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Throwing away a billion dollars: the cost of suboptimal exercise strategies in the swaptions market ☆

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Abstract

This paper studies the costs of applying single-factor exercise strategies to American swap options when the term structure is actually driven by multiple factors. Using a multifactor string market model of the term structure, we find that even when single-factor models are recalibrated to match the market at every exercise date, the exercise strategies they imply can be suboptimal. Based on estimates of notional amounts

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outstanding, the total present value costs of following single-factor strategies could be several billion dollars. These results illustrate the importance of using well-specified term structure models.

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1. Introduction

In the past few years, interest rate swaps have become one of the most important sectors in the global fixed income markets. The Bank for International Settlements estimates that the notional amount of interest rate swaps outstanding at the end of 1999 was $43.9 trillion, nearly eight times the $5.8 trillion notional value of Treasury debt outstanding. Given the size of the swap market, it is not surprising that options to enter into swaps, or to cancel existing swaps, represent one of the most widely used classes of fixed income derivatives. Known as swap options or swaptions, the total notional amount of these derivatives outstanding at the end of 1999 was on the order of $4.6 trillion, far exceeding the $15 billion notional value of all Chicago Board of Trade Treasury note and bond futures options combined.[1]

The importance of the swaptions market derives from the key role that swaptions play in corporate finance. Industry sources estimate that more than 50% of new agency and corporate debt issues are immediately swapped from fixed into floating or from floating into fixed. Debt issuers that swap their debt typically want the right to cancel the swap at future points in time. Similarly, debt issuers that do not immediately swap their debt often want the right to enter into a prespecified swap at later dates. Thus, swaptions arise as a natural outgrowth of efforts by debt issuers to preserve their flexibility throughout the financing cycle. Because of the long maturity of many debt issues, swaptions are often exercisable over horizons that span decades. Intuitively, swaptions can be viewed as calls or puts on coupon bonds, and our results are directly applicable to callable and puttable bonds. As with callable and puttable debt issues, a high percentage of swaptions have American-style exercise features.

Despite their importance, however, there are few areas in finance where there is a greater divergence between theory and practice than for American-style swaptions. On one hand, there is an extensive and well-established body of

[1] Estimated swaption notional amount based on summary statistics from the Bank for International Settlements and the International Swaps and Derivatives Association.

research showing that the term structure is driven by multiple factors.[2] On the other hand, many Wall Street firms use simple single-factor models in valuing, hedging, and exercising American-style swaptions. If term structure dynamics are driven by multiple factors, however, then the exercise strategies implied by single-factor models may be far from optimal, resulting in significant erosion of the value of the swaption to the optionholder as well as in hedging and dynamic replication errors.

This paper studies the costs of following single-factor exercise strategies for American-style swaptions in a realistic term structure framework with multiple factors. Our approach consists of first simulating paths of the term structure using the multi-factor model and solving for the value of the American-style swaption when the optimal exercise strategy is followed. Using the same paths of the term structure, we then solve for the value of the American-style swaption by recalibrating a single-factor model to the market at each exercise date and determining whether exercise is implied by the single-factor model at that exercise date. The difference between the values of the swaption under the optimal exercise strategy and the single-factor strategy directly measures the present value cost of following the suboptimal single-factor strategy. In repeatedly recalibrating the single-factor model, this approach closely parallels standard market practice in which single-factor models are continually recalibrated to a cross section of market prices to compensate for their inability to capture the dynamics of the term structure.

The benchmark term structure framework used in this study is a multifactor string market model similar to that used by Longstaff et al. (2001). String market models blend the market-model framework of Brace et al. (1997) and Jamshidian (1997) with the string-shock framework of Kennedy (1994, 1997), Goldstein (2000), Santa-Clara and Sornette (2001), Longstaff and Schwartz (2001), and others, and have the important advantages of being easily calibrated and providing rich multi-factor descriptions of the dynamic behavior of the term structure. Using a four-factor specification, we calibrate the string market model to match closely the market prices of an extensive set of European swaption and interest rate cap prices. The resulting benchmark four-factor string market model has the advantage of being fully time-homogeneous. Optimal exercise strategies and American-style swaption values in the four-factor model are easily determined using the least squares Monte Carlo (LSM) technique of Longstaff and Schwartz. As single-factor alternatives to the benchmark term structure model, we use the well-known and widely used Black et al. (1990) and Black and Karasinski (1991) models.

[2] For example, see Brown and Dybvig (1986), Stambaugh (1988), Litterman and Scheinkman (1991), Longstaff and Schwartz (1992), Chen and Scott (1993), Gibbons and Ramaswamy (1993), Knez et al. (1994), Pearson and Sun (1994), Balduzzi et al. (1996), Duffie and Singleton (1997), Dai and Singleton (2000a), Piazzesi (2000), Liu and Longstaff (2000), and Longstaff et al. (2001).

This study contributes to the growing literature on the economics of misspecified derivatives models in three ways.[3] First, we find that single-factor exercise strategies are suboptimal when the term structure is driven by a realistic multi-factor model. For many common American swaption structures, the present value loss from following a single-factor exercise strategy can be as large as 10–30 cents per $100 notional. While these losses are on the order of the size of the bid–ask spread, they are nonetheless economically significant for swaption holders who can avoid the losses by simply following the optimal strategy. Given the enormous size of the swaptions market, the aggregate present value costs to swaption holders from following single-factor exercise strategies are easily as large as several billion dollars.

Note that these present value costs reflect the average cost over all paths. For some paths, however, the costs can be much higher. To see this, note that for a large number of paths, the swaption is always out of the money and any model would tell the user not to exercise. Similarly, there are many paths where the swaption is so deep in the money that any reasonable model would tell the user to exercise. On the other hand, it is precisely when the exercise decision becomes a tough call that it is most important to have a good model; for these paths, the difference between single-factor and multi-factor exercise strategies can be very large. For example, the present value costs of exercising when the single-factor model signals exercise but the multi-factor does not can be as large as $1.25 per $100 notional. This adds an entirely new dimension to the potential effects of model risk.

Second, the results demonstrate that the popular practice of continually recalibrating a misspecified single-factor model to match a cross section of market prices does not fully compensate for its failure to capture term structure dynamics. Intuitively, this is clear since no matter how well a single-factor model is parameterized to match the cross section of market prices, it still implies that changes along the term structure are perfectly correlated and only allows the term structure to evolve in a very limited way. The dynamics of the term structure, however, are fundamental in determining the value of American swaptions because of the rich intertemporal nature of the optimal stopping problem.

Third, we show that the value of an American option given by a misspecified model can be a seriously biased measure of the actual present value of cash flows generated by following the model's exercise strategy. Furthermore, this bias can go in either direction. To illustrate, imagine that in a multi-factor

[3] Other recent papers focusing on the pricing and hedging effects of using misspecified derivatives models include Dybvig (1988), Chan et al. (1992), Derman (1996), Dybvig (1997), Bakshi et al. (1997), Backus et al. (1998), Dumas et al. (1998), Green and Figlewski (1999), Gupta (1999), Hull and White (1999), Jagannathan et al. (2000), Peterson et al. (1999), Brandt and Wu (2001), Buraschi and Corielli (2000), Dai and Singleton (2000b), Driessen et al. (2000), Hull and Suo (2000), and Buraschi and Jackwerth (2001).

world, the true value of an American option is $10. If an investor uses a single-factor model to make exercise decisions over time, however, the cash flows received by the investor might only have a present value of $9. Despite this, the value estimate that comes out of the misspecified single-factor model could easily be greater than $10 or less than $9 depending on how the model was calibrated. A particularly insidious case occurs when the single-factor model fortuitously gives a value estimate equal to the market price of $10. In this case, the investor might be reassured that his model is well specified and never be aware that the present value of his cash flows is less than the valuation given by his model. Note that in an efficient market, an American option is only worth its market value to an investor who follows the optimal strategy.

Although this paper addresses the exercise strategies of American-style swaptions, the insights we obtain from the analysis have more general applicability. In many other applications of financial and real options we make simplifying assumptions to reduce the dimensionality of the problem. Our analysis shows that caution is warranted, since in some cases, these simplifying assumptions can induce suboptimal behavior with significant economic implications. These results make clear the importance of using economically realistic models for financial and real options and illustrate some of the dangers of using overfitted, misspecified models.

