Risk and Valuation Under an Intertemporal Capital Asset Pricing Model

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Abstract

We analyze the risk characteristics and the valuation of assets in an economy in which the investment opportunity set is described by the real interest rate and the maximum Sharpe ratio. It is shown that, holding constant the beta of the underlying cash flow, the beta of a security is a function of the maturity of the cash flow. For parameter values estimated from U.S. data, the security beta is always increasing with the maturity of the underlying cash flow, while discount rates for risky cash flows can be increasing, decreasing or non-monotone functions of the maturity of the cash flow. The variation in discount rates and present value factors that is due to variation in the real interest rate and the Sharpe ratio is shown to be large for long maturity cash flows, and the component of the volatility that is due to variation in the Sharpe ratio is more important than that due to variation in the real interest rate.

1 Introduction

Despite the terminology, neither the single period Capital Asset Pricing Model[1] nor its intertemporal version (Merton, 1973), actually *prices* capital assets; rather, these models provide necessary conditions between the expected rates of change in asset prices and the covariances of those rates of change with other variables that must be satisfied in equilibrium. These necessary conditions can be combined with a rational expectations assumption (Lucas (1978)) to derive a partial differential equation which, given the appropriate boundary conditions, can be solved for asset prices as in Cox, Ingersoll and Ross (1985). A related approach to pricing assets relies on the multi-period Euler condition of a representative agent which equates the marginal utility of a dollar consumed today with the expected marginal utility of a dollar invested in a given asset for T periods as in Breeden and Litzenberger (1978). These two approaches are equivalent because of the Feyman-Kac Theorem, which allows the solution of a partial differential equation to be expressed as an expectation. However, the general lack of empirical success of consumption based asset pricing models[2] has meant that these two approaches are not much used in practice for valuing assets.

For practical purposes, assets are most often valued or priced by discounting their expected future cash flows, using discount rates that are calculated from a single period capital asset pricing model (CAPM).[3] If the investment opportunity set is stochastic, however, the single period capital asset pricing model holds only under highly restrictive

[1] Constantinides (1980) gives sufficient conditions for the single period CAPM to hold in a dynamic setting.

[2] 'The canonical consumption-based model has failed perhaps the most important test, the test of time $\cdots$ almost all applied work still uses portfolio-based models to correct for risk, to digest anomalies, to produce cost of capital estimates and so forth.' [Campbell and Cochrane (2000), p. 2864]

[3] Graham and Harvey (2001) survey 392 Chief Financial Officers of large corporations and report that "the CAPM is by far the most popular method of estimating the cost of equity capital: 73.5% of respondents always or almost always use the CAPM." Relatively small numbers of firms reported making further adjustments to discount rates for other risks such as interest rate, commodity price, inflation, or foreign exchange risk, or market-to-book ratio, size, or momentum. It is unclear how these adjustments were made. Cornell *et al.* (1997, p 12) report that "The only asset pricing model that has been applied widely in practice is the capital asset pricing model."

assumptions and, in any case, one period discount rates will depend on the investment opportunity set and its dynamics, as well as on the risk and maturity of the underlying cash flow, and it is then unclear how multi-period discount rates are to be determined. There is now strong evidence of stochastic variation in investment opportunities;[4] this makes an intertemporal version of the model (ICAPM) more appropriate for asset valuation than the classical single period model. However, implementation of the ICAPM faces two major obstacles: first, the relevant state variables must be specified; and secondly, the risk premia associated with these state variables must be determined.

In this paper, we argue that a natural minimal set of state variables consists of the interest rate and the slope of the capital market line, or Sharpe ratio. This yields the simplest version of the ICAPM that allows for correlated variation in the two parameters of the short run investment opportunity set, the interest rate and the Sharpe ratio. This model was originally derived in Brennan, Wang and Xia (2003) who show that it performs as well as, or better than, the Fama-French (1996) 3-factor model in pricing returns on portfolios sorted on size and book-to-market ratio, and on industry portfolios. We use the model to develop a closed form expression for the present value of a future cash flow, and estimate the parameters of the model up to a constant of proportion for the Sharpe ratio, using time series data on inflation and the yields on riskless nominal bonds with different maturities. The parameterized model is then used to analyze the determinants of security risk and the pricing of risky future cash flows.

Our investigation of risk under the ICAPM is motivated by the findings that, on the one hand, the betas of high growth firms tend to be higher than those of low growth firms and that, on the other hand, there seems to be relatively little relation between beta coefficients and the covariance matrix of the underlying cash flows.[5] Cornell (1999) argues that the betas of Amgen and other pharmaceutical companies are much too high to be explained

[4]See, for example, the findings of Perez-Quiros and Timmermann (2000) and Whitelaw (1994).
[5]See Beaver, Kettler and Scholes (1970).

by their cash flow betas alone and, following Campbell and Mei (1993),[6] suggests that it is common variation in expected returns that accounts for this. He finds that stock betas are positively associated with the "duration" of equity cash flows as measured by forecast earnings growth rates and (the negative of) dividend yields. Dechow, Sloan, and Soliman (2001) also find that a security's market beta is positively related to its estimated equity duration. Therefore, since the ICAPM explicitly allows for changing discount rates, it is natural to ask whether our parameter estimates are qualitatively consistent with the relation between betas and maturity that Cornell (1999) and Dechow et. al. (2001) find; we find that they are.

