which the firm now can modify not only its output rate but also its investment rate at any point in time, and by this means optimally increase output capacity or reduce costs. We can also see this model as an extension to Cortazar et al. (1997) in which investment instead of being a discrete one-time environmental investment which allows for a capacity expansion, now is a continuous-time process. Finally, this model can be considered also an extension to Pindyck (1988) which also was concerned with the optimal investment and production level, but presented a more restricted model with infinite resources, infinite investment rate and investments that only expanded capacity but did not alter cost functions.

The main results of our paper are as follows. First, it presents a more realistic model of a firm that has exhaustible resources and may modify its production and/or investment rate in response to changing conditions. We obtain both the value and the optimal policy of a firm that has investment projects that increase capacity and/or reduce costs. Second, it illustrates the variations on optimal policies, as resources and available investments decrease over the life of the firm. This leads to firm behavior that may start by exclusively investing, then go through a period in which it both invests and produces, to end with only producing.

The organization of this paper is as follows. In Section 2 we develop the model, including its numerical solution. In Section 3 the results of applying the model to several types of investment projects are presented. Finally, Section 4 concludes.

2. The Model

2.1. The Valuation Framework

In this section we develop a model for determining the value of a firm that produces an output which sells in a competitive market at a price that follows an exogenous stochastic process. To produce its output the firm transforms a finite resource by incurring a deterministic unit cost which may depend on the accumulated investment undertaken by the firm. The firm may optimally determine at each point in time the production and investment schedule, which in turn may be contingent on the output price, and the state of the firm in terms of the available resources and accumulated investment. The model considers the existence of a futures contract written on the output of the firm[3] which allows for determining the value of the firm as a derivative asset using option theory.

We start by setting our notation:

- S: Spot unit price of the firm output

- I: Accumulated capital invested by the firm, with an upper bound of $\bar{I}$

- Q: Accumulated resources available to the firm to produce output

- $V(S, I, Q)$: Value of the firm as a function of S, I and Q

- q^p: Production rate (between $[0, \overline{q^p}(I, Q)]$)

- q^I: Investment rate (between $[0, \overline{q^I}(I, Q)]$)

- $dI = q^I dt$ Change in accumulated investment

- $dQ = -q^P dt$ Change in accumulated resources available to the firm to produce output

- $a(I, Q)$ Cost of producing one unit of output

- r: Risk free rate of interest, assumed constant

- c: Convenience yield on holding one unit of output[4]

- σ: Instantaneous volatility of returns on holding one unit of output

- dW: Increment to a standard Gauss-Wiener process

- $F(S, \tau)$ Value of a futures contract on the output of the firm with maturity τ

We assume that the firm faces a competitive market for its output, with a spot price S that follows a Brownian motion. Equation (1) is the stochastic process for S under the equivalent martingale measure.

$$\frac{dS}{S} = (r - c)dt + \sigma dW \tag{1}$$

We also assume that there exists futures contracts, $F(S, \tau)$, written on the output of the firm. It can easily be shown that the differential equation that describes the value of $F(S, \tau)$ is:

$$\frac{1}{2}F_{SS}S^2\sigma^2 + SF_S(r - c) - F_\tau = 0 \tag{2}$$

subject to the following initial condition:

$$F(S, 0) = S \tag{3}$$

The solution to partial differential equation (2) with boundary condition (3) is:

$$F(S, \tau) = S \exp((r - c)\tau).$$

To derive the differential equation that governs the firm value we apply Itô's Lemma to $V(S, I, Q)$, obtaining:

$$dV(S, I, Q) = V_s dS + V_I dI + V_Q dQ + \frac{1}{2}V_{ss}S^2\sigma^2 dt \tag{4}$$

The owner of the firm receives the following instantaneous cash flow:

$$(q^P(S - a) - q^I)dt \tag{5}$$

Forming a portfolio between the firm and a short position on V_S/F_S futures contracts, we can hedge all price risk. Therefore, by arbitrage consideration the return on this portfolio

return must equal the risk free rate. Thus,

$$dV + (q^p(S - a) - q^I)dt - \frac{V_S}{F_S}dF = rVdt \tag{6}$$

By substitution we obtain the Bellman equation for the value of the firm under the optimal policy

$$MAX_{q^p, q^I}\left[\frac{1}{2}V_{SS}S^2\sigma^2 - p^pV_Q + q^IV_I\right.$$
$$\left. + (q^p(S - a) - q^I) + (r - c)SV_S - rV\right] = 0 \tag{7}$$

The optimal conditions of this maximization problem amounts to obtaining the value of the control variables q^p and q^I that maximize the value of the firm:

$$V_I - 1 = 0 \tag{8}$$

$$S - a - V_Q = 0 \tag{9}$$

Note that since the unit cost of production, a, is assumed to be a function only of the state variables I and Q, and not a function of the rates of production and investment, we obtain a bang-bang solution. That is, we either produce (invest) at the maximum possible rate, or do not produce (invest) at all. Equation (8) indicates that if the marginal benefit of investing an additional dollar is greater that the cost (i.e. one dollar) we will invest at the maximum possible rate. Equation (9) states that we will produce at the maximum possible rate as long as the marginal benefit of an additional unit of production $(S - a)$ is greater than the decrease in value due to the reduction of one unit of the accumulated resources.