The remainder of this paper is organized as follows. Section 2 provides an introduction to the swaptions market. Section 3 describes the string market model of the term structure. Section 4 explains how American-style swaptions are valued in the benchmark model as well as in the single-factor alternative models. Section 5 presents the results about the costs of following suboptimal strategies. Section 6 discusses how the value of an American option implied by a misspecified model can be a biased estimate of the present value of the cash flows generated by following the implied exercise policy. Section 7 summarizes the results and makes concluding remarks.

2. An introduction to swaptions

The underlying instrument for a swaption is an interest rate swap. In a standard swap, two counterparties agree to exchange a stream of cash flows over some specified period of time. One counterparty receives a fixed annuity and pays the other a stream of floating cash flows tied to the three-month Libor rate. Counterparties are identified as either receiving fixed or paying fixed in the swap.[4]

[4] For discussions about the economic role that interest rate swaps play in financial markets, see Bicksler and Chen (1986), Turnbull (1987), Smith et al. (1988), Wall and Pringle (1989), Macfarlane et al. (1991), Sundaresan (1991), Litzenberger (1992), Sun et al. (1993), Gupta and Subrahmanyam (2000), and Longstaff et al. (2001).

For a typical swaption, the underlying swap has a forward start date. In a forward swap with a start date of τ, fixed payments are made at time $\tau + 0.50, \tau + 1.00, \tau + 1.50, \ldots, T - 0.50$, and T, and floating rate payments are made at times $\tau + 0.25, \tau + 0.50, \tau + 0.75, \ldots, T - 0.25$, and T. Let $D(t, N)$ denote the value at time t of a discount bond with arbitrary maturity N. Abstracting from credit issues, the value of the floating leg equals par at the start date τ. Discounting this value at time τ back to time zero implies that the time-zero value of the floating cash flows is $D(0, \tau)$. Since the forward swap has a time-zero value of zero, the time-zero value of the fixed leg must also equal $D(0, \tau)$. This implies that the forward swap rate (required fixed coupon rate on the forward starting swap), $F(0, \tau, T)$, must satisfy

$$F(0, \tau, T) = 2\left[\frac{D(0, \tau) - D(0, T)}{\sum_{i=1}^{2(T-\tau)} D(0, \tau + i/2)}\right]. \tag{1}$$

After a swap is executed, the coupon rate on the fixed leg may no longer equal the current market swap rate and the value of the swap can deviate from zero. Let $V(t, \tau, T, c)$ be the value at time t to the counterparty receiving fixed in a swap with forward start date $\tau \geqslant t$ and final maturity date T, where the coupon rate on the fixed leg is c. The value of this forward swap is given by

$$V(t, \tau, T, c) = \frac{c}{2} \sum_{i=1}^{2(T-\tau)} D(t, \tau + i/2) + D(t, T) - D(t, \tau), \tag{2}$$

where the first two terms in this expression represent the value of the fixed leg of the swap, and the third term is the present value of the floating leg which will be worth par at time τ. For $t > \tau$, the swap no longer has a forward start date and the value of the swap on semiannual fixed coupon payment dates is given by the expression

$$V(t, \tau, T, c) = \frac{c}{2} \sum_{i=1}^{2(T-t)} D(t, \tau + i/2) + D(t, T) - 1. \tag{3}$$

Note that in either case, the value of the swap is just a linear combination of zero-coupon bond prices.

There are two basic types of European swaptions. The first is the option to enter a swap and receive fixed. For example, let τ be the expiration date of the option, c be the coupon rate on the swap, and T be the final maturity date on the swap. The holder of this option has the right at time τ to enter into a swap with a remaining term of $T - \tau$ and receive the fixed annuity of c. Since the value of the floating leg will be par at time τ, this option is equivalent to a call option on a bond with a coupon rate of c and a remaining maturity of $T - \tau$ where the strike price of the call is \$1. Alternatively, this can be viewed as a derivative with payoff at time τ equal to $\max(0, V(\tau, \tau, T, c))$. This option is generally called a τ into $T - \tau$ receiver's swaption, where τ is the maturity of the option and $T - \tau$ is the tenor of the underlying swap. This swaption is also known as a τ by T receiver's swaption. Note that if the option holder is paying fixed at rate c in a swap with a final maturity date of T, then exercising this option has the effect of canceling the original swap at time t since the two fixed and two floating legs cancel each other out. Observe, however, that when the option is used to cancel the swap at time τ, the current fixed for floating coupon exchange is made first.

The second type of European swaption is the option to enter a swap and pay fixed, and the cash flows associated with this option parallel those described above. An option that gives the option holder the right to enter into a swap at time τ with final maturity date at time T and pay fixed is generally termed a τ into $T - \tau$ or a τ by T payer's swaption. Again, this option is equivalent to a put option on a coupon bond where the strike price is the value of the floating leg at time τ of \$1. Alternatively, the payoff at time τ can be expressed as $\max(0, -V(\tau, \tau, T, c))$. A τ into $T - \tau$ payer's swaption can be used to cancel an existing swap with final maturity date at time T where the option holder is receiving fixed at rate c.

Although there are a number of different variations, the most common type of American-style swaption is the T noncall τ structure. A T noncall τ receiver's swaption gives the option holder the right to enter into a swap and receive fixed at any of the fixed coupon payment dates $\tau, \tau + 0.50$, $\tau + 1.00, \ldots,\ T - 1.00$, and $T - 0.50$. Similarly, a T noncall τ payer's swaption gives the option holder the right to enter a swap and pay fixed at the same coupon payment dates. As before, either of these structures can be used to cancel existing swaps at any of these coupon payment dates after making the coupon exchange for that payment date. These options are sometimes known as Bermuda swaptions, deferred American swaptions, or discrete American swaptions; for simplicity, we refer to them as American-style swaptions.

As with traditional American options, the value of an American-style swaption is greater than or equal to the value of its European counterpart. Since the underlying swap terminates at time T, the value of an American swaption converges to zero at time T. This implies that an American swaption should be exercised at time $T - 0.50$ if it is in the money. Hence, the value of an American swaption is identical to that of an equivalent European swaption at any time after the second to last coupon payment date for the underlying swap. Furthermore, consider a standard T noncall τ structure. Exercise dates for this swaption range from time τ to time $T - 0.50$. For each of these exercise dates, we can find a European swaption with the same exercise date on the same underlying swap. Designate the set of these European swaptions as the corresponding European swaptions. Standard no-arbitrage results can be used to show that the value of the American-style swaption must be greater than or equal to the maximum of the values of all corresponding European swaptions.

3. The valuation framework

As the benchmark term structure framework for this study, we use the string market model of Longstaff et al. (2001). This approach to modeling the term structure blends the market-model framework of Brace et al. (1997) and Jamshidian (1997) with the string-shock framework of Kennedy (1994, 1997), Goldstein (2000), Santa-Clara and Sornette (2001), Longstaff and Schwartz (2001), and others. String market models have the important advantages of being easily calibrated to the market prices of a wide variety of fixed income options while providing a rich multifactor description of the dynamics of the term structure.

Closely following Longstaff et al. (2001), we take the Libor forward rates out to 15 years, $F_i \equiv F(t, T_i, T_i + 1/2)$, $T_i = i/2$, $i = 1, 2, \ldots, 29$, to be the fundamental variables driving the term structure. As in Black (1976), we assume that the risk-neutral dynamics for each forward rate are given by

$$dF_i = \alpha_i F_i \, dt + \sigma_i F_i \, dZ_i, \tag{4}$$

where α_i is an unspecified drift function, σ_i is a deterministic volatility function, dZ_i is a standard Brownian motion specific to this particular forward rate, and $t \leqslant T_i$.[5] Note that while each forward rate has its own dZ_i term, these dZ_i terms are correlated across the forward rates.[6]

The correlation of the Brownian motions together with the volatility functions determine the covariance matrix of forwards Σ. To model the covariance structure among forwards in a parsimonious but economically sensible way, we make the assumption that the covariance between dF_i/F_i and dF_j/F_j is time-homogeneous in the sense that it depends only on $T_i - t$ and $T_j - t$. Although the assumption of time-homogeneity imposes additional structure on the model, it has the advantage of being more consistent with traditional dynamic term structure models in which interest rates are

[5] We assume that the initial value of F_i is positive and that the unspecified α_i terms are such that standard conditions guaranteeing the existence and uniqueness of a strong solution to Eq. (4) are satisfied. These conditions are described in Karatzas and Shreve (1988, Chapter 5). In addition, we assume that α_i is such that F_i is nonnegative for all $t \leqslant T_i$.

