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Convergence within the EU: Evidence from Interest Rates

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The economic and political changes which are taking place in Europe affect interest rates. This paper develops a two-factor model for the term structure of interest rates specially designed to apply to EMU countries. In addition to the participant country's short-term interest rate, we include as a second factor a 'European' short-term interest rate. We assume that the 'European' rate follows a mean reverting process. The domestic interest rate also follows a mean reverting process, but its convergence is to a stochastic mean which is identified with the 'European' rate. Closed-form solutions for prices of zero coupon discount bonds and options on these bonds are provided. A special feature of the model is that both the domestic and the European interest rate risks are priced. We also discuss an empirical estimation focusing on the Spanish bond market. The 'European' rate is proxied by the ecu's interest rate. Through a comparison of the performance of our convergence model with a Vasicek model for the Spanish bond market, we show that our model provides a better fit both in-sample and out-of sample and that the difference in performance between the models is greater the longer the maturity of the bonds.

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1. Introduction

The economic and political changes that are taking place in Europe are affecting financial markets, and interest rates are no exception. Changes in the European level of interest rates affect the domestic interest rates and we can no longer study these in isolation. In this paper, we develop a *convergence model* which takes into account the influence of the European rate on the behaviour of interest rates of European Monetary Union (EMU) countries.

Since the seminal papers by Merton (1973) and Vasicek (1977), many interest rate models have been developed. In the simplest form, interest rates have been modelled as one-factor Markovian processes where the term

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structure is dependent on the short-term rate which itself is normally distributed. Empirical research, however, has suggested that multi-factor models do significantly better than single-factor models in describing the behaviour of the term structure in the real world.[1]

Although one-factor models represent the short end of the term structure fairly well, they are inadequate to describe the behaviour of long-term rates. This has led to two-factor models, which include either the volatility (Longstaff and Schwartz, 1992) or the mean rate (Balduzzi, Das, Foresi and Sundaram, 1997, Balduzzi, Das and Foresi, 1998) as additional factors. Recently, some studies consider three or four factors, such as Balduzzi *et al.* (1996) and Chacko (1997). Both papers use, besides the short rate, stochastic mean and volatility as additional factors. While Balduzzi *et al.* (1996) conclude that volatility influences mainly short to medium-term yields and the mean rate affects long yields more strongly, Chacko (1997) states that the long-run mean is found to be the most important factor for the middle of the yield curve, the stochastic volatility having a minor impact on the yield curve.

In this paper, we develop and estimate a two-factor model of the term structure of interest rates. Following common practice, the first factor is identified with the level of the short-term rate. The second factor is identified with the central tendency of the short rate, which itself changes stochastically over time. We refer to this model as a stochastic mean reverting model or *convergence model.*

Stochastic mean reverting models have been successful in describing the process followed by short-term interest rates. Among the reasons for the use of stochastic mean models Balduzzi *et al.* (1997) note that there is considerable evidence of leptokurtosis ('fat tails') in the distribution of changes of interest rates, which can be a consequence of time variation in the mean level. This issue is specially important for the pricing of options where volatility plays a crucial role. Moreover, a model which imposes a constant mean level may overstate the volatility of the short interest rate because changes in the mean are included in the volatility parameter. Finally, there are macroeconomic based reasons, for example, changes in the level of inflation or exchange rates, which are likely to be reflected in mean shifts in interest rates.

These reasons make stochastic mean models a convenient framework to study the term structure of interest rates of European countries, given that for some years after forming the EMU, participant countries are allowed to issue debt in their home currency while simultaneously European debt will be issued in euros.[2]

[1] Some studies that test these models are Stambaugh (1988), Longstaff and Schwartz (1992), Litterman and Scheinkman (1991).

[2] By 2002, the euro will be the only currency but still participant countries will be allowed to issue domestic debt contemporaneously to the debt issued by the European Central Bank.

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EMU participant countries fixed their bilateral exchange rate against the euro in January 1999.[3] With fixed exchange rates, risk-free interest rates across countries should be the same. Even before the formation of the EMU block, we could observe how interest rates in the participating European countries were converging. Nevertheless, even with a single currency, risk-free interest rates across countries may be slightly different, reflecting different sovereign risks.

Because participant countries are allowed to issue domestic debt contemporaneously with the debt issued by the European Central Bank, small differences in domestic interest rates exist.[4] Since the level of interest rates will affect bond prices and prices of interest rates derivatives, developing a model that incorporates both the domestic rate and the central rate will improve the valuation and hedging of these instruments.

Our term structure model is exponentially affine and we obtain closed-form solutions for the prices of bonds and European options on discount bonds. American type options, options on coupon bonds and other exotic derivatives can be easily solved using numerical procedures.

It should be noted that, in the model presented here, the market price of risk for both stochastic processes is priced and can be estimated directly from the data, without any need for further assumptions.

The characterization of the term structure dynamics developed in this paper not only applies to a fixed exchange or single currency situation, but also to the situation in which markets anticipate the formation of the EMU block, when interest rates were induced to converge.

In this context, we test the model for the Spanish term structure. As a proxy for the central tendency, we use the ecu's interest rate. This is distinct from previous work (Balduzzi *et al.*, 1996) in which the stochastic mean is treated as an unobservable. Proxying the mean rate by an observable allows us to more efficiently investigate the pricing properties of the convergence model. The short-term interest rate of the ecu's deposits serves as a good proxy since after 4 January 1999 all debt in ecus became debt in euros, and the interest rate of this debt became the reference for all economies joining EMU.[5] The method used for estimation purposes is the generalized method of moments (GMM). This method provides a simple but flexible framework that is robust to

[3] The conversion rate, for each country, at which exchange rates was fixed is an interesting and polemic topic (Obstfeld, 1998), but it lies beyond the scope of this paper.

[4] For instance, during November 1999, the Spanish three-years interest rate was 4.37 per cent while the Italian one is 4.17 per cent and the German was 4.05 per cent.

[5] We are aware that the ecu is not an exact proxy for the euro because it included currencies like drachma and sterling that were not included in the euro. Another proxy for the euro rate could be the DEM rate, but during the sample period, it was not clear what weight the German or any other currency would have. In the absence of further information at that time, the use of the ecu rate seems highly suitable.