The critical spot prices $S_p^*(Q, I)$ and $S_I^*(Q, I)$ represent the prices above which it is optimal for the firm to produce or to invest, respectively.

To solve this problem we need to specify the initial and border conditions:

First, when available resources are zero the value of the firm must be zero.

$$V(S, I, 0) = 0 \tag{10}$$

Second, when accumulated investment is equal to its maximum the problem reduces to Brennan and Schwartz (1985) with value $W(S, Q)$:

$$V(S, \bar{I}, Q) = W(S, Q) \tag{11}$$

Third, when the spot price is zero, the firm value will also be zero.

$$V(0, I, Q) = 0 \tag{12}$$

Fourth, when the spot price tends to infinity, the firm will produce and invest at the maximum possible rate, so the option values of differing production and investment levels are zero, which implies that the value of the firm is linear in the spot price.

$$V_{SS}(\infty, I, Q) = 0 \tag{13}$$

2.2. *The Numerical Solution of the Model*

In this paper we present a real option model with three state variables and two control variables which captures in a more realistic way the operational options available to a firm of modifying its production or investment levels to deal with price uncertainty. In the following lines we present in a detailed way the numerical implementation of the implicit finite difference method used in the solution of the model. We choose the implicit finite difference method, as opposed to the explicit finite difference method, because of its robustness and superior stability properties (Geske and Shastri (1985)). A possible alternative would be to construct a binomial tree.

We start by constructing a three-dimensional grid on our three state variables: S, Q and I, and we use the following notation:

- $Smax$: Maximum spot price

- $Qmax$: Maximum accumulated production

- $Imax$: Maximum accumulated investment

- i: $(0, \ldots, n)$, index for S

- j: $(0, \ldots, m)$, index for I

- k: $(0, \ldots, p)$, index for Q

- $\Delta S = Smax / n$

- $\Delta I = Imax / m$

- $\Delta Q = Qmax / p$

Therefore, the state variables are defined as:

- $S = i \Delta S$

- $I = Imax - j \Delta I$

- $Q = k \Delta Q$

We use the following finite difference approximation of the partial derivatives:

$$H_S = \frac{H_{i+1,j,k} - H_{i-1,j,k}}{2 \Delta S} \tag{14}$$

$$H_{SS} = \frac{H_{i+1,j,k} - 2 H_{i,j,k} + H_{i-1,j,k}}{\Delta S^2} \tag{15}$$

$$H_I = -\frac{H_{i,j,k} - H_{i,j-1,k}}{\Delta I} \tag{16}$$

$$H_Q = \frac{H_{i,j,k} - H_{i,j,k-1}}{\Delta Q} \tag{17}$$

By replacing these derivatives into the differential equation (7), we obtain:

$$a_i H_{i-1,j,k} + b_i H_{i,j,k} + c_i H_{i+1,j,k} = d_i H_{i,j,k-1} + e_i H_{i,j-1,k} + f_i \tag{18}$$

$$a_i = \frac{1}{2} i^2 \sigma^2 - \frac{(r-c)i}{2}$$

$$b_i = -i^2 \sigma^2 - \frac{q^I}{\Delta I} - \frac{q^P}{\Delta Q} - r$$

$$c_i = \frac{1}{2} i^2 \sigma^2 + \frac{(r-c)i}{2}$$

$$d_i = -\frac{q^P}{\Delta Q}$$

$$e_i = -\frac{q^I}{\Delta I}$$

$$f_i = -(q^P (i \Delta S - a) - q^I)$$

Equation (18) can be represented as the following tridiagonal system of equations:

$$\begin{bmatrix} b_0 & c_0 & & & & & \\ a_1 & b_1 & c_1 & & & & \\ & a_2 & b_2 & c_2 & & & \\ & & & \cdots & & & \\ & & & & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & & & a_n & b_n \end{bmatrix} * \begin{bmatrix} H_{0,j,k} \\ H_{1,j,k} \\ H_{2,j,k} \\ \cdots \\ H_{n-2,j,k} \\ H_{n-1,j,k} \\ H_{n,j,k} \end{bmatrix}$$

$$= \begin{bmatrix} d_0 H_{0,k-1,j} + e_0 H_{0,k,j-1} + f_0 \\ d_1 H_{1,k-1,j} + e_1 H_{1,k,j-1} + f_1 \\ d_2 H_{2,k-1,j} + e_2 H_{2,k,j-1} + f_2 \\ \cdots \\ d_{n-2} H_{n-2,k-1,j} + e_{n-2} H_{n-2,k,j-1} + f_{n-2} \\ d_{n-1} H_{n-1,k-1,j} + e_{n-1} H_{n-1,k,j-1} + f_{n-1} \\ d_n H_{n,k-1,j} + e_n H_{n,k,j-1} + f_n \end{bmatrix} \tag{19}$$