[6] A more general approach would be to allow the volatility parameters σ_i to vary over time according to some stochastic process. This would be consistent with the growing body of empirical evidence documenting that interest rate volatility is stochastic. For example, see Brenner et al. (1996), Anderson and Lund (1997), Koedijk et al. (1997), and Ball and Torous (1999). In addition, time series of both implied cap and swaption volatilities display persistent variation in their values; see Longstaff et al. (2001). Although the extension to stochastic volatility is beyond the scope of this paper, we note that the string market model framework can easily accommodate stochastic volatility by either appending the dynamics for individual σ_i terms or by introducing additional factors driving common variation in the σ_i terms.

determined by the fundamental state of the economy.[7] Furthermore, since our objective is to apply the model to swaps that make fixed payments semiannually, we make the simplifying assumption that these covariances are constant over six-month intervals. With these assumptions, the problem of capturing the covariance structure among forwards reduces to specifying a 29×29 time-homogeneous covariance matrix Σ.

Although the string is specified in terms of the forward Libor rates, it is often more efficient to implement the model using discount bond prices. By definition,

$$F_i = \frac{360}{a}\left[\frac{D(t, T_i)}{D(t, T_i + 1/2)} - 1\right], \tag{5}$$

where a is the actual number of days during the semiannual coupon period. Thus, the forward rates F_i can all be expressed as functions of the vector of discount bond prices with maturities 0.50, 1.00, ..., 15.00. Conversely, these discount bond prices can be expressed as functions of the string of forward rates, assuming that standard invertibility conditions are satisfied. The primary condition is that the determinant of the Jacobian matrix for the mapping from discount bond prices to forward swap rates be nonzero. If this condition is satisfied, local invertibility is implied by the Inverse Function Theorem. Applying Itô's Lemma to the vector D of discount bond prices gives

$$\mathrm{d}D = rD\,\mathrm{d}t + J^{-1}\sigma F\,\mathrm{d}Z, \tag{6}$$

where r is the spot rate, $\sigma F\,\mathrm{d}Z$ is the vector formed by stacking the individual terms $\sigma_i(t, T_i)\ F_i\,\mathrm{d}Z_i$ in the forward rate dynamics in Eq. (4), and J^{-1} is the inverse of the Jacobian matrix for the mapping from discount bond prices to forward rates. Since each forward depends only on two discount bond prices, this Jacobian matrix has a simple banded diagonal form; see Longstaff, Santa-Clara, and Schwartz. The dynamics in Eq. (6) provide a complete specification of the evolution of the term structure. This string market model is arbitrage free in the sense that it fits the initial term structure exactly and the expected rate of return on all discount bonds equals the spot rate under the risk-neutral pricing measure.

Rather than specifying the covariance matrix Σ exogenously, we follow the approach of Longstaff, Santa-Clara, and Schwartz by solving for the implied matrix Σ that best fits the observed market prices of a set of market data. First, we estimate the historical covariance matrix of percentage changes in forward rates H from a time series of forward rates. Specifically, we obtain month-end

[7] Longstaff et al. (2001), Andersen and Andreasen (2000), and others argue that it may be important to incorporate time-homogeneity in term structure models. We note, however, that the results in this paper are robust to the assumption of time-homogeneity. Results very similar to those reported in this paper were obtained in an earlier version of this paper which used a model that was not time-homogeneous.

Libor and swap rates from Bloomberg for the period from January 1989 to June 1999. Using a cubic spline, we estimate discount bond prices for each date and compute forward rates. We estimate the sample covariance matrix from these forward rates. We then decompose the historical covariance matrix into its spectral representation $H = U\Lambda U'$, where U is the matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues. Finally, we make the identifying assumption that the implied covariance matrix is of the form $\Sigma = U\Psi U'$, where Ψ is a diagonal matrix with nonnegative elements. This assumption places an intuitive structure on the space of admissible implied covariance matrices. Specifically, if the eigenvectors are viewed as factors, then this assumption is equivalent to assuming that the factors that generate the historical covariance matrix also generate the implied covariance matrix, but that the implied variances of these factors may differ from their historical values. Viewed this way, the identification assumption is simply the economically intuitive requirement that the market will price interest rate options based on the factors that drive term structure movements.

Given this specification, the problem of finding the implied covariance matrix reduces to solving for the implied eigenvalues along the main diagonal of Ψ that best fit the market data. Recent evidence by Longstaff, Santa-Clara, and Schwartz, however, suggests that the implied covariance matrix for European swaptions is of rank four. Motivated by this, we estimate only the first four eigenvalues and set the remaining eigenvalues to zero. We solve for these four implied eigenvalues using a standard numerical optimization where the objective function is the root mean squared error (RMSE) of the percentage differences between the market price and the model price. In this parameterization we solve for the implied Σ that best fits the July 2, 1999 values of the six caps and the 42 European swaptions with $T \leqslant 15$ shown in Table 1, where equal weight is given to caps and swaptions in the objective function. This model fits the market quite well and the RMSE taken over all 48 prices is only 3.54%. This RMSE is significantly smaller than the typical bid/ask spread for the caps and swaptions, which is on the order of 6–8% of their value.

Our choice of four factors should be viewed as an attempt to balance the risk of overfitting with the benefits of capturing the dynamics of the term structure correctly. To the extent that our model includes too few factors, however, our results are likely to understate the costs of suboptimal single-factor strategies. There are also other reasons for considering lower-dimensional specifications of the covariance matrix. For example, lower-dimensional models have the advantage of being more econometrically efficient and there may be less risk of overfitting with a more parsimonious specification; see Dai and Singleton (2000a). We also replicate our results using data from a number of other dates during 1997, 1998, and 1999. The results are virtually the same as those reported. Finally, we used alternative specifications in which all of the weight is placed on the swaptions, or where caps are fitted exactly and the swaptions are

Table 1
Broker swaption and cap volatilities

This table shows mid-market implied volatilities for the indicated at-the-money-forward European swaptions and interest-rate caps for July 2, 1999 as reported by the Bloomberg system. Swaption maturity represents the number of years until expiration for the swaption. Swap tenor refers to the length in years of the swap that the swaption holder enters into if the swaption is exercised. Cap maturity refers to the number of years until the maturity date of the final caplet.

Swaption maturity	Swap tenor 1	2	3	4	5	7	10
0.5	16.8	16.8	16.6	16.4	16.3	16.2	16.2
1	16.9	16.7	16.5	16.2	16.0	15.8	15.7
2	17.0	16.7	16.5	16.2	15.8	15.6	15.4
3	16.9	16.5	16.3	16.0	15.6	15.2	15.0
4	16.8	16.4	16.1	15.7	15.4	14.9	14.6
5	16.6	16.1	15.7	15.3	15.0	14.4	14.0
7	15.2	14.4	14.0	13.7	13.3	12.9	12.5
10	13.1	12.4	11.9	11.5	11.2	10.8	10.3

	Cap maturity 2	3	4	5	7	10
Cap volatility	17.25	18.50	18.62	18.62	18.20	17.20

fitted as closely as possible. The results are robust to the choice of market prices used in the calibration of the model.

4. Valuing American-style swaptions

Our approach to estimating the costs of following suboptimal exercise strategies consists of first generating paths of the term structure using the benchmark four-factor string market model. From these paths, we solve for the value of an American-style swaption where the optimal four-factor exercise strategy is used. Using the same paths, we then solve for the value of the American-style swaption based on the exercise strategy implied by a continually recalibrated single-factor model. The difference between the two American-style swaption values represents the cost of following the suboptimal single-factor exercise strategy. In this section, we first describe how we solve for the optimal exercise strategy in the benchmark four-factor model. We then describe how the single-factor alternative models are recalibrated to the market at each exercise date and then used to define single-factor exercise strategies. This approach to estimating the costs of suboptimal exercise follows Green and Figlewski (1999) and Hull and Suo (2000) and is referred to as the 'full-fledged simulation experiment' by Andersen and Andreasen (2000). Although similar to the approach used in earlier versions of this paper, this approach has the advantage of more closely paralleling actual market practice in which single-factor models are continually recalibrated to match the market.