In the ICAPM that we develop, the pricing kernel is a linear combination of shocks to aggregate wealth (the market), to the real interest rate, and to the market Sharpe ratio or risk premium. We derive expressions for both the market beta and the pricing kernel beta of a security, and show that the market beta is a combination of three betas: the cash flow beta, and the real interest rate and risk premium betas. The relative importance of the three betas depends on the maturity of the cash flow underlying the security. Since under the ICAPM the risk premium is determined by the covariance of the security return with the pricing kernel and not with the market return alone, the market beta of a security is only one of the three betas that determine the instantaneous expected return. We use the estimated model parameters to calculate beta coefficients for securities that are claims to cash flows with different risk characteristics and maturities, and show that the market beta of a security increases with the maturity of the underlying cash flow. We also show that, in contrast to the market beta, the instantaneous expected return on a security, which depends on the beta of the security *with respect to the pricing kernel*, may be an increasing, decreasing, or non-monotone function of the maturity of the underlying cash flow.

[6] Campbell and Mei (1993) use a Vector Auto Regression to decompose the betas of industry portfolios into a cash flow beta, a beta associated with innovations in real interest rates, and a beta associated with innovations in the equity premium. They conclude that the real interest rate component of beta is very small, of the order of 0.012, and that for most industries the cash flow beta component is much smaller than the beta component that is associated with the equity premium. However, Wee (1996) casts some doubt on the robustness of these results.

We then analyze the relation between the discount rate that converts an expected future cash flow into a present value and the maturity of the cash flow. The classical CAPM is ill-suited to address this issue because of its one-period setting, and practitioners often use a rule of thumb that assigns a bigger discount rate to more distant cash flows (Cornell (1999)). To examine whether this rule of thumb is consistent with our ICAPM framework, we use the same model parameters to calculate the present values of risky future cash flows and the corresponding implied discount rates. We find that, like the instantaneous expected return, the discount rate can be an increasing, decreasing, or non-monotone function of the cash flow maturity. We also explore the effect on the discount rates of the risk of the underlying cash flow, and the current state of the investment opportunity set. We find that discount rates for risky cash flows vary considerably as the investment opportunity set changes over time.

While most of the analysis is conducted under the assumption that uncertainty about the cash flow is resolved at a constant rate over time, we also explore the effect of relaxing this assumption on the relation between discount rates and cash flow maturity. When the information arrival rate increases with time, the ratio of the conditional variance of the future cash flow to the remaining time to maturity also increases with time, tending to increase the discount rate for cash flow claims near maturity. This may cause an inverted term structure for discount rates in which discount rates are higher for shorter maturity cash flows. On the other hand, if the information arrival rate decreases with time, the reverse applies, and the term structure of discount rates is more likely to be upward sloping, which is consistent with the practitioner's rule of thumb.

Ang and Liu (2002) also explore the effect of time-variation in expected returns on cash flow discounting. However, they take the return beta of the cash flow claim as exogenously given whereas we derive it and show how it depends on the cash flow maturity. Campbell and Vuolteenhao (2002) (CV) use a VAR framework to decompose firm betas into a component which is associated with changing expected returns or discount rates, and a residual component which they attribute to changing cash flow expectations.

Unlike us, they do not distinguish between changes in the discount rate that are due to changes in the interest rate and changes that are due to the risk premium. This is a significant difference since we find that the risk associated with these two elements of the discount rate carry risk premia of *opposite* signs. They argue that discount rate betas should be associated with only a small risk premium while cash flow betas should earn a relatively high premium. While their point estimates are consistent with this, their estimate of the cash flow beta risk premium is not significant.

The remainder of the paper is organized as follows. Section 2 presents a simple two-period intertemporal asset pricing model and develop the two-state-variable version of the ICAPM. Section 3 considers the implications of the model for the market beta of cash flow claims. In Section 4 the model parameters are estimated using data on bond yields and inflation. Section 5 presents calculations of security betas, expected returns and discount rates as functions of the underlying cash flow maturity and risk, using the estimated model parameters. Section 6 concludes the paper.

2 Valuation Under the Two-Parameter ICAPM

To motivate the two-parameter ICAPM that is developed in this section, consider a two-period mean-variance economy in which investment opportunities are described by m state variables, $Z_1, \ldots, Z_m$, which follow a joint Markov process. Denote the investor's indirect utility function at time t, $t = 0, 1$, by $J(W, Z, t)$ where $Z = (Z_1, \ldots, Z_m)'$. At time 1 (the beginning of the second period), the investor faces a standard one period mean-variance optimization problem and, as shown in Figure 1, the investment opportunities at that time are fully described by the interest rate, $r(Z)$, and the slope of the capital market line, or market Sharpe ratio, denoted by $\eta(Z)$. Then the indirect utility function at time 1 can be written as $\widehat{J}(W, r(Z), \eta(Z), 1) \equiv J(W, Z, 1)$. It follows immediately from Merton (1973) that, in the first period equilibrium, security returns will be determined by their covariances with the market return, and with the sufficient statistics for the investment opportunity set at time 1, $r(Z)$ and $\eta(Z)$. That is, the intertemporal CAPM will hold in the first period with state variables $r(Z)$ and $\eta(Z)$. Moving back one period, the investor's indirect utility function at time 0 will in general depend on the full set of state variables Z. However, if $r(Z)$ and $\eta(Z)$ at time 0 are sufficient statistics for their joint distribution at time 1, the indirect utility function at time 0 may also be written as $\widehat{J}(W, r, \eta, 0) \equiv J(W, Z, 0)$: this condition will be satisfied, for example, whenever r and η follow a joint Markov process.