The initial and border conditions for the problem are the following:
Equations (10) and (11) are represented by the following expressions in the discrete grid.

$$H_{i,j,0} = 0 \quad i = 0, \ldots, n \quad j = 0, \ldots, m \tag{20}$$

$$H_{i,0,k} = W(i \Delta S, k \Delta Q) \, i = 0, \ldots, n \quad k = 0, \ldots, p \tag{21}$$

The condition that the value of the firm is zero when the spot price is zero represented by equation (12) in the continuous time approach is represented by the following conditions in the discrete grid.

$$\begin{aligned} b_0 &= 1 & d_0 &= 0 \\ c_0 &= 0 & e_0 &= 0 \\ a_0 &= 0 & f_0 &= 1 \end{aligned} \tag{22}$$

Making V_{SS} equal zero for the maximum value of S as specified in equation (13), requires the following values of a_n, b_n and c_n.

$$\begin{aligned} b_n &= b_n + 2c_n \\ c_n &= 0 \\ a_n &= a_n - c_n \end{aligned} \tag{23}$$

We start solving this problem when the cumulative investment and production have reached their upper limits, knowing the firm value with equations (20) and (21). Then to compute the value of the firm for lower levels of the variables Q and I, we use equation (19).

First we have the value of the firm when the cumulative investment has reached its upper limit ($k = 0, i = 0, \ldots, n$), and also we know the value when the investment is finished ($k = 0, \ldots, p, j = 0, i = 0, \ldots, n$). Then with equation (19) we find the value for $k = 1$, $j = 1$ and $i = 0, \ldots, n$ and with that value we again use equation (19) to find the value for $k = 1, j = 2$ and $i = 0, \ldots, n$. We repeat this scheme until we have the value for $k = 1$, $j = m$ and $i = 0, \ldots, n$.

We use the same procedure described to calculate the values for $k = 1$ to calculate the values for $k = 2, \ldots, p$. So when we have the value for $k = p, j = m$ and $i = 0, \ldots, n$, we would have calculated the value of the firm for each level of the state variables S, Q, I.

To solve equation (19) we must know S_p^* and S_I^* because the values of the coefficients of equation (19) depend on q^p and q^I which are determined by the optimal policy. That is, for values of S greater than S_p^* (S_I^*) production (investment) is at the maximum rate and below S_p^* (S_I^*) the rate of production (investment) is zero. For every arbitrary combination of these critical spot prices we can solve equation (19) and obtain a value of the firm. However, we are interested in the combination that will maximize the value of the firm, since this is the one that satisfies the optimality conditions (8) and (9). So, for each level of Q and I, we iterate with different values of S_p^* and S_I^* until we find the combination that maximizes the value of the firm. The iteration consists in fixing one of the critical spot prices and searching for the other until a maximum value is reached. Then, fixing the second critical

spot price and searching for the optimal first one. This process is repeated until convergence is obtained. Also, each time we solve equation (19) we replace equations (22) and (23) to satisfy the boundary conditions.

This completes the numerical solution to the model.

3. Results

Firms make capital outlays for different reasons, which range from substituting old production processes to satisfying environmental regulations. These investments can be seen as having the effect of increasing output capacity, reducing costs, or both. In our model the accumulated investment is represented by the state variable I, output capacity by q^p and unit costs by a. The interaction between these three variables will define the effect of investment on the firm. In all cases we will assume accumulated investment is bounded by $\overline{I}$.

In this section we illustrate the results of the model applied to three cases, one for each investment effect stated above. We use identical stochastic process for all three cases to highlight the effect of different investment projects on firm value and optimal policy. In all cases the real interest rate, r, is assumed to be 2%, the convenience yield, c, 2% and the instantaneous annual volatility, σ, 30%. For the numerical solution, the number of steps in each dimension was chosen such that further refinements did not change the results obtained.