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misspecification in the behaviour of the residuals, and is very suitable to estimate the system of equations we obtain, making use only of the certain moment conditions and avoiding oversimplifying assumptions.

For a cross-section of Spanish discount bonds during the period June 1990 to December 1997, we compare the fit of our convergence model with that of a Vasicek model which assumes a constant mean rate. The average in-sample root mean square error (RMSE) for the convergence model is 3.9 per cent, whereas for the Vasicek model it is 4.3 per cent. The average out-of-sample errors are also smaller for the convergence model than for the Vasicek model. The differences in favour of the convergence model are greater, the longer the maturity of the bonds for both in-sample and out-of-sample fit.

The results presented here should be interpreted with caution since the distribution of interest rates in Spain has been changing in the recent past, given that economic and political factors are forcing interest rates to converge. The performance of the model can be expected to improve in the future, since the reference rate will be more clearly defined and the distribution of interest rates more stable.

The paper is organized as follows. In section 2, we introduce the stochastic process followed by the European and the domestic interest rate. In section 3, we derive the closed-form solutions for zero-coupon discount bond prices, and for European options on the domestic bond. In section 4, as an example, we present the data and discuss the application of the model to the Spanish zero coupon bonds. Finally, we conclude in section 5.

2. A Convergence Model for Interest Rates

The basic element in the pricing of bonds and interest rates derivatives is the specification of the interest rate process.[6] In this section, we define the process followed by the domestic short-term interest rate, modelled as a two-factor process, and the process followed by the European rate which is taken as the benchmark process.

2.1. The Process followed by the Domestic Short Rate of Interest

Definition 1 *The domestic short-term interest rate follows a stochastic mean reverting process given by a stochastic differential equation (SDE) of the form*

[6] General equilibrium models that are consistent with these specifications can be constructed. Some examples are Longstaff and Schwartz (1992) for two-factor term structures, Goldstein and Zapatero (1996) for one-factor term structure.

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(1) $$dr_d = [a + b(r_e - r_d)]dt + \sigma_d dz_d$$

where a, b and σ_d are constants, r_d is the domestic rate and r_e the European rate. The mean reverting level is the European interest rate which itself evolves stochastically over time.

The presence of a in the mean reflects the fact that the convergence of interest rates within the EMU will take place at the average interest level of the core countries of the exchange rate mechanism (ERM), so the domestic interest rate does not need to replicate exactly the central interest level, and minor divergences may exist.

The difference $(r_e - r_d)$ represents the reversion of r_d towards r_e, b is the speed of adjustment coefficient, and dz_d is an increment to a Wiener process.

This two-factor model provides a richer pattern of both term structure movements and volatility structures than the one-factor models.[7]

2.2. The Process for the European Short Rate of Interest

Definition 2 *The SDE followed by the European interest rate is a mean reverting Ornstein-Uhlenbeck process,*

(2) $$dr_e = c(d - r_e)dt + \sigma_e dz_e$$

where c is the speed of adjustment coefficient and d is the long-run mean level of r_e and dz_e is an increment to a Wiener process.

The two processes are correlated with coefficient ρ

$$dz_d dz_e = \rho dt$$

Given that the errors are normal, the specifications of both processes, domestic and European, allow interest rates to become negative. It is well known that if the current short term is well above zero, there is only a very small probability of reaching a negative level.[8]

[7] For instance, we can reproduce volatility shapes, like the 'humped' shape that are not possible in one-factor models. For detail, see Hull and White (1994).

[8] For a detailed study of this topic, see Rogers (1995). One way to prevent the occurrence of negative interest rates is to assume that the short-term rate's diffusion coefficient is proportional to r^{α} when $\alpha > 0$. In some cases, additional technical conditions are required. Cox, Ingersoll and Ross (CIR) (1985) analyse the case where $\alpha = \frac{1}{2}$. All the results of this paper can be extended in a straightforward manner to the CIR model.

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3. Bond-pricing Equations and other Interest Rate Derivatives

We first proceed to derive the bond pricing formulas for the European (which depends on r_e) and domestic bonds. The equation for the European bond is well known since it corresponds exactly to the Vasicek case, so we discuss it briefly and skip the proofs.

The European market price of risk obtained here is an input to the equation for the domestic bonds.

Then we present the derivation of the closed-form solution to price *European options* on the domestic zero-coupon discount bond. Throughout the subsection, the term European (vs American) is used to denote the exercise condition of the option. We do not discuss the valuation of derivatives of the central interest rate since they are well known in the literature.

3.1. The European Bond

Proposition 1 *Let $P(r_e, \tau)$ be the price of a zero-coupon discount bond with face value 1 ECU and τ years to maturity when the interest rate is r_e and following process (2). Its price is given by*

(3) $$P(r_e, \tau) = \exp[F(\tau) - r_e G(\tau)]$$

where

$$G(\tau) = \frac{1 - \exp(-c\tau)}{c}$$

and

$$F(\tau) = \frac{(G(\tau) - \tau)(c(cd - \lambda_e \sigma_e) - \sigma_e^2/2)}{c^2} - \frac{\sigma_e^2 (G(\tau))^2}{4c}$$

3.2. The Domestic Bond

Proposition 2 *Let $P(r_d, r_e, \tau)$ be the price of a zero-coupon discount bond with face value 1 domestic currency unit and τ years to maturity when the domestic interest rate is r_d and the central rate is r_e. Its price is given by*

(4) $$P(r_d, r_e, \tau) = \exp[A(\tau) - r_d B(\tau) - r_e C(\tau)]$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are functions that depend on τ but not on r_d or r_e. The exact form of these functions is

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$$B(\tau) = \frac{1 - \exp(-b\tau)}{b}$$

$$C(\tau) = \frac{bB(\tau)(1 - \exp(-c\tau))}{c}$$

$$A(\tau) = \frac{1}{2}\tau[-2aB(\tau) + C(\tau)(-2cd + C(\tau)\sigma_e^2 + 2\sigma_e\lambda_e) + B(\tau)(2C(\tau)\rho\sigma_d\sigma_e$$
$$+ \sigma_d(2\lambda_d + B(\tau)\sigma_d))]$$

Proof of Proposition 2 See Appendix A.