This example suggests that r and η are fundamental state variables in an intertemporal asset pricing model. Any intertemporal model must include, either directly or indirectly, these two state variables which describe the short run investment opportunity set.[7] The simplest intertemporal model is one in which the only state variables are these two. Therefore, consider a continuous time economy in which asset prices follow diffusion processes and the process for the pricing kernel, m, is:

$$\frac{dm}{m} = -r dt - \eta dz_m \tag{1}$$

where r is the instantaneous real risk free rate, η is the volatility of the pricing kernel, and dz_m is the standard Brownian motion. The definition of the pricing kernel implies that V, the value of a claim to a future cash flow, satisfies the condition $\mathrm{E}\left[d(mV)\right] = 0$, so that the instantaneous risk premium of the claim, $\mu_V - r$, is given by:

$$\mu_V - r = -\mathrm{Cov}\left[\frac{dm}{m}, \frac{dV}{V}\right] = \eta \sigma_V \rho_{mV}, \tag{2}$$

where σ_V is the volatility of the return process for V, and ρ_{mV} is its correlation with the pricing kernel. Equation (2) implies that the Sharpe ratio of claim V is equal to $\eta\rho_{mV}$. The maximum Sharpe ratio for the economy is attained when the return process of a claim V is perfectly correlated with the pricing kernel. Therefore, η is the maximum or 'market' Sharpe ratio. We assume that both σ_V and ρ_{mV} are known constants so that the

[7] Nielsen and Vassalou (2000) provide a formal proof.

risk premium for claim V is proportional to η.

In a diffusion setting, the interest rate, r, and market Sharpe ratio, η, completely describe the instantaneous investment opportunity set. In order to ensure that these two state variables are sufficient statistics for present and future investment opportunities assume that they follow a joint Ornstein-Uhlenbeck process:

$$dr = \kappa_r(\bar{r} - r)dt + \sigma_r dz_r, \tag{3}$$

$$d\eta = \kappa_\eta(\bar{\eta} - \eta)dt + \sigma_\eta dz_\eta, \tag{4}$$

where dz_r and dz_η are correlated Brownian motions with a constant correlation coefficient $\rho_{r\eta}$.

An intertemporal asset pricing model is obtained by specifying that the innovation in the pricing kernel is a linear function of the innovations in the market return and the two state variables:

$$dz_m = \zeta_M dz_M + \zeta_\eta dz_\eta + \zeta_r dz_r, \tag{5}$$

where M denotes the market portfolio, dz_M is the Brownian motion which may be correlated with dz_r and dz_η, and the coefficients ζ_i ($i = M$, η, r) are constants that determine the risk premia in economy which we shall estimate in Section 5. Since $dz_m^2 = dt$, the coefficients satisfy the normalization:

$$\zeta_M^2 + \zeta_\eta^2 + \zeta_r^2 + 2\zeta_M\zeta_\eta\rho_{M\eta} + 2\zeta_M\zeta_r\rho_{Mr} + 2\zeta_\eta\zeta_r\rho_{\eta r} = 1 \tag{6}$$

The risk premium of a claim, V, is determined under the ICAPM by the covariation of its return with the innovations in the state variables r and η as well as with the market portfolio return:

$$\mu_V - r = \eta\sigma_V\left(\zeta_M\rho_{MV} + \zeta_\eta\rho_{\eta V} + \zeta_r\rho_{rV}\right) \tag{7}$$

The intertemporal pricing model reduces to the CAPM if $\zeta_\eta = \zeta_r = 0$ so that $\zeta_M = 1$.

Under these assumptions about the pricing kernel, Brennan, Wang and Xia (2003) show that the value at time t of a claim to a real cash flow x at time T, whose expectation, Y, follows the stochastic process $\frac{dY}{Y} = \sigma_Y dz_Y$, is given by:

$$V(Y,\tau,r,\eta) = \mathrm{E}_t^Q\left[Y_T \exp^{-\int_t^T r(s)ds}\right] = Y_t v(\tau,r,\eta) \tag{8}$$

where Q denotes the risk neutral probability measure, and

$$v(\tau,r,\eta) = \exp[A(\tau) - B(\tau)r - D(\tau)\eta] \tag{9}$$

with $B(\tau)$, $D(\tau)$ and $A(\tau)$ given in equations (A1 - A3) in Appendix A. The value of a real zero-coupon bond is obtained as a special case of equation (8) by setting $\sigma_Y = 0$.

If the price level, P, evolves according to:

$$\frac{dP}{P} = \pi dt + \sigma_P dz_P, \tag{10}$$

$$d\pi = \kappa_\pi(\overline{\pi} - \pi)dt + \sigma_\pi dz_\pi, \tag{11}$$

where π is the expected rate of inflation and the volatilities σ_P and σ_π are known constants, a nominal zero-coupon bond can be valued in a similar way. In this case, the real payoff on a zero-coupon nominal bond which matures at date T is $Y_T \equiv x_T = 1/P_T$, and the nominal and real prices of the bond at time $t < T$ are given by Theorem 2 of Brennan *et. al.* (2003):

$$N(P,r,\pi,\eta,\tau) \equiv Pn(r,\pi,\eta,\tau) = \exp[\widehat{A}(\tau) - B(\tau)r - C(\tau)\pi - \widehat{D}(\tau)\eta] \tag{12}$$

where $\tau = T - t$, $B(\tau)$ is the same as in equation (9) and given by equation (A1), while $C(\tau)$, $\widehat{D}(\tau)$, and $\widehat{A}(\tau)$ are defined in equations (A6 - A8) in Appendix A.

Equation (12) implies that the yield of the nominal zero-coupon bond is a linear function of the instantaneous real interest rate, r, the current expected rate of inflation, π, and the market Sharpe ratio, η:

$$-\frac{\ln N}{\tau} = -\frac{\widehat{A}(\tau)}{\tau} + \frac{B(\tau)}{\tau}r + \frac{C(\tau)}{\tau}\pi + \frac{\widehat{D}(\tau)}{\tau}\eta, \tag{13}$$

This expression provides the basis for the model calibration in Section 5.