3.1. The Capacity Expansion Investment Case

To model the capacity expansion investment case we will consider the following Cobb-Douglas function:

$$\overline{q^p}(I) = \left(\frac{I}{\overline{I}}\right)^{\alpha} (\overline{q} - \underline{q}) + \underline{q} \tag{24}$$

in which $\underline{q}$ and $\overline{q}$ represent the production capacity when accumulated investment is zero or $\overline{I}$, respectively. This formulation allows for a flexible specification of the marginal effect of investment depending on the value of α. For this case we assume that the unit cost of production **a** does not depend on accumulated investment. The parameter values used in the example are given in Table 1.

The Optimal Production and Investment Schedule

The optimal production and investment schedule is defined by the critical spot prices over which it is optimal to produce and/or invest, as a function of the state variables spot price, **S**, accumulated investment, **I**, and resources available, **Q**.

Figure 1 plots the critical production spot price for an accumulated investment of $5 million. The discreteness in the figure comes from the discreteness of the numerical solution.

Table 1. Capacity expansion plan parameters. The table shows the parameters used to illustrate the capacity expansion investment case.

Unit cost of production	a	0.5 US$/lb
Maximum Accumulated Investment	$\overline{I}$	10 Million US$
Maximum rate of Investment	$\overline{q^I}$	2 Million US$/yr
Capacity when $I = \overline{I}$	$\overline{q}$	12 Million lbs/yr
Capacity when $I = 0$	$\underline{q}$	10 Million lbs/yr
Convexity of the Investment function	α	1

As can be seen from the figure the lower the available resources (e.g. Q is smaller) the higher is the critical production spot price. The intuition for this result is that, given that the production capacity is limited, the opportunity costs of depleting the resource (waiting for higher prices to occur) increases when resources are scarcer. Actually, as resources grow, critical production spot prices converge to the marginal production costs, which in this example is US $0.5. On the other side, if production capacity were infinite, the critical production spot price is independent of the available resources and can be computed as the optimal exercise price of a perpetual American call option (see Cortazar and Schwartz (1993)).

In the capacity expansion investment case an increase in the accumulated investment induces a higher production capacity. Then, for given available resources, higher accumulated investment implies higher production critical spot prices since available resources can be depleted faster. In essence, what matters for determining the opportunity cost of depleting the resources is the relation of available resources to production capacity.

Figure 2 shows that the critical investment spot price also increases with less resources available. The reason is that the benefits of expanded capacity induced by the investment is smaller for less resources. Figure 3 illustrates that critical investment spot prices increase with accumulated investment. The reason is that the marginal benefit of an additional unit of capacity includes the direct benefit of that extra unit plus the option value of being able to achieve additional units. Given that as accumulated investment increases it approaches the investment maximum $\overline{I}$, the extra unit value and the option value of extra units of capacity decreases, thus the critical investment spot price increases.

Firm Value

As can be expected, firm value increases with output spot price and the available resources. It also increases with accumulated investment since a firm with higher accumulated investment has higher production capacity and therefore can exploit higher prices more effectively.

The economic significance of modeling the firm with variable production and investment rates can be elicited by comparing the value of the firm using this model with the values
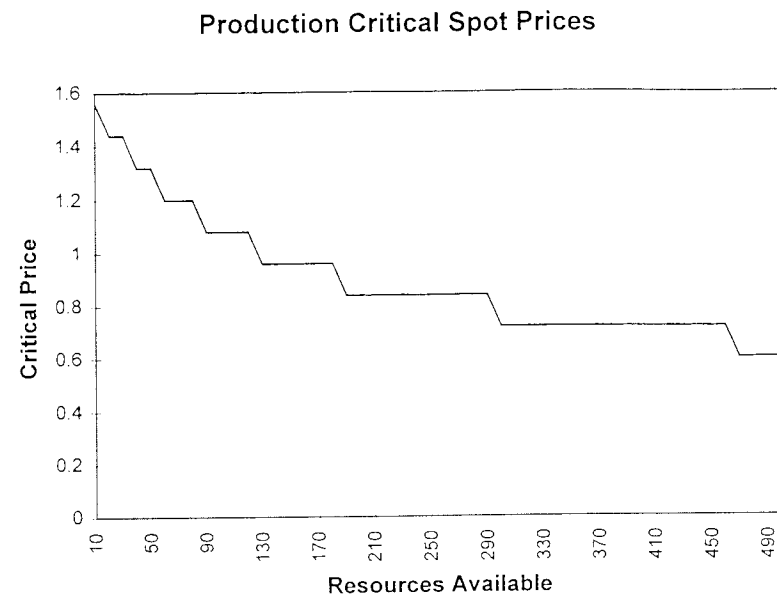
Production Critical Spot Prices

Figure 1. Optimal production policy for capacity expansion for an accumulated investment of $5 million.

determined using alternative ones which do not take this flexibility into account. First, we compute the value of a firm having a fixed capacity of 10 Million lb./yr that does not have the ability to expand investment. This firm can be evaluated by a numerical solution to the Brennan and Schwartz (1985) model. This result is labeled: *No Investment*. Second, we value a firm that performs the capacity expansion investment project at the same investment rate of 2 Million US$/yr., but without the option of optimally stopping and resuming the investment. This firm is valued using our model, but imposing a critical investment spot price of zero which effectively induces the firm to always invest until the completion of the investment project. This solution is labeled: *Investment without Timing*.