3.3. Options on Domestic Discount Bonds

From (4), we can write the stochastic process followed by bond prices $P(r_d, r_e, \tau)$ as:

$$\frac{dP(r_d, r_e, \tau)}{P} = \mu(r_d, r_e, \tau)d\tau + \upsilon_d(\tau)dz_d + \upsilon_e(\tau)dz_e$$

where $\mu(r_d, r_e, \tau)$, $\upsilon_d(\tau)$, and $\upsilon_e(\tau)$ are known functions of r_d, r_e, τ and the parameters of the interest rate processes (υ_d, and υ_e are only functions of time).

Using Ito's lemma, the volatility of $P(r_d, r_e, \tau)$ is

$$\sigma^2(\tau) = \sigma_d^2 B(\tau)^2 + \sigma_e^2 C(\tau)^2 + 2\rho\sigma_d\sigma_e B(\tau)C(\tau)$$

and we have used the fact that $E[dz_d, dz_e] = \rho dt$.

Since this is independent of the level of r_d, r_e, the distribution of a bond price at any given time conditional on its price at an earlier time must be lognormal.

Consider a European call option $C(r_d, r_e, t, T_C)$ on a discount bond with exercise price K. Suppose that the current time is t, the option expires at T_C, and the bond expires at time T ($t \leqslant T_C \leqslant T$).

Given that $C(r_d, r_e, T_C)$ depends on the same random variables r_d and r_e, it too must satisfy the equation (A.1). The only difference is that the terminal value for the option is

$$C(r_d, r_e, t, T_C) = \max(P(r_d, r_e, T_C, T) - K, 0)$$

Merton (1973) extends the Black-Scholes option pricing model to accommodate a stochastic term structure. His model applies to the process followed by $P(r_d, r_e, \tau)$ since the drift coefficient can be of general specification, while the diffusion coefficient must be equal to a deterministic function times the current bond price.

From the lognormal property, and the results in Merton (1973) and Langetieg (1980), it follows that the option price C is given by

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$$C = P(r_d, r_e, t, T)N(h) - KP(r_d, r_e, t, T_C)N(h - \sigma_P)$$

where

$$h = \frac{1}{\sigma_P} \log \frac{P(r_d, r_e, t, T)}{P(r_d, r_e, t, T_C)K} + \frac{\sigma_P}{2}$$

and σ_P^2 is the variance of the logarithm of the price of the underlying bond at the option expiration date

$$\sigma_P^2 = \int_t^{T_C} \sigma^2(\tau) d\tau$$

or

$$\sigma_P^2 = \int_t^{T_C} \{\sigma_d^2[B(\tau, T) - B(\tau, T_C)]^2 + \sigma_e^2[C(\tau, T) - C(\tau, T_C)]^2$$
$$+ 2\rho\sigma_d\sigma_e[B(\tau, T) - B(\tau, T_C)][C(\tau, T) - C(\tau, T_C)]\} d\tau$$

and finally,

$$\sigma_P^2 = \nu_d(t, T_C)^2 B(T_C, T)^2 + \nu_e(t, T_C)^2 C(T_C, T)^2$$
$$+ 2\rho\nu_d(t, T_C)\nu_e(t, T_C)B(T_C, T)C(T_C, T)$$

where $\nu_d^2(t, T_C)$, $\nu_e^2(t, T_C)$ are the variances of r_d and r_e, respectively, and are given by

$$\nu_e^2(t, T_C) = \sigma_e^2 \frac{1 - \exp[-2c(T_C - t)]}{2c}$$

$$\nu_d^2(t, T_C) = \sigma_e^2 \frac{1 - \exp[-2c(T_C - t)]}{2c}(1 - \exp[-b(T_C - t)])^2$$
$$+ \sigma_d^2 \frac{1 - \exp[-2b(T_C - t)]}{2b}$$

Appendix B provides the derivation of $\nu_d^2(t, T_C)$ and $\nu_e^2(t, T_C)$.

The price of a European put can be easily found from the call-put parity.

Unlike the case with a single stochastic factor, where we can decompose an option on a coupon-bearing bond into a portfolio of options on discount bonds (Jamshidian, 1989), in our two-factor model, it is not possible to mimic this procedure. The reason is that with two interest rates, it is not possible to build a one-to-one correspondence between strike interest rates and prices. However, we can still find a numerical solution.[9] These solutions can also be applied to value American options.

[9] Possible approaches are Hull and White (1994) or Broadie and Glasserman (1997).

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4. Example: Empirical Implementation for the Case of Spanish Bonds

We now provide an empirical application of the convergence model. We first present the econometric methodology employed in the estimation. Then we describe the data used. The empirical implementation takes the Spanish short-term interest rate as the domestic rate and the ECU's short-term interest rate as a proxy for the stochastic central tendency. Finally, we discuss the results obtained and compare them with those obtained using a one-factor constant mean Vasicek model.

4.1. Methodology

The econometric approach used in estimating the parameters of the interest rate models is the generalized method of moments (GMM). This technique is robust to misspecifications in the behaviour of the residuals since it allows us to use certain moment conditions without specifying the full density function.[10] Some studies that already use this technique are Balduzzi *et al.*, (1997) and Chan *et al.*, (1992).

Following the usual practice in these type of studies, we estimate the parameters of the continuous-time model using a discrete-time version of process

$$r_{st-1} - r_{st} = [a + b(r_{et} - r_{st})]\Delta t + \sigma_s \xi_{st} \sqrt{\Delta t} \tag{5}$$

and

$$r_{et+1} - r_{et} = c(d - r_{et})\Delta t + \sigma_e \xi_{et} \sqrt{\Delta t} \tag{6}$$

where the ξ_{st} and the ξ_{et} are correlated i.i.d. draws from a standard normal distribution. The r_{st} are observations of the Spanish short-term interest rates at each moment of time and r_{et} are observations of the European short-term interest rates at each moment in time.