The valuation model for the single cash-flow claim can be extended easily to value a share of common stock which pays a continuous (nominal) dividend at the rate X. If the expected dividend growth rate follows an Ornstein-Uhlenbeck process so that :

$$\frac{dX}{X} = gdt + \sigma_X dz_X, \tag{14}$$

$$dg = \kappa_g(\bar{g} - g)dt + \sigma_g dz_g, \tag{15}$$

then, $S(X, r, \pi, \eta, g)$, the value of a share of common stock at time t is given by:

$$S(X, r, \pi, \eta, g) = \mathrm{E}^Q\left[\int_t^\infty \frac{X_s}{P_s} e^{-\int_t^s r(u)du} ds\right] = Y_t \int_t^\infty \tilde{v}(s-t, r, \pi, \eta, g)ds \tag{16}$$

where $Y_t \equiv X_t/P_t$ is the real dividend, Q denotes the risk neutral probability measure, and

$$\tilde{v}(s, r, \pi, \eta, g) = \exp[\tilde{A}(s-t) - B(s-t)r - C(s-t)\pi - \tilde{D}(s-t)\eta + F(s-t)g] \tag{17}$$

with expressions of $\tilde{A}$ to F given in Appendix A. Thus, the return on the share depends on innovations in five state variables: the two state variables that describe the nominal dividend rate, X and g, the expected rate of inflation, π, and the two investment opportunity set state variables, r and η. Expression (16) can not, however, be further simplified, and numerical or approximation techniques must be used to value the security.

3 Risk under the Two-Parameter ICAPM

The stochastic process for the value of a cash flow claim, V, is obtained by applying Ito's Lemma to equation (8):

$$\frac{dV}{V} = [r + \eta\,(D_\tau(\tau) + D(\tau)\kappa_\eta)]\,dt + \sigma_Y dz_Y - B(\tau)\sigma_r dz_r - D(\tau)\sigma_\eta dz_\eta, \tag{18}$$

where $D_\tau \equiv \frac{\partial D(\tau)}{\partial \tau}$ is given in equation (A4) in Appendix A. Similarly, equation (16) implies that the return process for a share of common stock, dS/S, which is essentially the return process on a portfolio of single cash flow claims with time to maturity varying from 0 to ∞, is given by:

$$\frac{dS}{S} = f(\cdot)dt + \frac{dY_s}{Y_s} - \mathcal{B}\sigma_r dz_r - \mathcal{C}\sigma_\pi dz_\pi - \mathcal{D}\sigma_\eta dz_\eta + \mathcal{F}\sigma_g dz_g$$

where $f(\cdot)$ is a function of the state variables and the cash flow durations. The sensitivity of the equity return to changes in the real interest rate, $\mathcal{B}$, is a weighted average of the sensitivity of single time-s cash flow claims ($s \in [t, \infty)$), $B(s-t)$:

$$\mathcal{B} \equiv \int_t^\infty \frac{\exp[\tilde{A}(s-t) - B(s-t)r - C(s-t)\pi - \tilde{D}(s-t)\eta + F(s-t)g]}{\int_t^\infty \exp[\tilde{A}(s-t) - B(s-t)r - C(s-t)\pi - \tilde{D}(s-t)\eta + F(s-t)g]ds} B(s-t)ds,$$

which is analogous to the calculation of coupon bond durations. The definitions for $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{F}$ are similar.

The expected real excess return of the security, $\eta\,(D_\tau(\tau) + D(\tau)\kappa_\eta)$, is proportional to the maximum Sharpe ratio, η, and depends on the risk of the underlying real cash flow,[8] as well as its maturity, $\tau = T - t$. The stochastic component of the return has three elements that are due, respectively, to cash flow risk, $\sigma_Y dz_Y$, real interest rate risk, $B(\tau)\sigma_r dz_r$, and risk premium risk, $D(\tau)\sigma_\eta dz_\eta$. The loadings on the innovations in both r and η are functions of the maturity of the underlying cash flow. It will be shown that, subject to parameter restrictions that are satisfied by our empirical estimates, $B(\tau)$ and

[8]The risk of the cash flow depends on the parameters of the joint stochastic process for the cash flow expectation, Y, and the pricing kernel, which are embedded in $D(\tau)$.

$D(\tau)$ are increasing in τ, so that both interest rate risk and risk premium risk are more important for claims to long maturity cash flows.

As mentioned in the Introduction, there is strong evidence that the market betas of securities are positively associated with the duration or maturity of the underlying cash flows. To analyze the importance of time variation in r and η for the betas of securities with different cash flow maturities, define $\beta_{VM} \equiv \text{Cov}\left(\frac{dV}{V}, \frac{dM}{M}\right)/\sigma_M^2$ as the beta of a security with respect to the market portfolio, and $\beta_{Vm} \equiv \text{Cov}\left(\frac{dV}{V}, -\frac{dm}{m}\right)/\eta^2$ as the beta of a security with respect to the pricing kernel. Then:

$$\begin{aligned} \beta_{VM} &= \beta_{YM} - D(\tau)\beta_{\eta M} - B(\tau)\beta_{rM}, \quad (19) \\ \beta_{Vm} &= \beta_{Ym} - D(\tau)\beta_{\eta m} - B(\tau)\beta_{rm}, \quad (20) \end{aligned}$$

where $B(\tau)$ and $D(\tau)$ are given in equations (A1) and (A2), and $\beta_{rM} \equiv \text{Cov}\left(dr, \frac{dM}{M}\right)/\sigma_M^2$ etc. Note that β_{Vm} is inversely proportional to η and follows a complicated stochastic process:

$$\begin{aligned} \beta_{Vm} &= \left[\sigma_Y \rho_{Ym} - D(\tau)\sigma_\eta \rho_{m\eta} - B(\tau)\sigma_r \rho_{mr}\right]\frac{1}{\eta} \equiv \frac{\alpha(\tau)}{\eta} \\ \frac{d\beta_{Vm}}{\beta_{Vm}} &= \left[\kappa_\eta - \kappa_\eta \bar{\eta}\frac{\beta_{Vm}}{\alpha(\tau)} + \sigma_\eta^2 \frac{\beta_{Vm}}{\alpha(\tau)}\right]dt - \frac{\beta_{Vm}}{\alpha(\tau)}\sigma_\eta dz_\eta \quad (21) \end{aligned}$$

In contrast, the market beta, β_{VM}, is non-stochastic given our assumptions that r and η follow Gaussian processes with constant volatilities, and that σ_M is constant. If either r or η followed a square root process or σ_M were stochastic, then β_{VM} and β_{Vm} would both be stochastic.[9] The market beta for a share of common stock is similar to expression (19) with B and D replaced by $\mathcal{B}$ and $\mathcal{D}$. Since the intuitions derived from the simpler single cash flow claim valuation model can be easily carried over to the common stock valuation model, we concentrate on the simpler model results given in equation (18) and (19).