In valuing American-style swaptions in the benchmark four-factor term structure model, we use the least squares Monte Carlo (LSM) simulation technique of Longstaff and Schwartz (2001). There are several key reasons for using this valuation methodology. For example, standard binomial or finite difference techniques are not computationally feasible because of the high dimensionality of the string model. Furthermore, Longstaff and Schwartz demonstrate that the LSM algorithm is accurate and computationally efficient. Finally, the general convergence properties of the LSM algorithm have recently been demonstrated by Tsitsiklis and Van Roy (2001).

The key to the LSM approach is the fact that at any exercise date, the optimal stopping strategy for an American option is determined by comparing the value of immediate exercise with the value of continuing to keep the option alive. From standard option pricing theory, however, this continuation value can be expressed as a conditional expectation under the risk-neutral measure. The LSM approach estimates this conditional expectation using the cross-sectional information about the term structure in the simulation. Specifically, the LSM approach regresses the discounted ex post cash flows from continuing along each path onto functions of the current values of the state variables. The fitted value from this regression is an efficient estimator for the conditional expectation function. Exercising the option whenever the immediate exercise value is greater than the estimated value of continuation defines a simple stopping time rule, which Longstaff and Schwartz (2001) show closely approximates the optimal stopping rule. In applying the LSM technique, we use the first three powers of the value of the underlying swap, the first three powers of each of the corresponding forward swaps, and the cross products of the values of the current swap and the forward swaps up to degree three as conditioning variables in forming conditional expectations. We explore numerous alternative forms for the basis functions, but the results were virtually identical to those reported for the specification we use.[8]

Since the optimal exercise strategy at any exercise date is determined by the conditional expectation function, we can gain some intuition about the differences between single-factor and multifactor models by examining their implications for the conditional expectation function. One well-known property of single-factor term structure models is their implication that changes in the term structure are perfectly instantaneously correlated across

[8]The LSM algorithm has also been applied in solving dynamic investment portfolio problems. For example, see Longstaff (2001) and Brandt et al. (2001). Simulation techniques are also used by Carr and Yang (2001) in valuing interest-rate options.

maturities. It is important to observe that this feature is due entirely to the fact that the time-series properties of the term structure are driven by a single Brownian motion, and has nothing to do with how the single-factor model is fitted (or overfitted) to match the cross section of current market prices. A direct implication of this perfect correlation is that once we condition on the current value of a single point along the term structure, all other information about the term structure becomes redundant in forming the conditional expectation. Thus, a single point along the term structure becomes a sufficient statistic for making exercise decisions in a single-factor model.

If the term structure is actually driven by multiple factors, then other information besides the current value of the underlying swap becomes useful in forming the conditional expectation. In a multifactor setting, the conditional expectation depends not only on the current value of the underlying swap, but on all forward values of the swap as well. Thus, because of the perfect correlation inherent in single-factor models, single-factor exercise decisions are made conditional on only a subset of the relevant term structure information. This is the sense in which single-factor models are myopic. In contrast, the optimal multifactor strategy is determined by the conditional expectation function which is based on all of the term structure information.

Intuitively, this result parallels Merton (1973) who shows that in a single-factor setting, the optimal early exercise decision for an American option is determined entirely by whether the value of the underlying asset exceeds a critical threshold. Thus, by its nature, a single-factor model implies that the exercise boundary is one-dimensional. In contrast, when the term structure is driven by multiple factors, the optimal exercise boundary is actually a multidimensional surface. In general, the multidimensional optimal exercise boundary cannot be well approximated by the one-dimensional exercise boundary implied by a single-factor model.

An alternative way of seeing this is from the perspective of the hedging or replication strategy. Because of the perfect correlation inherent in single-factor models, single-factor models imply that an American-style swaption can be perfectly hedged by trading in the underlying swap. If there are multiple factors, however, then an American swaption cannot be completely hedged by any single-security hedging portfolio, no matter how elaborately the single-factor model is fitted to the cross section of market prices. Fitting a model to a cross section of European swaption prices does not guarantee that the model will capture the time-series properties or dynamics of the term structure. Because of the intertemporal nature of the American swaption exercise problem, the optimal exercise decision depends crucially on the dynamics of the term structure. Thus, by failing to capture the dynamic behavior of the term structure, single-factor models inherently miss a key determinant of the optimal exercise strategy.

In studying the costs of using single-factor models, we implement the well-known and widely used Black et al. (1990) and Black and Karasinski (1991) models as the single-factor alternatives within the four-factor benchmark term structure framework. The Black, Derman, and Toy model is the default valuation model for American-style swaptions in the popular Bloomberg system, and the Black and Karasinski model is the alternative model in the system. The Bloomberg system also allows swaptions to be valued using the single-factor Hull and White (1990) model, but the user needs to provide additional calibration information.

In the Black, Derman, and Toy model, the short-term rate r follows the dynamic process

$$d \ln r = \left(\mu(t) + \frac{s'(t)}{s(t)} \ln r \right) \mathrm{d}t + s(t)\,\mathrm{d}Z, \tag{7}$$

where $\mu(t)$ is a drift function, $s(t)$ is a volatility function, and Z is a standard Brownian motion. This model is calibrated by fitting the drift function $\mu(t)$ and the volatility function $s(t)$ to match both the initial term structure and some set of fixed income option volatilities. Often $s(t)$ is chosen to be constant, and the model is calibrated to match the price of the τ into $T - \tau$ European swaption corresponding to the first exercise date of the American-style swaption. In the Black and Karasinski model, the short-term rate follows the slightly more general dynamic process

$$d \ln r = (\mu(t) - \beta(t) \ln r)\,\mathrm{d}t + s(t)\,\mathrm{d}Z, \tag{8}$$

where $\mu(t)$, $\beta(t)$, and $s(t)$ are again functions of time that are calibrated to match the current term structure and some set of fixed income option prices.

To illustrate how the single-factor models are implemented in our multifactor framework, we focus specifically on the case of the Black, Derman, and Toy model; the approach for the Black and Karasinski model is almost identical. In the first step, we use the string market model to simulate paths of the discount function. Let $D(i, i+t; j)$, where $i = 0.50, 1.00, 1.50, \ldots, 14.50$, and $t = 0.50, 1.00, 1.50, \ldots, 15.00 - i$, denote the vector of simulated discount bond prices for time i along path j. We next solve for the set of at-the-money interest caplet and European swaption prices that would be observed in the market at time i along path j given the four-factor model. Specifically, we solve for the prices of at-the-money caplets with exercise dates ranging from $i + 0.50$ to the final maturity date of the underlying swap and for the set of at-the-money European swaptions with exercise dates ranging from $i + 0.50$ to $T - 0.50$ and where the final maturity date of the underlying swap is T. In doing this, we parallel the LSM algorithm in computing these values by regressing the discounted ex post cash flows from these European style derivatives on the LSM set of conditioning variables described above. The fitted values from these regressions, computed at the values of the conditioning variables at time i and path j, provide unbiased measures of the values of these derivatives at time i and path j.

At time i along path j, we wish to recalibrate the Black, Derman, and Toy model to match the current term structure $D(i, i+t; j)$ and the prices of fixed income derivatives that would be observed at that point. From Eq. (7), this recalibration is equivalent to defining the functions $\mu(t; i, j)$ and $s(t; i, j)$ where the dependence on the date i and path j is made explicit. Conditional on the function $s(t; i, j)$, the function $\mu(t; i, j)$ is solved numerically by imposing the condition that the model match the current term structure $D(i, i+t; j)$. The next step is to solve for the volatility function $s(t; i, j)$ that best fits these derivative values in a RMSE sense. Note that since there are typically more caplets and swaptions than there are distinct values of $s(t; i, j)$, the single-factor Black, Derman, and Toy model cannot fit all of these fixed income derivative prices exactly. Given the numerically intensive nature of having to solve for the $s(t; i, j)$ functions for each i and j, we use the following parsimonious but economically sensible algorithm to minimize the computational requirements. For each i and j, we solve first for the $s(t; i, j)$ function that exactly fits the time i prices of the interest rate caplets. We then multiply this $s(t; i, j)$ function by a constant amount, where the constant is chosen to minimize the RMSE of the percentage differences between the estimated European caplet and swaption market prices and the prices implied by the Black, Derman, and Toy model. As before, the caps and swaptions are given equal weight in this minimization. The advantage of this approach is that we obtain a solution for $s(t; i, j)$ that provides the best overall fit to the market prices of caps and swaptions while preserving the general shape of the volatility function. Since we fit the current term structure $D(i, i+t; j)$ exactly at any exercise date for the American swaption, the underlying European swaption with the same exercise date and final maturity date is fitted exactly.