Third, we value the firm using our model, which allows for optimally timing the investment. This solution is labeled *Investment with Timing*

In Table 2 we compare the three values of the firm obtained, reporting under *Timing Option* the percentage of the firm value as computed by our model that would be lost if we were forced to undertake the investment project without the option to delay investment. We also report under *Investment Option* the percentage of the firm value as computed by our model that would be lost if we did not recognize the existence of a capacity expansion investment project.

Figure 4 plots the option percentages as a function of spot prices. It can be seen that as spot prices increase the timing option is less relevant. The reason is that for high prices it will always be valuable to invest immediately and not to wait. The investment option, on
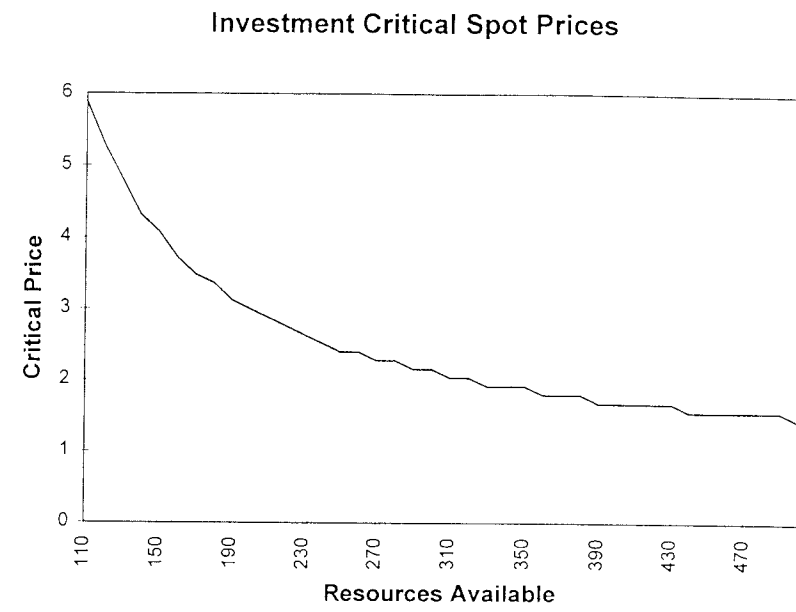
Investment Critical Spot Prices

Figure 2. Optimal investment policy for capacity expansion for an accumulated investment of $5 million.
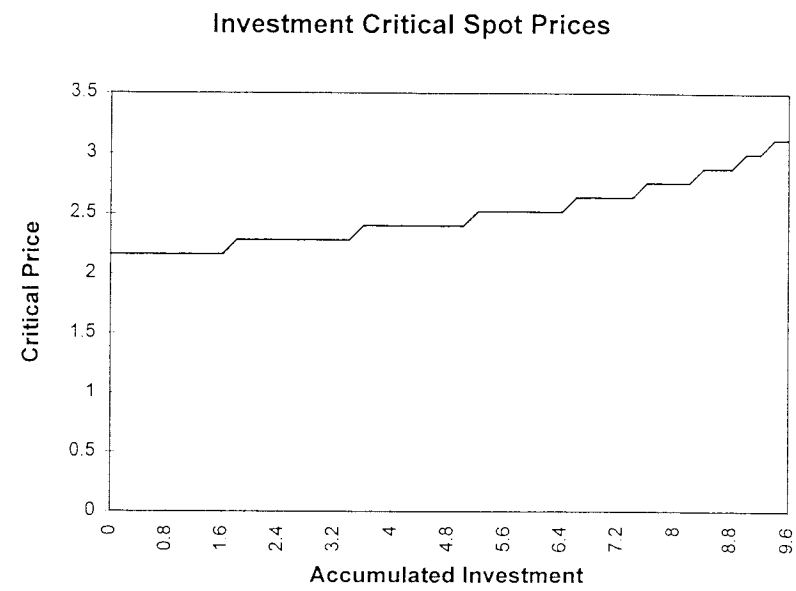
Investment Critical Spot Prices

Figure 3. Optimal investment policy for capacity expansion for available resources of 250 million pounds.