[9] Ang and Liu (2002) assume an exogenous AR(1) process for β_{VM}.

Equation (19) decomposes a security's market beta into three elements: the cash flow beta, β_{YM}, which we have assumed to be exogenously given and independent of the cash flow maturity; the product of the beta of η, $\beta_{\eta M}$, and the sensitivity of the security value to η, $D(\tau)$; and the product of the beta of r, β_{rM}, and the sensitivity of the security value to r, $B(\tau)$. Expression (19) provides a semi-structural model of Campbell and Mei's (1993) empirical decomposition of asset betas into components attributable to news about future cash flows, real interest rates, and equity premia, and it allows us to analyze the effect on the market betas of securities of the maturity of the underlying cash flow.

Examination of the expressions in the Appendix for $D(\tau)$ and $B(\tau)$ shows that the only *cash flow specific* determinants of the security's market beta are the beta of the cash flow itself, β_{YM}, and the covariance of the cash flow with the *pricing kernel*, m. This means that, if the CAPM holds so that the pricing kernel is perfectly correlated with the return on the market, then the security market beta depends only on the market beta of the underlying cash flow. More generally, in a multi-factor setting the market beta will depend on the betas of the cash flow with respect to the market and with respect to all of the factors that enter the pricing kernel.

Note first that $B(\tau) \equiv \left(1 - e^{-\kappa_r \tau}\right)/\kappa_r$ is always positive and increasing in maturity, τ. On the other hand, the sign of $D(\tau)$ depends on parameter values. A *sufficient* condition for $D(\tau)$ to be positive and increasing in τ is that both $\rho_{Ym} > 0$ and $\rho_{mr} < 0$: the first condition implies that the cash flow risk carries a positive risk premium, while the second implies that real bonds, whose returns covary negatively with r, have positive interest rate risk premia. If, in addition, the beta of r and the beta of η are both negative (i.e., $\rho_{Mr} < 0$ and $\rho_{M\eta} < 0$), then the beta of the security is greater than the beta of the underlying cash flow, and is increasing in the cash flow maturity. To summarize,

$$\begin{aligned} \frac{\partial \beta_{VM}}{\partial \tau} &= -\beta_{\eta M}\left[\sigma_Y \rho_{Ym} e^{-\kappa_\eta^* \tau} + \frac{\sigma_r \rho_{mr}}{\kappa_\eta^* - \kappa_r}\left(e^{-\kappa_\eta^* \tau} - e^{-\kappa_r \tau}\right)\right] - \beta_{rM} e^{-\kappa_r \tau} \quad (22) \\ &> 0 \quad \text{if } \rho_{mr}, \beta_{M\eta}, \beta_{Mr} < 0, \text{ and } \rho_{Ym} > 0. \end{aligned}$$

where $\kappa_\eta^* = \kappa_\eta + \sigma_\eta \rho_{m\eta}$ is the mean reversion parameter for η under the risk neutral measure. Looking ahead, Table 1 reports that the conditions $\rho_{mr} < 0$ and $\beta_{\eta M}, \beta_{rM} < 0$ are satisfied by our empirical estimates from the U.S. data. Thus, β_{VM} will be an increasing (but nonlinear) function of τ provided that the underlying cash flow risk premium is positive ($\rho_{Ym} > 0$).

It follows from equation (18) that the expected excess real return on a security can be written as

$$\mu_V - r = \eta \left(D_\tau + D(\tau)\kappa_\eta\right) \tag{23}$$

Since both D_τ and D are functions of the underlying cash flow maturities, the instantaneous expected excess return on a security is a nonlinear (and possibly non-monotonic) function of the cash flow maturity. The instantaneous expected excess return can also be written as

$$\mu_V - r = -\mathrm{Cov}\left(\frac{dV}{V}, \frac{dm}{m}\right) = \eta^2 \beta_{Vm}. \tag{24}$$

so that it is an increasing linear function of the *pricing kernel* beta with a time-varying slope, η^2, but this does not imply that it is an increasing function of the *market* beta.

4 Model Calibration

In order to quantify the effect of time variation in r and η on market betas and discount rates, and the relation of betas and discount rates to cash flow maturity, we estimate the parameters of the simple ICAPM presented in Section 3, using data on zero-coupon Treasury yields and inflation. Since the state variables r, π, and η are unobservable, we use a Kalman filter algorithm to estimate the time series of these variables, and the parameters of their joint stochastic process, from monthly data on inflation and nominal interest rates. The interest rate data consist of yields on nine constant maturity zero coupon U.S. treasury bonds with maturities of 3, 6 months, and 1, 2, 3, 4, 5, 10, and 15 years. The sample period, which is from January 1983 to December 2000,[10] was chosen to avoid the period of the Federal Reserve policy experimentation around 1980. Panel A of Table 1 provides summary statistics for the data. The mean bond yields increase slightly with maturity, while their standard deviations are relatively constant across maturities. The inflation rate during the same sample period is calculated from the CPI and has an annualized sample mean of 3.22% and a sample standard deviation of 0.72%, which is less than half that of the bond yields, suggesting considerable variation in real interest rates.