We define

$$\epsilon_{st} = r_{st+1} - r_{st} - [a + b(r_{et} - r_{st})]\Delta t$$

from (5), and

$$\epsilon_{et} = r_{et+1} - r_{et} - c(d - r_{et})\Delta t$$

from (6). Then the moment equations for our model are given by

[10] Additionally, it is useful in models where the diffusion varies with the level of interest rates. This fact would allow us to perform comparisons with models like CIR (Cox *et al.*, 1985) in a unified framework.

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$$E[\epsilon_{st}] = 0$$

$$E[\epsilon_{st}^2] = \sigma_s^2 \Delta t$$

$$E[(r_{et} - r_{st})\epsilon_{st}] = 0$$

$$E[\epsilon_{et}] = 0$$

$$E[\epsilon_{et}^2] = \sigma_e^2 \Delta t$$

$$E[r_{et}\epsilon_{et}] = 0$$

$$E[\epsilon_{st}\epsilon_{et}] = \rho\sigma_s\sigma_e\Delta t$$

In addition, for the estimation we also use the information contained in the third moments,

$$E[\epsilon_{et}^3] = 0 \qquad \text{and} \qquad E[\epsilon_{st}^3] = 0$$

The value obtained for the quadratic form after convergence, that under the null hypothesis (that the third moments are zero, i.e. our variables are symmetric) follows a χ^2 distribution with two degrees of freedom, the number of overidentifying moments.

Estimations using also the fourth moments of the residuals were performed and the results turned out to be similar to the ones with just the third moments. However, the convergence properties of the estimates involving the fourth moments tended to be more unstable and sensitive to the starting values, due to the increasingly complex behaviour of the objective function as higher-order moments are included. For this reason, we do not report those estimates.

Were we to estimate the two regressions separately, we could use maximum likelihood to obtain more efficient results, since the model implies that ϵ_{st} and ϵ_{et} are normally distributed. The estimate of the correlation coefficient, however, would not be efficient. GMM allows us to provide asymptotically efficient estimates of the seven parameters. Assuming that ϵ_{st} and ϵ_{et} follow a bivariate normal distribution (BVN), maximum likelihood would allow to efficiently estimate the seven parameters; but bivariate normality is not implied by the model.[11]

[11] The model just implies two variables that are normal separately; it is not necessary that their joint distribution behaves as a bivariate normal. Nevertheless, we estimated the system by a two-step maximum likelihood procedure (the joint likelihood of the seven parameters is not globally concave, and the joint estimation of the seven parameters is highly unstable). Thus, we first estimated the covariance matrix using the OLS residuals. Then we used this estimated covariance matrix in the bivariate normal likelihood to obtain estimates of the intercepts and slopes. Results (both after a first iteration, already efficient, and after several iterations of the two-step procedure with strong convergence criteria) were generally consistent with those of GMM, except for some subsets of the data. This discrepancy, and the fact that BVN is not implied by the model, led us to stick to the GMM results, which are more robust to nonnormal behaviour of the errors.

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It is important to acknowledge that the discrete version of the process in (5) and (6) is only an approximation of the continuous-time specification. However, for short time sampling intervals such as the one we use in our study (one week), this approximation is almost exact; see for example Schwartz (1997).

We also estimate a one-factor Vasicek model for the domestic term structure. The results of this estimation are used as a benchmark to compare them with the results of the convergence model. We also use the GMM method for consistency with the previous estimation.

In the Vasicek case, the discrete-time version is

$$r_{st+1} - r_{st} = c(d - r_{st})\Delta t + \sigma_s \xi_t \sqrt{\Delta t}$$

where, as before, ξ_t are i.i.d. draws from a standard normal distribution and the r_{st} are observations of the Spanish short-term interest rates at each moment in time. Defining

$$\epsilon_{st} = r_{st+1} - r_{st} - c(d - r_{st})\Delta t$$

the moment equations are

$$E[\epsilon_{st}] = 0 \qquad E[\epsilon_{st}^2] = \sigma_s^2 \Delta t \qquad E[r_{st}\epsilon_{st}] = 0$$

We again use the information in the third moment, $E[\epsilon_{st}^3] = 0$. In this case, under the null hypothesis (the third moment is zero), the quadratic form after convergence will follow a χ^2 distribution with one degree of freedom.

To use the pricing models developed, we still need to obtain parameter values for the markets price of risk (λ_s and λ_e). To do so, for each model, we search over different values of λ until we minimize the prediction error i.e. root mean squared error (RMSE), over a sample of Spanish and European bonds. This is computed by

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[P_i - A_i]^2}$$

where N is the number of observations, P_i is the model predicted price and A_i is the actual market price for the ith observation.

4.2. *Data*

We follow standard practice and treat the one-month interest rate as a proxy for the instantaneous rate.

The short-term interest rates used are the one-month interbank middle (between bid and ask) rate in the case of the Spanish market and the middle interest rate for one-month deposits in the ECU one. We use weekly data from

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September 1990 to December 1997. Therefore we have 382 weekly observations of the interest rates. The period chosen was determined by the availability of data and the need to have enough observations to perform the estimations. The database used is Datastream.

Tables 1 and 2 provide the descriptive statistics of the interest rate data, and Figure 1 shows the evolution of both rates.

To obtain the weekly prices for European and Spanish zero-coupon discount bonds, we use the estimation of the yield curve[12] with maturities 1, 2, 3, 5 and 10 also available in Datastream. Discount bond prices given in terms of the yields are

$$P(t, T) = \exp[-(T - t)Y(t, T)]$$

where $P(t, T)$ is the price at t of the discount bond maturing at T, and $Y(t, T)$ is its corresponding yield.

We have a total of 1910 discount bond prices (five observations per week during 382 weeks) for each class of bonds (European and Spanish).

The first four months of 1998 are used for prediction purposes. Again, we have five observations per week corresponding to bonds with 1, 2, 3, 5 and 10 years to maturity which gives a total of 90 observations. Note that the prediction period is not used for the computations of the parameters of the interest rates processes, nor of the market prices of interest rate risk.