Once the $\mu(t; i, j)$ and $s(t; i, j)$ functions are specified at time i for path j, we construct a standard Black, Derman, and Toy binomial tree for time i and path j and value the American-style swaption using this tree. If the continuation value of the American-style swaption implied by the tree is less than the value of immediate exercise, the single-factor model then implies that the swaption should be exercised at time i along path j. Repeating this procedure for all times i and all paths j provides a complete specification of the single-factor exercise strategy. The value of the American-style swaption is then given by averaging over all paths the discounted cash flows given by applying the single-factor exercise strategy. Because multiple optimizations are needed at each date i and path j, computing the costs of suboptimal exercise for a single American swaption is very time consuming, taking as long as 60 hours on a Pentium III 750 mHz processor.

5. The cost of suboptimal strategies

In this section, we study the costs of following single-factor exercise strategies for American-style swaptions when the term structure is driven by multiple factors. We first report the results when the Black et al. (1990) model is used to make exercise decisions. We then report the case with the Black and Karasinski model used as the single-factor model.

To evaluate the present value costs of using the single-factor Black, Derman, and Toy model over time to make exercise decisions, we follow the procedure described in the previous section. Specifically, we recalibrate the Black, Derman, and Toy model to match the market at each exercise date for an American-style swaption and then use the corresponding binomial tree to check whether exercise is optimal at that exercise date according to the recalibrated Black, Derman, and Toy model. The cost of suboptimal exercise is then estimated as the difference between the value of the discounted cash flows of the swaption obtained by following the optimal multifactor strategy and the value of discounted cash flows of the swaption obtained by following the strategy implied by the Black, Derman, and Toy model.

Define the American exercise premium to be the difference between the T noncall τ American swaption value and the value of the corresponding τ into $T-\tau$ European swaption. Table 2 reports the American exercise premia given by the one-factor Black, Derman, and Toy model and the four-factor string market model, the difference between these premia, and the difference between these premia expressed as a percentage of the single-factor American exercise premium.

Several important results are shown in Table 2. First, the value of the swaption is always higher when the optimal strategy is followed than when the single-factor strategy is followed. This demonstrates clearly that the single-factor exercise strategy cannot approximate the optimal strategy accurately enough to avoid some erosion in the value of swaption cash flows to an optionholder. Second, the size of the difference between the American exercise premia can be large in economic terms. For example, the difference between the American exercise premia often exceeds ten cents per \$100 notional and can even exceed 30 cents for long-dated swaptions. Since a large percentage of American-style swaptions are created in conjunction with the issuance of corporate or agency debt, it is typical for these swaptions to have longer maturities. Finally, the percentage differences between the American exercise premia can also be significant. In some cases, these differences are close to 10%.

To provide additional intuition about the costs of suboptimal exercise, it is also useful to compare the optimal and single-factor exercise strategies directly in terms of their implications for the timing of the exercise decision. Table 3 provides summary statistics for the risk-neutral probabilities of exercise and

Table 2
Summary statistics for American exercise premia given by following the exercise strategies implied by the Black et al. and string market models

This table reports the values in units of $1 per $100 notional value of the American exercise premia for the indicated T noncall τ American at-the-money-forward swaptions implied by the single-factor Black et al. (1990) model as well as the four-factor string market model. The American exercise premium is the difference between the American swaption value and the corresponding European swaption value. The percentage difference is computed relative to the Black, Derman, and Toy American exercise premium. Values are based on 5,000 simulated paths of the term structure.

T	τ	Swaption type	Single-factor Black, Derman, and Toy American exercise premium	Four-factor string market model American exercise premium	Difference	Percentage difference
5	1	Recvr	0.538	0.560	0.022	4.1
10	1	Recvr	1.881	1.931	0.050	2.7
10	2	Recvr	1.303	1.329	0.026	2.0
15	1	Recvr	3.252	3.391	0.139	4.3
15	3	Recvr	1.854	1.974	0.120	6.5
15	5	Recvr	1.129	1.179	0.050	4.5
5	1	Payer	0.700	0.724	0.024	3.4
10	1	Payer	2.546	2.657	0.111	4.4
10	2	Payer	1.687	1.783	0.096	5.7
15	2	Payer	4.400	4.710	0.310	7.0
15	3	Payer	2.504	2.713	0.209	8.3
15	5	Payer	1.521	1.607	0.086	5.7

the frequency with which the single-factor Black, Derman, and Toy model implies that the American swaption should be exercised earlier, at the same time, or later than implied by the optimal strategy. Table 3 also reports the present value costs of exercising when the single-factor model implies that exercise is optimal but the multifactor model does not, and the present value costs when the single-factor implies that exercise is not optimal but the multifactor does.

Table 3 shows that American-style swaptions are often more likely to be exercised when the Black, Derman, and Toy exercise strategy is followed than when the optimal multifactor strategy is followed. Exceptions include some of the longer-dated swaptions with 15-year horizons. The reason for this is clear from the frequency with which the single-factor strategy tends to provide an exercise signal earlier than the optimal strategy. The single-factor model leads to earlier than optimal exercises between 10% and 22% of the time. Alternatively, the single-factor model leads to delayed exercises between 1% and 14% of the time. Interestingly, the two models provide the same exercise strategy 62 88% of the time.

Table 3
Comparison of single-factor Black, Derman, and Toy and four-factor string market model exercise strategies

This table reports summary statistics for the single-factor Black et al. (1990) and optimal four-factor string market model exercise strategies. Probability of exercise represents the total percentage of paths for which the swaption is exercised. The table also reports the percentage of paths for which the single-factor model results in an exercise decision earlier, at the same time, or later than the exercise decision given by the four-factor string market model. The present value cost of exercising when the single-factor model implies exercise is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the optimal strategy, averaged over all paths where the single-factor model implies exercise earlier, and similarly for the present value cost of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. The present value costs are expressed in units of $1 per $100 notional value. All values are based on 5,000 simulated paths of the term structure.

			Probability of exercise		Probability single-factor model results in an exercise decision			Present value costs of exercising when single-factor model implies that exercise is optimal	
T	τ	Swaption type	Single-factor	Four-factor	Earlier	Same time	Later	Earlier	Later
5	1	Recvr	71.36	68.80	10.78	88.62	0.60	−0.226	0.113
10	1	Recvr	72.62	71.76	10.92	83.14	5.94	−0.377	−0.197
10	2	Recvr	72.24	71.40	10.08	85.24	4.68	−0.271	−0.053
15	1	Recvr	74.26	74.62	17.22	70.50	12.28	−0.480	−0.522
15	3	Recvr	73.60	73.82	14.78	76.54	8.68	−0.286	−1.053
15	5	Recvr	71.78	71.26	12.14	82.08	5.78	−0.237	−0.629
5	1	Payer	68.12	65.76	12.78	85.32	1.90	−0.201	−0.069
10	1	Payer	71.06	71.22	17.42	72.54	10.04	−0.635	−0.334
10	2	Payer	69.68	68.92	17.14	75.28	7.58	−0.597	−0.401
15	1	Payer	74.50	75.28	22.58	62.78	14.66	−1.254	−0.943
15	3	Payer	72.32	72.58	22.06	66.00	11.94	−1.090	−0.638
15	5	Payer	69.70	68.84	19.06	74.78	6.16	−0.678	−0.463

The present value costs shown in Table 2 represent the average cost over all paths. These paths, however, include many paths for which the swaption is always out of the money and for which no model would give an exercise signal. Similarly, there are many paths where the American swaption is so deep in the money that almost any reasonable model would imply that immediate exercise was optimal. In some sense, the real test of a model is how well it performs for those paths where the exercise decision is not at all clear-cut. To this end, we focus first on the paths where the single-factor model implies exercise at an earlier date than does the multifactor model. For these paths, we contrast the cash flows obtained by exercising at the date implied by the single-factor model with the discounted cash flows obtained by following the optimal exercise policy. The difference between the two directly measures the conditional present value cost of exercising when the single-factor model signals exercise but the multifactor model does not. From Table 3, these conditional costs can be quite large, ranging from about 25 cents to as much as \$1.25 per \$100 notional.