Details of the Kalman filter algorithm in similar applications, which are presented in Brennan, Wang and Xia (2003), are omitted here. We summarize the assumptions made in the estimation. The nine observation equations relating constant maturity zero-coupon bond yields at time t, $y_{\tau_j,t}$, on bonds with maturities τ_j, $j = 1, \cdots, 9$, to state variables are the same as given by equation (13) except for the measurement error terms, ϵ_{τ_j}:

$$y_{\tau_j,t} \equiv -\frac{\ln N(t, t+\tau_j)}{\tau_j} = -\frac{\widehat{A}(t,\tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j} r_t + \frac{C(\tau_j)}{\tau_j}\pi_t + \frac{\widehat{D}(\tau)}{\tau}\eta_t + \epsilon_{\tau_j}(t). \quad (25)$$

The measurement errors, $\epsilon_{\tau_j}(t)$, are assumed to be serially and cross-sectionally uncorrelated, and to be uncorrelated with the innovations to the three state variables. The variance of these measurement errors is assumed to be of the form: $\sigma^2(\epsilon_{\tau_j}) = \sigma_b^2/\tau_j$ where σ_b is a parameter to be estimated. The final observation equation uses the realized rate of inflation, $\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}}$, as a signal of the expected rate of inflation, π:

$$\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}} = \pi\Delta t + \epsilon_P(t), \quad (26)$$

where $\mathrm{Var}\,(\epsilon_P)$ is assumed to be the same as the sample variance of the realized rate of inflation.

The estimation was carried out after setting the long run means of the three state

[10] We thank Robert Bliss for providing the data.

variables exogenously: $\bar{r}$ was set to 2.8%, which is the difference between the sample mean of the 3-month nominal interest rate and the CPI inflation, $\bar{\pi}$ to the CPI inflation sample mean of 3.2%, and $\bar{\eta}$ was set to 0.7, which is approximately equal to the sample mean of the Sharpe ratio of the CRSP value-weighted portfolio.[11] To reduce the number of parameters to be estimated, innovations to the realized rate of inflation were assumed to be orthogonal to all three state variables so that $\rho_{rP} = \rho_{\pi P} = \rho_{\eta P} = 0$. Since the estimate of ρ_{mP} was not significantly different from zero in the first round estimation, the results reported here were obtained after also setting ρ_{mP} to zero, which implies that unexpected inflation is not priced.

Panel B of Table 1 reports the parameter estimates, which are qualitatively similar to those reported in Brennan, Wang, Xia (2003) for the period 1952-2000. The main difference is that the estimates reported here imply less persistence and more volatility in the Sharpe ratio, η. The estimate of σ_b implies that the standard deviation of the yield measurement error varies from 45 basis points for the 3-month bond to 6 basis points for the fifteen-year bond, so that the model fits the bond yield data well. The volatility of expected inflation is only 0.55% per year while the volatility of the real interest rate is about 1% per year. The volatility of the Sharpe ratio is 0.42, which compares with the long run mean of 0.7. The mean-reversion coefficient for expected inflation is close to zero and is not statistically significant. In comparison, the estimated mean-reversion coefficient for the real interest rate is 0.069, which is highly significant and implies a half life of around ten years. The Sharpe ratio, η, has a mean-reversion coefficient of 0.103, which is also highly significant and implies a half life of 6.7 years. A Wald test of $H_0 : \kappa_\eta = \sigma_\eta = 0$ is rejected at 1% significance level, strongly supporting our conjecture that risk premia are time-varying. The strong persistence in both the real interest rate and the maximum Sharpe ratio suggests that shocks to these two state variables will have large effects on the values of long maturity cash flow claims. The estimated correlation between innovations to the

[11]Note that $\bar{\eta}$ is only a normalization. It may be seen from equation (2) that risk premia are unchanged if $\bar{\eta}$ is multiplied by a constant, k, and all correlations with the pricing kernel are multiplied by $1/k$.

real interest rate and the pricing kernel, ρ_{mr}, is -0.652 and highly significant: this implies that assets such as long term real bonds whose prices are inversely related to the real interest rate will have positive interest rate risk premia. On the other hand, the estimated correlation between innovations in η and the pricing kernel, $\rho_{m\eta}$, is 0.518 and also highly significant: implies that long term assets whose returns load negatively on η shocks will have negative η risk premia. The estimated correlation between innovations to expected inflation and the pricing kernel is not significant, implying that expected inflation risk is not priced. Innovations to η are significantly and negatively correlated with innovations to r so that shocks to the two components of expected returns are partially offsetting. Consistent with the results in Fama and Gibbons (1982), innovations to r and π are also negatively correlated.

Figure 2 plots the estimated state variables r and η. Both variables exhibit low frequency fluctuations which is consistent with their low mean reversion coefficients. The levels of r and η are slightly negatively correlated (-0.17). There is a pronounced downward trend in the level of η during the 1990s, and its *estimated* value becomes negative towards the end of the decade. Using the CRSP value-weighted portfolio as a proxy for the market portfolio, M, the regression of monthly market excess returns on estimates of the standardized innovations in r and η, dz_r and dz_η, yields negative coefficients on both innovations, and the coefficient on dz_η is highly significant:

$$\begin{array}{llll} dM/M = & 0.002- & 1.475dz_r- & 0.089dz_\eta \\ & (0.83) & (1.97) & (5.15) \end{array} \qquad \bar{R}^2 = 9.4\%$$

where t-ratios are reported in parentheses below the coefficients. The regression is consistent with the intuition that higher interest rates and risk premia reduce asset prices, and suggests that about 10% of stock market volatility is attributable to non-cash flow news.[12]

[12] To the extent that the state variable innovations are estimated with error the estimated R^2 (10%) is likely to be a lower bound on the proportion of stock volatility accounted for by variation in investment opportunities.