Table 1: Descriptive Statistics

	r_s	dr_s	r_e	dr_e	$r_e - r_s$
Mean	0.10	−0.00026	0.071357	−0.00014	−0.0292
Standard deviation	0.0319	0.0046	0.0244	0.00226	0.0129
Skewness	0.197	−2.0087	0.2228	0.2556	−1.01
Kurtosis	−1.12	70.48	−1.4743	14.4	2.28
Maximum	0.1812	0.0431	0.12687	0.01437	−0.0055
Minimum	0.048	−0.0544	0.04	−0.015	−0.0929

Notes: Weekly data: 9/6/90–31/12/97

Table 2: Correlations

	dr_e	$r_e - r_s$		r_s
dr_s	0.08	0.1948	r_e	0.929

Notes: Data from 9/6/90–31/12/97

[12] We use the curve to the power of three.

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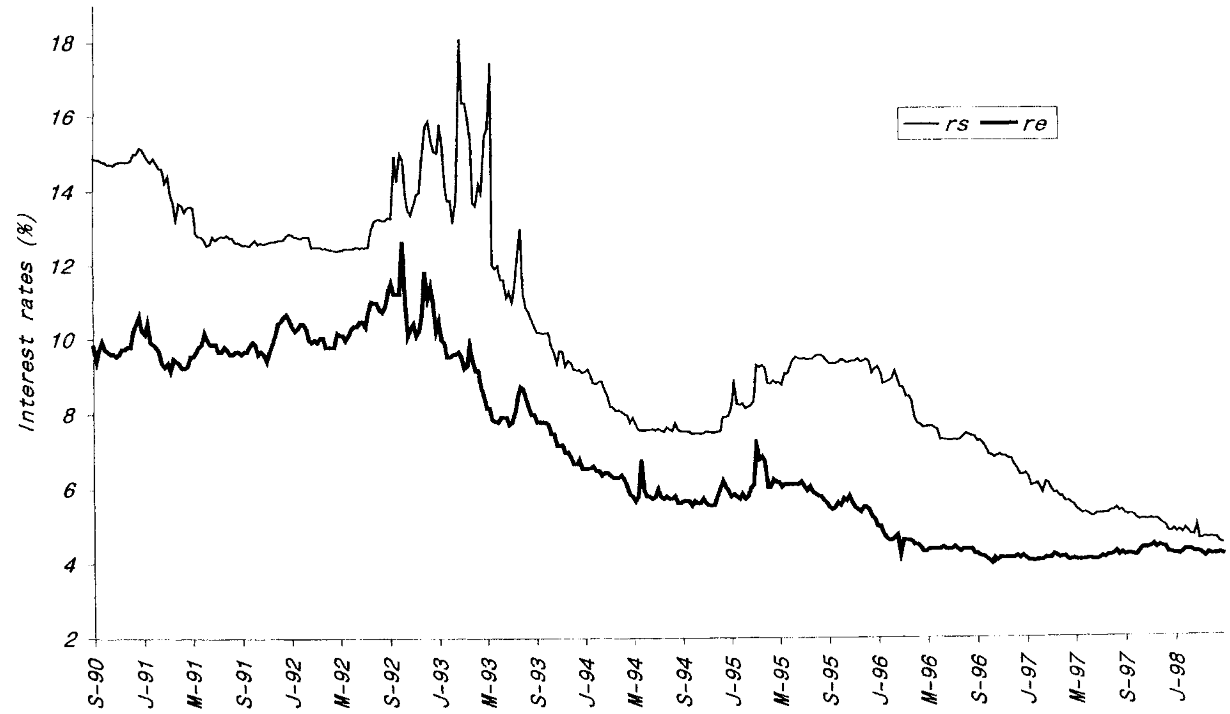

Figure 1: Short-term interest rates evolution

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4.3. *Results*

The results of the estimation for the parameters of the interest rates processes for the convergence model are presented in Table 3. The coefficients are more significant in the case of r_s than in the case of r_e. The coefficient that measures the speed of adjustment, b, turns out to be the most significant parameter, thus confirming the reversion character of the Spanish interest rates towards the European ones. As shown in the table, we cannot reject the null hypothesis that the third moments are zero.

The standard errors of the intercepts and slopes obtained with GMM are similar to the White-corrected errors (those corrected for heteroscedasticity). This result does not hold for the standard errors of the variances and covariance: with GMM estimation the standard errors estimated differ substantially from the maximum likelihood or least squares ones, and are only valid asymptotically. This is the reason why we do not report the t-ratio of the standard deviations and the correlation.

Figure 2 graphs the weekly changes in the Spanish interest rate and its weekly drift over the sample period. Excluding the period from September 92 to September 93 which corresponds to the European monetary crisis, changes in the Spanish interest rate followed closely the drift of the convergence model.[13]

Using a cross-section of ECU bond prices with 1, 2, 3, 5 and 10 years to maturity during the period September 1990 to December 1997, Table 4 shows

Table 3: Method of Moment Estimation for Convergence Model specified in equations 5 and 6

	a	b	σ_s	c	d	σ_e	ρ	χ^2_2
Coeff.	0.0938	3.67	0.032	0.2087	0.035	0.016	0.219	0.0003
t-stat	1.72	1.85		0.43	0.79			

Notes: Weekly data: September 1990 December 1997; 382 obs; annualized coefficients; 52 weeks per year

Table 4: Estimation of the European Market Price of Risk Corresponding to the Parameters in Table 3

Years to maturity	No. obs	$\hat{\lambda}_e$	RMSE	ME
1, 2, 3, 5, and 10	1910	−0.655	0.0241	0.017

Notes: This table reports the price of risk using weekly data of a panel of European bonds. We report the in-sample pricing errors: mean sum of squared errors, (MSSE), root mean squared errors (RMSE), mean error (ME). Estimation period: September 1990 to December 1997.

[13] This can be taken as evidence of model misspecification, regime shifts or stochastic volatility; there is already some research in this direction concerning the Spanish market. Regarding regime shifts Gómez-Biscarri (1999) can be consulted. Evidence about stochastic volatility can be found in Corzo and Gómez-Biscarri (1999).