Table 2. Capacity expansion investment case. The table shows the value of the firm for different spot prices under three possible scenarios: with no expansion investment options, with forced immediate expansion, and with the optimal timing of the expansion option.

| SPOT	No Investment	Investment without Timing	Investment with Timing	Timing Option	Investment Option
0.3	43.90	36.95	45.21	18.27%	2.90%
0.6	110.23	107.08	113.53	5.68%	2.91%
0.9	188.71	189.87	194.50	2.38%	2.98%
1.2	273.22	279.41	282.50	1.09%	3.28%
1.5	360.59	372.25	374.28	0.54%	3.66%
1.8	449.58	466.96	468.34	0.30%	4.01%
2.1	539.56	562.81	563.78	0.17%	4.30%
2.4	630.20	659.42	660.13	0.11%	4.53%
2.7	721.29	756.56	757.08	0.07%	4.73%
3	812.71	854.06	854.46	0.05%	4.89%
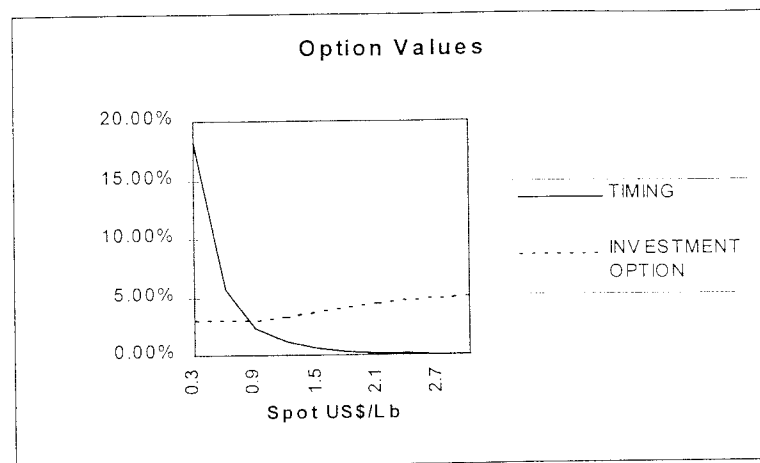
Figure 4. Options values for capacity expansion investment case. The timing option is the loss in firm value with forced immediate expansion and the investment option is the loss in value with no investment expansion option.

the other hand, is more valuable the higher the spot price because of the increase in value of capacity expansion investment projects.

Table 3. Sensitivity analysis. The table shows values of the firm
for different spot prices due to changes in key parameters of the
model.

SPOT US\$/LB	BASE CASE	$\bar{q} = 13$	$c = 3\%$	$\alpha = 1.1$	$\bar{I} = 11$
		FIRM VALUE (US\$ MILLIONS)			
0.3	45.21	46.02	27.78	45.19	45.15
0.6	113.53	115.55	79.12	113.48	113.38
0.9	194.50	198.07	144.54	194.41	194.24
1.2	282.50	288.14	216.83	282.36	282.09
1.5	374.28	382.34	292.87	374.10	373.71
1.8	468.34	478.88	371.05	468.12	467.62
2.1	563.78	576.83	450.55	563.54	562.94
2.4	660.13	675.71	530.89	659.86	659.17
2.7	757.08	775.20	611.82	756.78	756.03
3	854.46	875.14	693.15	854.13	853.32

Sensitivity Analysis

Table 3 reports the values of the firm obtained by changing some of the key parameters of the model one by one, for different spot prices. Column 2 reproduces the Base Case from column 4 in Table 2. The other columns report the values of the firm for changes in the maximum production capacity, the convenience yield, the convexity parameter and the maximum accumulated investment.

As expected, an increase in the production capacity (from 12 to 13 million lbs/yr) increases the value of the firm. This increase is larger the higher is the spot price since it is likelier that this capacity will be used. Also, at higher spot prices this excess capacity will be more valuable.

An increase in the convenience yield (from 2 to 3%) substantially reduces the value of the firm. The reason for this is that, from equation (1), when the convenience yield is higher future spot prices will tend to be lower.

Changing the convexity parameter from 1.0 to 1.1 has a negligible (negative) effect on the value of the firm. The convexity effect operates through the production capacity function (24).

Finally, increasing the maximum accumulated investment from 10 to 11 million dollars slightly reduces the value of the firm since more investment is required to get to the maximum production capacity.

Table 4. Cost reduction plan parameters. The table shows the parameters used to illustrate the cost reduction investment case.

Maximum Production Capacity	q^P	10 Million lb/yr
Maximum Accumulated Investment	$\overline{I}$	10 Million US$
Maximum rate of Investment	$\overline{q^I}$	2 Million US$/yr
Unit cost of production when $I = \overline{I}$	$\underline{a}$	0.35 US$/lb
Unit cost of production when $I = 0$	$\overline{a}$	0.5 US$/lb
Convexity of the Investment function	α	1

3.2. The Cost Reduction Investment Case

In this section we consider a firm with an investment project that does not increase production capacity, but reduces unit costs described with the following Cobb-Douglas function:

$$a(I) = \left(\frac{I}{\overline{I}}\right)^{\alpha} (\underline{a} - \overline{a}) + \overline{a}$$

in which $\overline{a}$ and $\underline{a}$ represent the unit cost of production when accumulated investment is zero or $\overline{I}$, respectively. This formulation allows for a flexible specification of the marginal effect of investment depending on the value of α. The parameter values used in this case are given in Table 4.