In order to determine ζ_i ($i = M$, η, r), the coefficients of the ICAPM pricing kernel, equation (5), we match the estimated correlations of r and η with the pricing kernel which are reported in Table 1, and impose the normalization condition (6). This yields three linear equations for the three unknowns:

$$\begin{aligned}
\rho_{mr} &= \zeta_M \rho_{Mr} + \zeta_\eta \rho_{\eta r} + \zeta_r, && (27)\\
\rho_{m\eta} &= \zeta_M \rho_{M\eta} + \zeta_\eta + \zeta_r \rho_{\eta r}, && (28)\\
1 &= \zeta_M^2 + \zeta_\eta^2 + \zeta_r^2 + 2\zeta_M\zeta_\eta\rho_{M\eta} + 2\zeta_M\zeta_r\rho_{Mr} + 2\zeta_\eta\zeta_r\rho_{\eta r} && (29)
\end{aligned}$$

The correlations between r, η and the proxy market portfolio, M, $\rho_{Mr} = -0.14$ and $\rho_{M\eta} = -0.31$, were calculated using the time series estimates of r and η. These correlation coefficients, together with the estimates of ρ_{mr}, $\rho_{m\eta}$ and $\rho_{r\eta}$ reported in Table 1, imply $\zeta_M = 0.742$, $\zeta_\eta = 0.640$, and $\zeta_r = -0.352$. Writing the stochastic process for the market return as

$$\frac{dM}{M} = \mu_M dt + \sigma_M dz_M, \tag{30}$$

the values of ζ_M, ζ_η and ζ_r imply an average equity premium of 6% per year for $\sigma_M = 14.5\%$ and $\eta = 0.7$, its long run mean.

5 Risk, Valuation and Cash Flow Maturity

In this section we use the calibrated ICAPM to investigate first the implied relation between the traditional market beta and the maturity of the underlying cash flow. We then analyze the implied term structure and volatility of discount rates, and discuss the implications for stock market volatility.

5.1 Beta and Cash Flow Maturity

Although the theoretically correct measure of risk when there is time variation in the investment opportunities is β_{Vm}, the beta with respect to the pricing kernel, defined in equation (20), attention has been focused on the traditional market beta, β_{VM}, and its puzzling dependence on the maturity of the underlying cash flows. Therefore we focus on the determinants of β_{VM}. We have already shown that, when the underlying cash flow carries a positive risk premium, a sufficient condition for β_{VM} to increase with the cash flow maturity is that $\rho_{Mr}, \rho_{M\eta}, \rho_{mr} < 0$. The estimates of these parameters reported in Table 1 are -0.14, -0.31 and -0.65, respectively, so that the sufficient condition is satisfied by our parameter estimates.

Figure 3 shows the effect of cash flow maturity on the market betas of securities calculated from equation (19), for different values of the market risk of the underlying cash flow, β_{YM}, which is defined from the process for the cash flow expectation:

$$\frac{dY}{Y} = \beta_{YM}\sigma_M dz_M + \sigma_e dz_e \tag{31}$$

where $\sigma_M dz_M$ is the innovation in the market return, and dz_e is a Brownian motion that is assumed for simplicity to be orthogonal to the pricing kernel. The market beta of the cash flow, β_{YM}, is taken to be constant. Then ρ_{Ym} is also constant, since the correlation of the cash flow with the pricing kernel, ρ_{Ym}, is proportional to the market beta of the cash flow:

$$\sigma_Y \rho_{Ym} = \beta_{YM}\sigma_M \left(\zeta_M + \zeta_r \rho_{Mr} + \zeta_\eta \rho_{M\eta}\right). \tag{32}$$

Figure 3a shows that the maturity effect on security market betas is material for all three values of the cash flow beta, β_{YM}. Moreover, the effect is increasing in the cash flow beta: for $\beta_{YM} = 0$, the difference between the market beta of a thirty-year maturity cash flow claim and the beta of an immediately maturing claim is 0.34, while the difference goes to 0.58 for $\beta_{YM} = 1$. This maturity effect is consistent with Cornell's

(1999) empirical finding that security betas are positively related to earnings growth rates. Figure 3a also shows that the security market beta is very close to the cash flow beta when the horizon is short, and that the other components of the market beta become important for long horizons. For example, the sum of the real interest rate and risk premium betas is close to 0.6 when $\beta_{YM} = 1.0$ and the horizon is 30 years.

If the CAPM holds, then the pricing kernel beta coincides with the market beta, and the expected return increases with beta, which in turn increases with the cash flow maturity. But under the ICAPM parameter estimates reported in Table 1, the fact that market beta increases with cash flow maturity does not imply that the expected returns also increase with maturity. In fact, depending on β_{YM} (or equivalently ρ_{Ym}), the instantaneous expected return can be an increasing, decreasing, or non-monotone function of the cash flow maturity. This is because the instantaneous expected excess return is proportional to $\beta_{Vm} \equiv \beta_{Ym} - D(\tau)\beta_{\eta m} - B(\tau)\beta_{rm}$. Both B$(\tau)$ and $D(\tau)$ are increasing with maturity, but $\beta_{\eta m}$ and β_{rm} are of opposite signs.[13] Therefore, if $D(\tau)$, the sensitivity of the claim value to changes in risk premia, increases with τ sufficiently rapidly relative to $B(\tau)$, the expected excess return can decrease with the maturity of the cash flow. This is more likely the higher is the systematic risk of the cash flow (β_{Ym}), because the values of claims with higher systematic risk are more sensitive to η and this sensitivity increases more rapidly with τ, as illustrated in equation (A4) in Appendix A.