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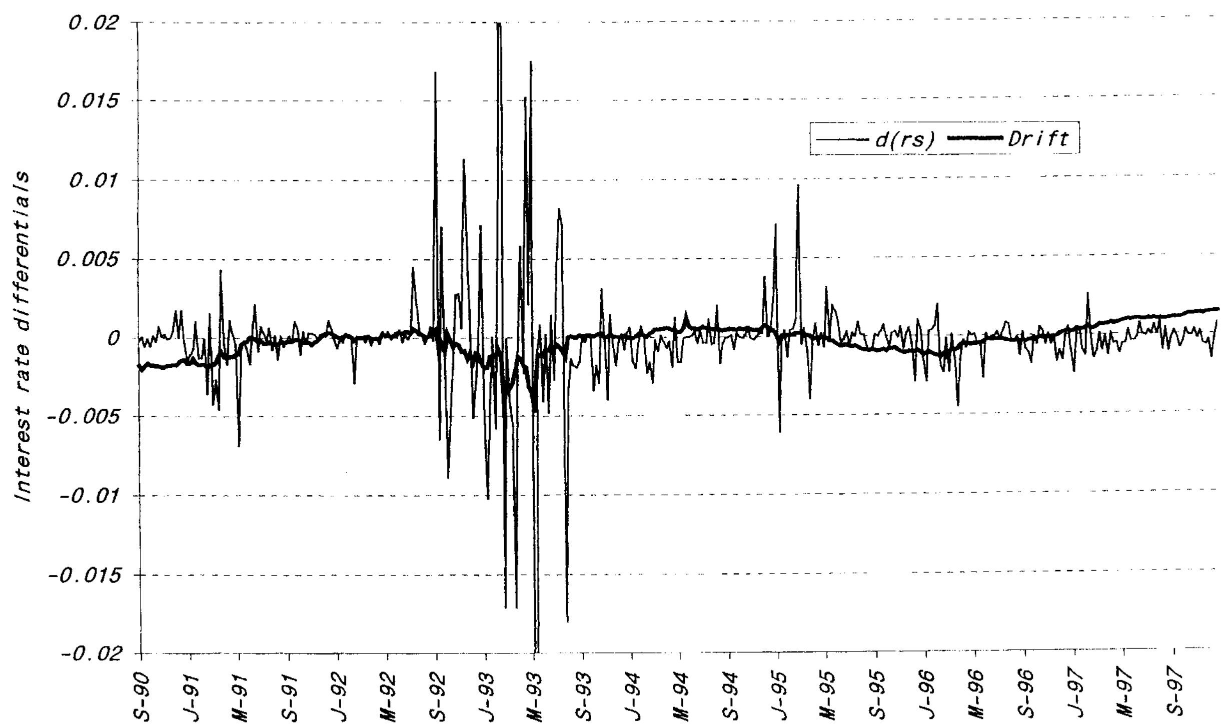

Figure 2: Interest rate variations around drift level; convergence model

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that the best fit was achieved for a market price of European interest rate risk of $\lambda_e = -0.655$.

Using a cross-section of Spanish bond prices with 1, 2, 3, 5 and 10 years to maturity, the estimation of the Spanish price of risk and the errors incurred can be found in Tables 5 and 6. Using the previously estimated $\lambda_e = -0.655$, we obtain $\lambda_s = 3.315$. The in-sample RMSE is 3.9 per cent and the mean error is 0.14 per cent. RMSE raises from 0.95 per cent for 1-year bonds to 6.1 per cent for 10-year bonds.

Tables 7 and 8 show the same set of parameter estimates as Table 3 when we divide the sample period into two equal subperiods. This allows us to see – by looking at the t-statistic of b – that the convergence has been more significative during the last three and a half years, although it has also been important during the first subperiod. The parameter a losses importance at the end reflecting that the rates are drawing closer so we are not able to estimate with confidence the level of a, although we are able to do it for the whole period.

Table 5: Estimation of the Spanish Market Price of Risk Corresponding to the Parameters in Table 3

		$\lambda_e = -0.655$		
Years to maturity	No. Obs	λ_s	RMSE	ME
1	382	3.465	0.0095	−0.000007
2	382	3.020	0.0215	0.000204
3	382	3.105	0.033	0.000464
5	382	3.39	0.048	0.000900
10	382	3.391	0.061	0.002860
1, 2, 3, 5 and 10	1910	3.315	0.039	0.001446

Notes: This table reports the prices of risk using weekly data. In-sample pricing errors. Period: September 1990 to December 1997.

Table 6: Out-of-sample Test; Convergence Model

		$\lambda_e = -0.655$	$\lambda_s = 3.315$
Years to maturity	No. obs	RMSE	ME
1	18	0.0182	−0.01818
2	18	0.03967	−0.03963
3	18	0.0649	−0.0649
5	18	0.11357	−0.11348
10	18	0.1766	−0.1764
1, 2, 3, 5 and 10	90	0.100	−0.0825

Notes: Prediction period: 18 weeks, January 1998–May 1998 (estimation period: September 1990–December 1997)

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Table 9 reports the results of the estimation for the Vasicek model. The speed of adjustment coefficient, though positive, is not significant. Again, we cannot reject that the third moment is zero. In Table 10, we report the estimation of the Spanish market price of interest rate risk and the errors. In this case, $\lambda_s = -0.4316$, the in-sample RMSE is 4.3 per cent and the mean error 0.19 per cent. The fit of the model is worse than the one obtained with the convergence model. It can also be seen that the errors worsen in the Vasicek model as the time to maturity of the bond grows. For 1-year bonds, the RMSE is 0.93 per cent for the convergence model and 0.95 per cent for the

Table 7: Estimation of the Convergence Model

	a	b	σ_s	c	d	σ_e	ρ	χ_2^2
Coeff.	0.1877	6.0639	0.0457	0.1869	0.0346	0.0198	0.20	9.15e−06
t-stat	1.5	1.54		0.3	0.4			

Notes: September 1990–April 1994; 191 observations; annualized coefficients

Table 8: Estimation of the Convergence Model

	a	b	σ_s	c	d	σ_e	ρ	χ_2^2
Coeff.	−0.0085	0.687	0.0087	1.01	0.045	0.0101	0.31	0.0058
t-stat	0.95	1.76		1.26	0.98			