Similarly, we focus on the paths where the multifactor model implies exercise earlier than does the single-factor model. The difference between the cash flows obtained by exercising at the date implied by the multifactor model and the discounted cash flows obtained by following the single-factor strategy is again very large. These conditional costs range from less than ten cents to more than \$1 per \$100 notional. These results demonstrate clearly that the cost of following suboptimal exercise strategies can be much higher for some paths than others.

As an alternative single-factor model, we use the popular Black and Karasinski model. The approach is very similar to that described for the Black, Derman, and Toy model with the exception that the mean-reversion function $\beta(t)$ needs to be specified. To do this, we define the function $\beta(t; i, j)$ that makes the mean-reversion function for horizon t depend explicitly on the date i and path j. At date i, we estimate the function $\beta(t; i, j)$ by regressing the change in $\ln r$ from time $t - 0.50$ to t on the value of $\ln r$ at time i, where the slope coefficient on $\ln r$ is represented as a linear combination of the same conditioning variables used to estimate the LSM regression at time i. The fitted value of this linear combination of state variables is an estimate of the conditional mean-reversion function $\beta(t; i, j)$. Since the regression is done using the actual paths generated by the multifactor string market model, the conditional mean-reversion function has the advantage of being based on the actual distribution of the economy. In this sense, this gives a slight advantage to the Black and Karasinski model since the estimate of $\beta(t; i, j)$ uses additional information that would not be available to an investor who used only the Black and Karasinski model with current option data to make decisions.

Once the mean-reversion function is estimated, we then use the same procedure as before to fit exactly the vector of zero-coupon bonds $D(i, i + t; j)$ through the drift function $\mu(t; i, j)$ and solve for the volatility function $s(t; i, j)$ that best fits the at-the-money caplets and swaptions prices that would be observed at date i and path j. Once calibrated to the market at date i, we then build a trinomial tree using an approach described by James and Webber (2000) and evaluate whether exercise is optimal in the Black and Karasinski model. The cost of suboptimal exercise is again the difference between the value of the American-style swaption obtained by following the optimal multifactor strategy and the value obtained by following the single-factor Black and Karasinski strategy. The results are shown in Table 4.

Table 4
Summary statistics for American exercise premia given by following the exercise strategies implied by the Black and Karasinski and string market models

This table reports the values in units of $1 per $100 notional value of the American exercise premia for the indicated T noncall τ American at-the-money-forward swaptions implied by the single-factor Black and Karasinski (1991) model as well as the four-factor string market model. The American exercise premium is the difference between the American swaption value and the corresponding European swaption value. The percentage difference is computed relative to the Black and Karasinski American exercise premium. Values are based on 5,000 simulated paths of the term structure.

T	τ	Swaption type	Single-factor Black and Karasinski American exercise premium	Four-factor string market model American exercise premium	Difference	Percentage difference
5	1	Recvr	0.551	0.560	0.009	1.6
10	1	Recvr	1.870	1.931	0.061	3.3
10	2	Recvr	1.299	1.329	0.030	2.3
15	1	Recvr	3.197	3.391	0.194	6.1
15	3	Recvr	1.861	1.974	0.113	6.1
15	5	Recvr	1.134	1.179	0.045	4.0
5	1	Payer	0.702	0.724	0.020	2.8
10	1	Payer	2.552	2.657	0.105	4.1
10	2	Payer	1.699	1.783	0.084	4.9
15	2	Payer	4.409	4.710	0.301	6.8
15	3	Payer	2.513	2.713	0.200	8.0
15	5	Payer	1.510	1.607	0.097	6.4

The results in Table 4 are very similar to those obtained using the Black, Derman, and Toy model. As before, the American exercise premia based on the four-factor string market model are all greater than the single-factor exercise premia. Although there are significant theoretical differences between the Black and Karasinski model and the Black, Derman, and Toy model, the estimated present value costs of following the single-factor exercise strategy are very similar. Once again, the present value costs of suboptimal exercise can be as large as 30 cents per $100 notional. A close examination of Tables 2 and 4 suggests that the present value costs are generally slightly less using the Black and Karasinski model. This is not surprising since the Black and Karasinski model has far more parameters that can be fitted to market data.

Turning to the probabilities and timing of exercise, summary statistics are presented in Table 5. In contrast to the results in Table 3, the probability of exercise following the Black and Karasinski strategy is generally lower than under the optimal strategy. Furthermore, the Black and Karasinski strategy results in fewer early exercises relative to the optimal strategy than does the

Table 5
Comparison of single-factor Black and Karasinski and four-factor string market model exercise strategies

This table reports summary statistics for the single-factor Black and Karasinski (1991) and optimal four-factor string market model exercise strategies. Probability of exercise represents the total percentage of paths for which the swaption is exercised. The table also reports the percentage of paths for which the single-factor model results in an exercise decision earlier, at the same time, or later than the exercise decision given by the four-factor string market model. The present value cost of exercising when the single-factor model implies exercise is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the optimal strategy, averaged over all paths where the single-factor model implies exercise earlier, and similarly for the present value cost of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. The present value costs are expressed in units of \$1 per \$100 notional. All values are based on 5,000 simulated paths of the term structure.

			Probability of exercise		Probability single-factor model results in an exercise decision			Present value costs of exercising when single-factor model implies that exercise is optimal	
T	τ	Swaption type	Single-factor	Four-factor	Earlier	Same time	Later	Earlier	Later
5	1	Recvr	68.68	68.80	2.44	95.40	2.16	−0.308	−0.076
10	1	Recvr	71.44	71.76	9.90	83.10	7.00	−0.314	−0.441
10	2	Recvr	70.78	71.40	6.40	86.60	7.00	−0.227	−0.249
15	1	Recvr	74.56	74.62	21.58	68.28	10.14	−0.610	−0.549
15	3	Recvr	73.06	73.82	14.12	76.70	9.18	−0.398	−0.699
15	5	Recvr	70.58	71.26	8.96	83.50	7.54	−0.085	−0.644
5	1	Payer	66.02	65.76	6.14	90.36	3.50	−0.218	−0.298
10	1	Payer	69.90	71.22	13.56	74.36	12.08	−0.757	−0.268
10	2	Payer	68.20	68.92	12.90	77.50	9.60	−0.558	−0.432
15	1	Payer	73.78	75.28	20.80	63.12	16.08	−1.402	−0.785
15	3	Payer	71.44	72.58	19.24	67.42	13.34	−1.085	−0.710
15	5	Payer	68.32	68.84	15.62	75.96	8.42	−0.705	−0.697

Black, Derman, and Toy strategy. On the other hand, the Black and Karasinski strategy results in delayed exercises a little more frequently than the Black, Derman, and Toy strategy. Finally, the conditional costs of using the single-factor exercise strategy can again be very substantial. Conditional on not receiving an exercise signal from the optimal strategy, the cost of exercising when the single-factor model implies that exercise is optimal can be as large as \$1.40 per \$100 notional.

In summary, these results demonstrate that even when a single-factor model is recalibrated to match the market at each exercise date, the exercise strategy or stopping policy implied by the single-factor model can be significantly suboptimal. Since many American-style swaption structures have final maturities of 10, 15, or more years, these results suggest that the present value costs of following single-factor strategies over time can easily exceed 10 or 20 cents per $100 notional. Given that the total notional amount of American-style swaptions outstanding is in the trillions of dollars, a back-of-the-envelope calculation shows that the total present value costs to swaption holders from following single-factor strategies could easily be several billion dollars. While one could argue that costs on the order of ten cents per $100 notional approximate the size of the bid–ask spread for these derivative products, it is clear that these costs are economically relevant to swaption holders, who could avoid them by simply following the optimal exercise strategy.

Finally, we note that while only results using the Black, Derman, and Toy and Black and Karasinski models are presented here, we have also done similar tests using alternative single-factor models. The results from these alternative single-factor models closely parallel those reported in this section, demonstrating that our results are robust to the choice of the single-factor model. This makes intuitive sense since it is the implication that term structure movements are perfectly correlated across maturities, common to all single-factor models, that is the primary reason for the suboptimality of their corresponding exercise strategies. While we focus on American-style swaptions, it is also important to observe that the differences between single-factor and multifactor term structure models could be even more dramatic for other types of derivatives such as options on interest rate spreads where values are heavily dependent on correlation assumptions.