The Optimal Production and Investment Schedule

For this case the effect of resource availability (Q) on critical production and critical investment spot prices is similar to the one described in our capacity expansion investment project detailed in the previous section. That is, higher resource availability lowers the critical spot prices for production and investment. Also, the effect of accumulated investment on critical production spot prices is similar: higher accumulated investment lowers critical prices since marginal costs are lower. The effect of accumulated investment on critical investment spot prices, however, might differ. The intuition is that the higher the accumulated investment, the lower the unit cost, making it more profitable to invest. This effect might be more relevant than the option value lost by additional accumulated investment described in the previous section, which induced a higher critical investment spot price.

Firm Value

As in the previous case the value of the firm increases with the spot price, the available resources and the accumulated investment. Table 5 presents firm values with and without timing and investment options, and Figure 5 plots the relevance of these options with respect to spot price.

Table 5. Results for cost reduction investment case. The table shows the value of the firm for different spot prices under three possible scenarios: with no cost reduction investment options, with forced immediate investment, and with the optimal timing of the investment option.

| SPOT	No Investment	Investment without Timing	Timing with Timing	Timing Option	Investment Option
0.3	43.90	39.83	45.94	13.30%	4.44%
0.6	110.23	112.81	115.36	2.21%	4.45%
0.9	188.71	196.46	197.41	0.48%	4.41%
1.2	273.22	284.74	285.17	0.15%	4.19%
1.5	360.59	374.98	375.20	0.06%	3.89%
1.8	449.58	466.25	466.38	0.03%	3.60%
2.1	539.56	558.15	558.22	0.01%	3.34%
2.4	630.20	650.45	650.50	0.01%	3.12%
2.7	721.29	743.02	743.06	0.00%	2.93%
3	812.71	835.79	835.82	0.00%	2.76%
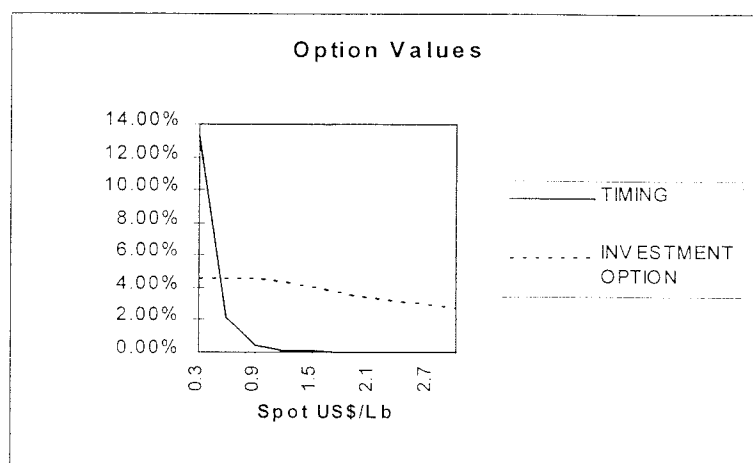
Figure 5. Option values for cost reduction investment case. The timing option is the loss in firm value with forced immediate expansion and the investment option is the loss in value with no investment expansion option.
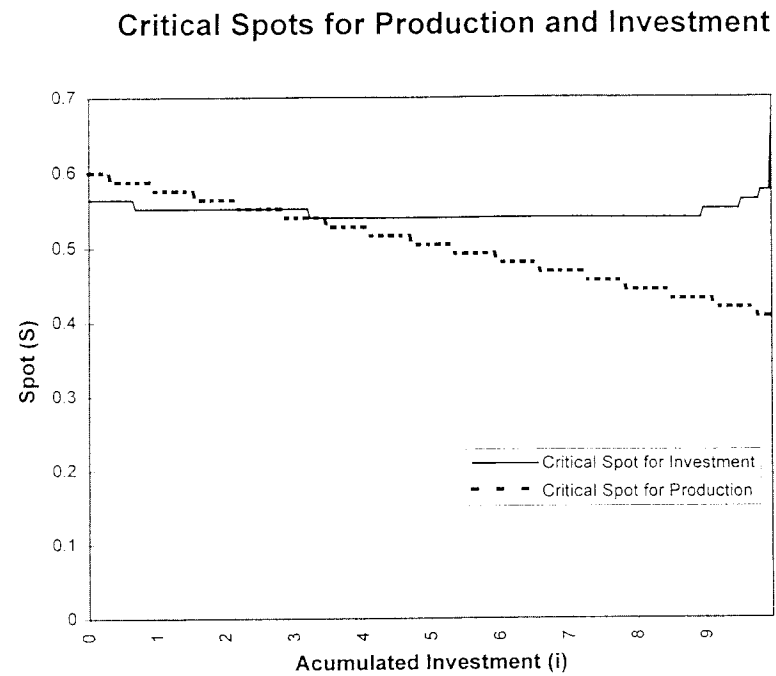
Critical Spots for Production and Investment