Notes: May 1994–December 1997; 191 observations; annualized coefficients

Table 9: Estimation of a Vasicek Model for the Spanish Term Structure

	c	d	σ_s	χ_1^2
Coeff.	0.524	0.0747	0.032	0.0003
t-stat	1.47	1.29		

Notes: Weekly data September 1990–December 1997; 382 obs.; annualized coefficients; 52 weeks per year

Table 10: Estimation of the Spanish Market Price of Risk Corresponding to c, d and σ_s

Years to maturity	No. obs	λ_s	RMSE	ME
1	382	−0.39	0.0093	0.00012
2	382	−0.395	0.022	0.00039
3	382	−0.395	0.0348	0.00093
5	382	−0.407	0.0525	0.00172
10	382	−0.449	0.069	0.00297
1, 2, 3, 5 and 10	1910	−0.4316	0.0433	0.00191

Notes: This table reports the prices of risk using weekly data; in-sample pricing errors

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Vasicek model, but for 10-year bonds, the RMSE with Vasicek are 6.9 per cent and with the convergence model are 6.1 per cent. The convergence model seems to be more adequate to value long-term interest rates derivatives.

The average out-of-sample RMSE (Table 11) is 10.0 per cent for the convergence model and 10.5 per cent for the Vasicek one. Once more, the differences between both models is greater as the time to maturity of the bonds grows longer.

Table 11: Out-of-sample Test, Vasicek Model

		$\lambda_s = -0.4316$	
Years to maturity	No. obs	RMSE	ME
1	18	0.0155	−0.0155
2	18	0.0415	−0.0415
3	18	0.0698	−0.0698
5	18	0.1193	−0.1192
10	18	0.1829	−0.1827
1, 2, 3, 5 and 10	90	0.105	−0.0857

Notes: Out-of-sample pricing errors for Spanish bonds
Prediction period: 18 weeks, 7/January/98–30/April/98
Estimation period: September 1990–December 1997

5. Conclusions

We have developed a two-factor term structure of interest rates model that applies to EMU countries, and provided closed-form solutions for the prices of bonds and European options on zero-coupon discount bonds.

The model is a stochastic mean reverting model. The first factor is identified with the level of the short-term interest rate of the country participating in EMU, and the second factor is identified with a 'central rate' or a rate of the debt issued by the European Central Bank. The economic intuition underlying the model is provided by the fact that after EMU is established, each country will be able to issue national debt contemporaneously with the debt issued by the Central Bank. With a fixed exchange mechanism or with a single currency, economic theory and empirical evidence indicate that the interest rates of both debts will have to be very close, but not necessarily equal. The fact that domestic interest rates are converging to the European rate should be taken into account for valuation purposes, especially for the case of interest rates derivatives.

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One important contribution of the model is that the prices of risk of both factors can be estimated from easily accessible data.

To assess the model's performance with some degree of confidence, we need to wait until the EMU has been in place for some time. Nevertheless, we do a preliminary estimation for the case of the Spanish term structure taking the one-month interbank interest rate as a proxy for the first factor and the one-month interest rate on the ECU deposits as the stochastic central mean. We obtain a good in-sample and an out-of-sample fit. We also provide a comparison with the results of a Vasicek model estimation for the Spanish interest rate. It is shown that, when valuing zero-coupon bonds, the errors are smaller with our convergence model, and that this improvement is especially marked for long-term bonds.

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APPENDIX A

Proof of Proposition 2

If $P(r_d, r_e, \tau)$ is the price of a discount bond with face value 1 domestic currency and τ years to maturity dependent on $r_d(r_d, r_e, \tau)$ (1) by Ito's lemma, P must follow the SDE:

$$\frac{dP(r_d, r_e, \tau)}{P} = P_{r_d} dr_d + P_{r_e} dr_e + P_\tau d\tau + \frac{1}{2}\sigma_d^2 P_{r_d r_d} d\tau + \frac{1}{2}\sigma_e^2 P_{r_e r_e} d\tau$$

$$+ \rho\sigma_d\sigma_e P_{r_d r_e} d\tau$$

where P_{r_d}, P_{r_e}, P_τ, $P_{r_d r_d}$, $P_{r_e r_e}$, $P_{r_d r_e}$ denote partial derivatives.

Standard no-arbitrage conditions will lead us to obtain the following differential equation for the bond price

$$E(DP) - r_d P - \lambda_d \sigma_d P_{r_d} - \lambda_e \sigma_e P_{r_e} = 0$$

where D denotes the Dynkin differential operator, λ_d is the market price of the domestic interest rate risk and λ_e is the market price of the European interest rate risk. Alternatively, we can write:

(A.1)

$$[a + b(r_e - r_d) - \lambda_d\sigma_d]P_{r_d} + [c(d - r_e) - \lambda_e\sigma_e]P_{r_e} + \frac{1}{2}\sigma_d^2 P_{dd}$$

$$+ \frac{1}{2}\sigma_e^2 P_{ee} + \rho\sigma_d\sigma_e P_{de} - P_t - r_d P = 0$$

The boundary condition for this PDE is $P(r_d, r_e, 0) = 1$

We guess a solution of the form (4).[14] These class of solutions are often

[14] This is quite standard, as we know that assuming affine functions for the drift and variance terms in the processes for our state variables assures an affine solution; see Duffie and Kan (1996).

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denoted as the 'exponential-affine' form, following the work by Duffie and Kan [1996].