6. Valuation and misspecified models

In the previous section, we estimated the cost of following suboptimal single-factor strategies. It is important to stress that the correct way to study the cost of following single-factor strategies in a multifactor world is by comparing the present values of the cash flows obtained by following the stopping rules on a common set of paths of the term structure. If the market is driven by multiple factors, the present value of the cash flows generated by following a single-factor exercise strategy will be less than when the optimal exercise strategy is followed.

While this latter point may seem straightforward, it is occasionally claimed that American-style swaptions can be 'worth more in a single-factor model than in a multifactor model' (e.g., Andersen and Andreasen (2000) and Brace and Womersley (2000)). The reason for this claim stems from an interesting apparent paradox. Specifically, if the holder of an American swaption uses a single-factor model to value the option as well as make exercise decisions, the value implied by the single-factor model can exceed the value implied by the multifactor model. Thus, the swaption holder could be led to believe that the swaption was actually more valuable using a single-factor model to make exercise decisions.

This paradox can be resolved, however, by first noting that in comparing values implied by a single-factor model with those implied by a multifactor model, the dynamics of the term structure are not held fixed. Because of this, the comparison between the two models is meaningless; the two calibrations are not comparable. A second, but potentially even more important, point is that American option values implied by a misspecified model need not equal the present value of the actual cash flows generated from following the exercise strategy implied by the misspecified model. Hence, the American swaption value implied by a misspecified single-factor model is an illusory number that can be a severely biased estimate of the present value of cash flows actually received by a swaption holder who follows the single-factor strategy.

To illustrate the point that a misspecified model will give a value for American options that need not equal the present value of cash flows obtained by following the implied exercise strategy, consider the following example. Let X and Y denote the prices of two risky assets. One-period and two-period European exchange options are available in the market with payoffs of $\max(0, X_1 - Y_1)$ and $\max(0, X_2 - Y_2)$, respectively. If $r = 0.10$, the two assets have a correlation of 0.98, each has a current price of 100, and the volatility of both assets is 0.10, then the standard Margrabe (1978) model implies prices for these options of 0.80 and 1.13, respectively. Now assume that we want to value an American exchange option that is exercisable at times $t = 1$ and 2. It is easily shown that early exercise of this American exchange option is not optimal, and that the correct value is 1.13.

Now assume, however, than an investor has the mistaken belief that X is constant over time and that market prices of these options are driven by the single factor Y. This investor now views the one-period and two-period exchange options as simple put options on Y with strike prices of X_0, and the investor can match their market prices by assuming that the volatility of Y is 0.100 during the first year and 0.152 during the second year. Given this calibration, the single-factor model implies that the value of the American exchange option is 1.37. When this investor arrives at $t = 1$, the value of X may have changed, and he will need to recalibrate the single-factor model to reflect the new price of the remaining European exchange option. It is easily shown that once this calibration is performed, the investor will never find it optimal to exercise early at time $t = 1$. Thus, the actual cash flows received by the investor will have a present value of 1.13 since they are the same as under the optimal strategy. The bottom line is that although the misspecified single-factor model values the American exchange option at 1.37, the actual present value of the cash flows is only about 82% of the amount that the investor would have been willing to pay to acquire the American exchange option. Note that at both $t = 0$ and 1, the single-factor model is calibrated to match the market prices of all European exchange options. Despite this, the misspecified single-factor model results in a valuation for the American exchange option that is severely upward biased.

This example suggests that if the term structure is driven by multiple factors, the American option values given by a misspecified single-factor model are unreliable since they do not equal the present value of the cash flows generated by following the single-factor, multifactor, or any other strategy. Thus, comparing the value implied by a single-factor model with the value implied by a multifactor model is a questionable exercise. Single-factor and multifactor models can only be compared in terms of the present value of the cash flows generated by their exercise strategies while holding fixed the term structure model.

The fact that American option values implied by a misspecified model are biased poses some subtle but important risks to option holders. For example, consider an investor who believes that the term structure is driven by a single-factor model and calibrates his model to exactly match a cross section of European swaptions. Now imagine that the single-factor model just happens to match the current market price of an American swaption as implied by the true multifactor model. This investor might well conclude that since his model matches both the European and American swaption prices, his single-factor model is adequate. In fact, however, the cash flows generated by following the single-factor strategy would have a lower present value than the market price of the American swaption. The only clues that this investor might have that there was a problem with his model would be the frequent need to recalibrate, persistent hedging errors, and a general tendency for his portfolio to underperform expectations. This underperformance would appear inexplicable to the investor since the valuations implied by the single-factor model would match the market. The problem, of course, is that the American swaption is only worth the market price to an investor who follows the optimal multifactor strategy. Purchasing an American swaption at the market price is a negative NPV investment to an investor following a suboptimal single-factor strategy.

7. Conclusion

This paper studies the costs of following single-factor exercise strategies for American swaptions when the term structure is actually driven by multiple factors. A number of important contributions emerge from this study.

- We estimate the present value costs to the holder of an American swaption who uses a single-factor model to make exercise decisions when the term structure is driven by multiple factors. These present value costs are substantial and can be avoided by swaption holders through simply following the optimal strategy. Based on current market statistics, the total present value costs of following suboptimal strategies implied by single-factor models could be on the order of several billion dollars.

- These results illustrate that if the dynamic specification of a model does not match actual market dynamics, the American exercise strategy implied by the model will be suboptimal. This is true no matter how extensively the model is recalibrated to match a cross section of current option prices. Our results demonstrate that the common practice of continually recalibrating a single-factor model to match the market does not fully compensate for its failure to capture the actual dynamics of the term structure; overfitting a misspecified model does not eliminate its weaknesses.
- Furthermore, if the dynamics of a model are misspecified, then American option values implied by the model will be biased estimates of the actual present value of cash flows generated by following the exercise strategy implied by the model.

These results make a strong case for moving beyond simplistic single-factor models to more realistic (and easier to calibrate) multifactor string market models in fixed income markets. The results also make clear the importance of using economically realistic models in derivatives applications and point out some of the subtle dangers of overfitting misspecified models. These lessons are important to consider given the widespread industry practice of frequent recalibration of simplistic models as an alternative to the rigorous econometric modeling of fundamental market variables. The results also have clear implications for the issue of model risk in hedging and dynamic replication of derivatives; the process of having to recalibrate a model drives a wedge between the value of a derivative and the value of the dynamic hedging portfolio that attempts to replicate its payoff. Furthermore, the hedging and dynamic replication of derivatives may be less forgiving of model misspecification than the optimal exercise decision for American options.[9]

Finally, although we have focused on fixed income markets, these findings are relevant to many other markets and applications. For example, it is a common practice in many options markets to continually update implied Black-Scholes volatilities to correct for the failure of the model to capture the stochastic behavior of volatility. Even if the Black-Scholes model is continually adjusted to fit the current market prices of European options, however, the

[9] One interesting issue which we leave for future research is the extent to which single-factor exercise strategies can be partially 'corrected' of their biases by using the multifactor information in the term structure. For example, can single-factor strategies be improved by overriding the exercise decision in a way that depends on multiple points along the term structure? We are grateful to the referee and John Uglum for suggesting this possibility.

single-factor exercise strategy implied by the model for American options may be far from optimal in a market where volatility is stochastic. Binomial tree approaches will suffer from the same problem.

References

Andersen, L., Andreasen, J., 2000. Factor dependence of Bermudan swaption prices: fact or fiction? Unpublished working paper, General Re Financial Products.

Anderson, T., Lund, J., 1997. Estimating continuous time stochastic volatility models of the short-term interest rate. Journal of Econometrics 77, 343–377.

Backus, D., Foresi, S., Zin, S., 1998. Arbitrage opportunities in arbitrage-free models of bond pricing. Journal of Business and Economic Statistics 16, 13–26.

Bakshi, G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. The Journal of Finance 52, 2003–2049.

Balduzzi, P., Das, S., Foresi, S., Sundaram, R., 1996. A simple approach to three factor affine term structure models. Journal of Fixed Income 6, 43–53.

Ball, C., Torous, W., 1999. The stochastic volatility of short-term interest rates: some international evidence. The Journal of Finance 54, 2339–2359.

Bicksler, J., Chen, A.H., 1986. An economic analysis of interest rate swaps. Journal of Finance 41, 645–655.

Black, F., 1976. The pricing of commodity contracts. Journal of Financial Economics 3, 167–179.

Black, F., Derman, E., Toy, W., 1990. A one-factor model of interest rates and its application to Treasury bond options. Financial Analysts Journal Jan./Feb. 33–39.