Figure 6. Optimal policy for the mixed capacity expansion-cost reduction investment case. The figure shows the critical spot prices for investment and production in this case.

In the cost reduction case, both the investment and the timing option decrease in relevance (in percentage terms) with increases in spot price. The reduction in the timing option is similar to that described in the capacity expansion case, while the investment option, even though decreasing in percentage terms, increases in value in dollar terms.

3.3. The mixed Capacity Expansion-Cost Reduction Investment Case

In this section we illustrate the possible operative behavior of a firm which faces an investment project which simultaneously has both effects described in the past two sections, namely the ability to expand capacity from 10 to 12 million lb./year and to reduce costs from 0.5 US$/lb. to 0.35 US$/lb. with the same US$ 10 million in investment.

Figure 6 plots the critical investment and production spot prices as a function of accumulated investment when the firm has 500 million lb. left in resources.

It can be seen that when the firm starts its operation and available resources and investment projects are high, the firm may be investing but not producing (for example if spot prices are US$ 0.55). Later, when accumulated investments increase, the firm will be either both

producing and investing or shutting down the operation (critical investment and production prices are the same). Finally, as availability of investment projects decrease (accumulated investment increases), the firm is more reluctant to invest than to produce. This behavior is consistent with the investment life cycle of a typical firm.

For this case we performed sensitivity analysis to assess the impact of some key parameters on the critical prices for production and investment. Increases in the maximum production capacity increases both critical prices since available resources can be depleted faster and the opportunity cost of extraction increases. Increases in the convenience yield decreases both critical prices. The reason for this is that the probability of prices going up in the future decreases with increasing convenience yield. As expected, increases in volatility increases critical prices.

4. CONCLUSIONS

In this paper we construct and solve a model of a firm that has exhaustible resources which, by incurring a deterministic cost, produces an output that can be sold at a stochastic price. The firm may alter its investment and/or production rate in response to changing conditions. The model is based on option theory, and assumes the existence of assets that allows the firm to hedge output price risk.

The model is applied to capacity expansion and/or cost reduction investment projects and optimal policies and firm value are explored. The model may induce an investment/production life cycle that is consistent with observed firm behavior, in which investment is concentrated in earlier stages and production in later ones.

The model presented is more comprehensive than others found in the literature, capturing a richer set of responses to uncertainty by firms. To obtain solutions numerical methods are implemented.

The model has empirical implications with respect to aggregate investment behavior. For example, in our model an increase in the volatility of the spot price will increase the investment critical spot price. This implies that in times of more uncertainty (when prices are more volatile) firms require higher prices to initiate expansion and cost reduction investments. Thus, everything else equal, aggregate investment will decline.

In the partial equilibrium model we have developed, the stochastic process for the spot price is assumed to be exogenous. In a more general equilibrium model the spot price process should be endogenous. When prices are relatively high, higher cost producers enter the market putting a downward pressure on prices and conversely when prices are low. Demand substitution produces similar effects. These supply and demand adjustments induce mean reversion in spot prices. A stochastic convenience yield positively correlated with spot prices would be one possible way to induce mean reversion in spot prices.[5]

There are several directions in which the analysis presented in this article can be extended. In many practical situations there are costs associated with starting and stopping production or investment programs. The incorporation of these costs into the model would not change the qualitative results of the analysis. However, the numerical solution would be more complicated since there would be a higher critical spot price to start production (or investment) and a lower one to stop production (or investment).

Another possible extension would be to introduce stochastic available resources. For example, in a typical mine as the mineral is extracted new information about the available resources is obtained. This learning process can be modeled by a stochastic process for the available resources. Though possible, this extension would increase the dimensionality of the problem and substantially increase the computational time required to solve it.

Notes

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2. For a complete review of the literature see Trigeorgis (1993).

3. Actually we do not require the existence of a futures contract, but the existence of a portfolio of assets that is perfectly correlated with the price of the output.

4. Convenience yield is the flow of services accruing to the owner of the spot commodity but not to the owner of a futures contract.

5. See Schwartz (1997) for an extensive discussion on this subject as it applies to copper and oil prices.

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