Substituting the derivatives of the posited guess into equation (A.1), and then simplifying by separating terms as coefficients of r_s and r_g, we arrive at the following transformation of the PDE:

$$r_d B(\tau)b + B_t - 1] + r_e[C(\tau)c - B(\tau)b + C_t] + \tag{A.2}$$

$$[-aB(\tau) + B(\tau)\lambda_d\sigma_d - C(\tau)cd + C(\tau)\lambda_e\sigma_e$$

$$+ \frac{1}{2}\sigma_d^2 B^2(\tau) + \frac{1}{2}\sigma_e^2 C^2(\tau) + C(\tau)B(\tau)\rho\sigma_d\sigma_e - A_t] = 0$$

For (A.2) to be uniformly satisfied over the support of r_d and r_e, all three of the terms in brackets must be equal to zero. The solutions for $A(\tau)$, $B(\tau)$, and $C(\tau)$ in (4) are found by integrating a system of ODEs:

$$B(\tau)b + B_t - 1 = 0$$

$$C(\tau)c - B(\tau)b + C_t = 0$$

$$-aB(\tau) + B(\tau)\lambda_d\sigma_d - C(\tau)cd + C(\tau)\lambda_e\sigma_e + \frac{1}{2}\sigma_d^2 B^2(\tau)$$

$$+ \frac{1}{2}\sigma_e^2 C^2(\tau) + C(\tau)B(\tau)\rho\sigma_d\sigma_e - A_t = 0$$

subject to three boundary conditions: $B(0) = 0$, $C(0) = 0$ and $A(0) = 0$. ■

APPENDIX B

Derivation of $v_d^2(t, T_C)$, $v_e^2(t, T_C)$.
Given the specifications

$$dr_e = c(d - r_e)dt + \sigma_e dz_e$$

and

$$dr_d = [a + b(r_e - r_d)]dt + \sigma_d dz_d$$

the processes followed by $r_e(\tau)$ and $r_d(\tau)$, with $\tau = Tc - t$, are respectively

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$$r_e(\tau) = \mathrm{e}^{-c\tau} r(t) + cd\int_0^\tau \mathrm{e}^{-c(\tau-u)}du + \sigma_e\int_0^\tau \mathrm{e}^{-c(\tau-u)}dz_e(u)$$

$$r_d(\tau) = \mathrm{e}^{-b\tau}[r_d(t) + a\int_0^\tau \mathrm{e}^{bs}ds + b\int_0^\tau \mathrm{e}^{bs}(\mathrm{e}^{-c\tau}r(t) + cd\int_0^\tau \mathrm{e}^{-c(\tau-u)}du$$

$$+ \sigma_e\int_0^\tau \mathrm{e}^{-c(\tau-u)}dz_e(u))ds + \sigma_d\int_0^\tau \mathrm{e}^{bs}dz_d(s)]$$

In the case of $r_e(\tau)$ it is well known (Jamshidian, 1989) that the variance of the process is

$$\nu_e^2(\tau) = \sigma_e^2\frac{1-\mathrm{e}^{-2c(\tau)}}{2c}$$

In the case of $r_d(\tau)$ the random part can be written as

$$b\sigma_e\int_0^\tau \mathrm{e}^{-b(\tau-s)}\int_0^\tau \mathrm{e}^{-c(\tau-u)}dz_e(u)ds + \sigma_d\int_0^\tau \mathrm{e}^{-b(\tau-s)}dz_d(s)$$

and, by Fubini's Theorem,

$$b\sigma_e\int_0^\tau \mathrm{e}^{-b(\tau-s)}\int_0^\tau \mathrm{e}^{-c(\tau-u)}dz_e(u)ds = b\sigma_e\int_0^\tau \mathrm{e}^{-c(\tau-u)}\int_0^\tau \mathrm{e}^{-b(\tau-s)}ds\,dz_e(u)$$

or,

$$\sigma_e(1-\mathrm{e}^{-b(\tau)})\int_0^\tau \mathrm{e}^{-c(\tau-u)}dz_e(u)$$

So, the variance of $r_d(\tau)$ is

$$\nu_d^2(\tau) = \sigma_e^2\frac{1-\mathrm{e}^{-2c(\tau)}}{2c}(1-\mathrm{e}^{-b(\tau)})^2 + \sigma_d^2\frac{1-\mathrm{e}^{-2b(\tau)}}{2b}.$$

Non-technical Summary

The economic and political changes that are taking place in Europe are affecting financial markets, and interest rates are no exception. Changes in the European level of interest rates affect the domestic rates and we can no longer study these in isolation. The paper develops and estimates a two-factor model, called the *convergence model*, for the term structure of interest rates specially designed to apply to EMU countries. It takes into account the influence of the European rate on the behaviour of interest rates of EMU countries.

EMU participant countries fixed their bilateral exchange rate against the euro in January 1999. With fixed exchange rates, risk-free interest rates across countries should be the same. Nevertheless, even with a single currency, risk-

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free interest rates across countries may be slightly different, reflecting different sovereign risks. Because participants' countries are allowed to issue domestic debt contemporaneously with the debt issued by the European Central Bank, small differences in domestic interest rates exist. Since the level of interest rates will affect prices of interest rates derivatives, developing a model that incorporates both the domestic rate and the central rate will improve the valuation and hedging of these instruments.

The convergence model is a stochastic mean reverting model. Following common practice, the first factor is identified with the level of the short-term rate. The second factor is identified with the central tendency of the short rate, which itself changes stochastically over time. It is assumed that the central or 'European' rate follows a standard mean reverting process.

The term structure model is exponentially affine and closed-form solutions for the prices of bonds and European options on discount bonds are obtained. In the model presented, the market price of risk for both stochastic processes is priced and can be estimated directly from the data, without any need for further assumptions.

In this context, the model is tested for the Spanish term structure. The short-term interest rate of the ecu's deposits is used as a proxy for the central tendency. This is distinct from previous work in which the stochastic mean is treated as an unobservable. Proxying the mean rate by an observable allows us to more efficiently investigate the pricing properties of the convergence model.

The method used for estimation purposes is the generalized method of moments. For a cross-section of Spanish discount bonds during the period June 1990 to December 1997, the average in-sample root mean square error (RMSE) for the convergence model is smaller than that of a Vasicek Model (3.9 per cent versus 4.3 per cent). The average out-of-sample errors is also smaller for the convergence model. The differences in favour of the new model are greater, the longer the maturity of the bonds for both in-sample and out-of-sample fit.

The characterization of the term structure dynamics developed in this paper not only applies to a fixed-exchange or single-currency situation, but also to any situation in which markets anticipate that interest rates are forced to converge.

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