Black, F., Karasinski, P., 1991. Bond and option pricing when short rates are lognormal. Financial Analysts Journal Jul./Aug. 52–59.

Brace, A., Gatarek, D., Musiela, M., 1997. The market model of interest rate dynamics. Mathematical Finance 7, 127–155.

Brace, A., Womersley, R., 2000. Exact fit to the swaption volatility matrix using semidefinite programming. Unpublished working paper, University of New South Wales.

Brandt, M., Goyal, A., Santa-Clara, P., 2001. Dynamic portfolio choice with simulation methods. Unpublished working paper, University of California, Los Angeles.

Brandt, M., Wu, T., 2001. Cross-sectional tests of deterministic volatility functions. Unpublished working paper, University of Pennsylvania.

Brenner, R., Harjes, R., Kroner, K., 1996. Another look at models of the short-term interest rate. Journal of Financial and Quantitative Analysis 31, 85–107.

Brown, S., Dybvig, P., 1986. The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates. Journal of Finance 41, 617–630.

Buraschi, A., Corielli, F., 2000. Staying ahead of the curve: model risk and the term structure. Unpublished working paper, London Business School.

Buraschi, A., Jackwerth, J., 2001. The price of a smile: hedging and spanning in options markets. The Review of Financial Studies 14, 495–527.

Carr, P., Yang, G., 2001. Simulating Bermudan interest rate derivatives. Quantitative Analysis of Financial Markets 2, forthcoming.

Chan, K.C., Karolyi, A., Longstaff, F., Sanders, A., 1992. An empirical comparison of alternative models of the short-term interest rate. Journal of Finance 47, 1209–1227.

Chen, R.R., Scott, L., 1993. Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates. Journal of Fixed Income 3, 14–31.

Dai, Q., Singleton, K., 2000a. Specification analysis of affine term structure models. The Journal of Finance 55, 1943–1978.

Dai, Q., Singleton, K., 2000b. Expectations puzzles, time-varying risk premia, and dynamic models of the term structure. Unpublished working paper, Stanford University.
Derman, E., 1996. Model risk. In: Quantitative Strategies Research Notes, Goldman Sachs, New York, NY.
Driessen, J., Klaassen, P., Melenberg, B., 2000. The performance of multi-factor term structure models for pricing and hedging caps and swaptions. Unpublished working paper, Tilburg University.
Duffie, D., Singleton, K., 1997. An econometric model of the term structure of interest rate swap yields. The Journal of Finance 52, 1287–1321.
Dumas, B., Fleming, J., Whaley, R., 1998. Implied volatility functions: empirical tests. The Journal of Finance 53, 2059–2106.
Dybvig, P., 1988. Inefficient portfolio strategies or how to throw away a million dollars in the stock market. The Review of Financial Studies 1, 67–88.
Dybvig, P., 1997. Bond option pricing based on the current term structure. In: Dempster, M., Pliska, S. (Eds.), Mathematics of Derivative Securities. Cambridge University Press, Cambridge.
Gibbons, M., Ramaswamy, K., 1993. A Test of the Cox, Ingersoll, and Ross model of the term structure. Review of Financial Studies 6, 619–658.
Goldstein, R., 2000. The term structure of interest rates as a random field. The Review of Financial Studies 13, 365–384.
Green, C., Figlewski, S., 1999. Market risk and model risk for a financial institution writing options. The Journal of Finance 54, 1465–1499.
Gupta, A., 1999. An examination of the static and dynamic accuracy of interest rate option pricing models in the cap-floor market. Unpublished working paper, Case Western Reserve University.
Gupta, A., Subrahmanyam, M., 2000. An empirical investigation of the convexity bias in the pricing of interest rate swaps. Journal of Financial Economics 55, 239–279.
Hull, J., Suo, W., 2000. A methodology for assessing model risk and its application to the implied-volatility function model. Unpublished working paper, University of Toronto.
Hull, J., White, A., 1990. Pricing interest rate derivative securities. The Review of Financial Studies 3, 573–592.
Hull, J., White, A., 1999. Forward rate volatilities, swap rate volatilities, and the implementation of the Libor market model. Unpublished working paper, University of Toronto.
Jagannathan, R., Kaplin, R.A., Sun, G., 2000. An evaluation of multi-factor CIR models using LIBOR, swap rates, and cap and swaption prices. Unpublished working paper, Northwestern University.
James, J., Webber, N., 2000. Interest Rate Modeling. Wiley, New York.
Jamshidian, F., 1997. Libor and swap market models and measures. Finance and Stochastics 1, 290–330.
Karatzas, I., Shreve, S., 1988. Brownian Motion and Stochastic Calculus. Springer, New York.
Kennedy, D., 1994. The term structure of interest rates as a Gaussian random field. Mathematical Finance 4, 247–258.
Kennedy, D., 1997. Characterizing Gaussian models of the term structure of interest rates. Mathematical Finance 7, 107–118.
Knez, P., Litterman, R., Scheinkman, J., 1994. Explorations into factors explaining money market returns. Journal of Finance 49, 1861–1882.
Koedijk, K., Nissen, F., Schotman, P., Wolff, C., 1997. The dynamics of short-term interest rate volatility reconsidered. European Finance Review 1, 105–130.
Litterman, R., Scheinkman, J., 1991. Common factors affecting bond returns. The Journal of Fixed Income 1, 54–61.
Litzenberger, R., 1992. Swaps: plain and fanciful. The Journal of Finance 42, 831–850.

Liu, J., Longstaff, F., 2000. The market price of credit risk: an empirical analysis of interest rate swap spreads. Unpublished working paper, University of California, Los Angeles.
Longstaff, F., 2001. Optimal portfolio choice and the valuation of illiquid securities. The Review of Financial Studies 14, 407–431.
Longstaff, F., Santa-Clara, P., Schwartz, E.S., 2001. The relative valuation of caps and swaptions: theory and empirical evidence. The Journal of Finance, forthcoming.
Longstaff, F., Schwartz, E.S., 1992. Interest rate volatility and the term structure: a two-factor general equilibrium model. The Journal of Finance 47, 1259–1282.
Longstaff, F., Schwartz, E.S., 2001. Pricing American options by simulation: a simple least squares approach. The Review of Financial Studies 14, 113–147.
Macfarlane, J., Ross, D., Showers, J., 1991. The interest rate swap market: yield mathematics, terminology, and conventions. In: Beidleman, C. (Ed.), Interest Rate Swaps. Business One Irwin, Homewood, Illinois.
Margrabe, W., 1978. The value of an option to exchange one asset for another. Journal of Finance 33, 177–186.
Merton, R.C., 1973. The theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141–183.
Pearson, N., Sun, T.-S., 1994. Exploiting the conditional density in estimating the term structure: an application to the Cox, Ingersoll, and Ross model. Journal of Finance 49, 1279–1304.
Peterson, S., Stapleton, R., Subrahmanyam, M., 1999. The valuation of American-style swaptions in a two-factor spot-futures model. Unpublished working paper, New York University.
Piazzesi, M., 2000. An econometric model of the yield curve with macroeconomic jump effects. Unpublished working paper, University of California, Los Angeles University.
Santa-Clara, P., Sornette, D., 2001. The dynamics of the forward interest rate curve with stochastic string shocks. The Review of Financial Studies 14, 149–185.
Smith, C.W., Smithson, C.W., Wakeman, L.M., 1988. The market for interest rate swaps. Financial Management 17, 34–44.
Stambaugh, R., 1988. The information in forward rates: Implications for models of the term structure. Journal of Financial Economics 21, 41–70.
Sun, T., Sundaresan, S.M., Wang, C., 1993. Interest rate swaps: an empirical investigation. Journal of Financial Economics 36, 77–99.
Sundaresan, S.M., 1991. Valuation of swaps. In: Khoury, S. (Ed.), Recent Developments in International Banking and Finance, Vol. 5. Elsevier Science Publishers, New York.
Tsitsiklis, J., Van Roy, B., 2001. Regression methods for pricing complex American-style options. IEEE Transactions on Neural Networks, 12 694–703.
Turnbull, S., 1987. Swaps: zero-sum game? Financial Management 16, 15–21.
Wall, L.D., Pringle, J.J., 1989. Alternative explanations of interest rate swaps: a theoretical and empirical analysis. Financial Management 16, 15–21.