Figure 3b plots the instantaneous expected excess return on cash flow claims for different values of β_{YM} and the cash flow maturity when $\eta = \bar{\eta}$. The expected excess returns are calculated using equation (23). When $\beta_{YM} = 0$, the instantaneous expected return increases from 0% to almost 2% as the cash flow maturity increases from zero to thirty years. When $\beta_{YM} = 0.5$, the expected excess return remains close to 3% for all maturities. When $\beta_{YM} = 1$ the expected excess return first decreases with maturity, from 6% with a zero maturity to less than 4% when the maturity is about 10 years, and

[13] Brennan *et al.*(2003) confirm this result in asset pricing tests using equity portfolios.

then increases modestly for longer maturities. This is consistent with the discussion in the previous paragraph since a high value of β_{YM} implies a high value of β_{Ym}. It is also consistent with the well documented finding that growth stocks, which have long duration cash flows, tend to have lower returns than value stocks which tend to have short duration cash flows. For all three values of β_{YM}, most of the maturity effect is concentrated in the first ten years because both r and η are mean-reverting.

5.2 Cash Flow Maturity, Valuation and Discount Rates

To examine the effect of cash flow maturity on asset valuation, we use equation (9) to calculate the value of a claim to a unit of expected real cash flow, $v(r,\eta,\tau) \equiv V(Y,r,\eta,\tau)/Y$, when the cash flow expectation follows the process given in equation (31). The present value factor for a risky cash flow with maturity τ, $v(r,\eta,\tau)$, also depends on the risk characteristics of the cash flow, $\sigma_Y\rho_{Ym}$ or, equivalently, on β_{YM}, as shown in equation (32).

The (continuously compounded) risk-adjusted discount rate, $\phi(\beta_{YM},\tau)$, defined as the rate which, when applied to the expected cash flow Y, yields the present value V so that $e^{-\phi\tau} = v$, is given by:

$$\phi(\beta_{YM},\tau) \equiv -\frac{\ln v}{\tau} = -\frac{A(\tau)}{\tau} + \frac{B(\tau)}{\tau}r + \frac{D(\tau)}{\tau}\eta, \tag{33}$$

where $A(\tau)$, $B(\tau)$, and $D(\tau)$ are defined in Appendix A.

Equation (33) implies that the discount rate $\phi(\beta_{YM},\tau)$ increases with the interest rate, r, since $B(\tau)$ is always positive. It also increases with η if $D(\tau) > 0$, which is satisfied when the interest rate risk premium and the cash flow risk are both positive, i.e., when $\rho_{mr} < 0$ and $\rho_{Ym} > 0$. Note that the interest rate risk premium condition is satisfied by the parameter estimate in Table 1. The discount rate is also a function of the cash flow beta and maturity. The derivative of ϕ with respect to τ is:

$$\frac{\partial\phi(\tau)}{\partial\tau} = -\frac{A'\tau - A}{\tau^2} + \frac{B'\tau - B}{\tau^2}r + \frac{D'\tau - D}{\tau^2}\eta, \tag{34}$$

where $A' \equiv \partial A/\partial\tau$ etc. In general, $\frac{B'\tau-B}{\tau^2} < 0$, since $B(\tau)$ increases *less than proportionally* with time to maturity. However, the sign of $\frac{D'\tau-D}{\tau^2}$ and $-\frac{A'\tau-A}{\tau^2}$ depend on the magnitude of the cash flow risk relative to other parameter values. Thus, $\phi(\tau)$ can be increasing, decreasing, or non-monotone in τ, depending on the parameter values and the state variables r and η, as we shall see in Figures 4 and 5.

Figure 4 plots the term structure of (real) discount rates calculated from equation (33) for $\beta_{YM} = 0$, 0.5, and 1 when the state variables r and η are set to their long run mean values. For very short maturities, security betas are close to the underlying cash flow betas, and the discount rates are roughly the same as those given by a simple CAPM with a risk free rate of 2.8% and a market risk premium of 6% as implied by $\bar{\eta} = 0.7$. As a result, the discount rates vary from around 3% when $\beta_{YM} = 0$ to more than 8% when $\beta_{YM} = 1$. However, discount rates tend to converge for long maturities. For $\beta_{YM} = 0.5$ or 1.0, the discount rate decreases with maturity - it decreases from 8.5% to 4.5% for $\beta_{YM} = 1.0$ - while it increases for $\beta_{YM} = 0.0$. The decrease in the discount rate with maturity is caused by the same factors that make the expected excess return on a claim decrease with maturity under certain conditions as discussed above. It contrasts with the practitioner's rule of thumb that long maturity cash flows should be discounted at *higher* rates.

Figure 5 shows the effects of one standard deviation perturbations in the current values of the state variables, r_0 and η_0, on the term structure of discount rates, for a constant rate of information arrival, σ_Y. A perturbation in r_0 changes the discount rates for short maturities by approximately the same amount, and the effect decays slowly as the maturity increases because of the persistence of r. The effect of a perturbation in η_0 increases with β_{YM}. For example, when the maturity is one year, the effect of a one-standard deviation increase in η_0 is to increase the discount rate from 6% to 9% when $\beta_{YM} = 0.5$ and to increase it from 8% to 15% when $\beta_{YM} = 1.0$. Although the effect of a perturbation in η_0 decays more rapidly with maturity than that of a perturbation in r_0, it is apparent that variation in η_0 is considerably more important for the discount rates of cash flows with $\beta_{YM} > 0.0$ than the effect of variation in r_0. At the ten year horizon, the discount rate for a cash flow with $\beta_{YM} = 1.0$ increases from around 2.9% when η_0 is one standard deviation below its mean value to around 9% when η_0 is one standard deviation above the mean.

The pattern of decreasing discount rates for $\beta_{YM} = 0.5$ and 1.0 also depends on the assumption that σ_Y, the rate at which new information about future cash flow is generated, is independent of maturity. Samuelson (1965) has shown that when there is a mean-reverting component in the spot price of a commodity, the volatility of the conditional expectation of the future spot price increases as the maturity decreases, which implies that σ_Y would be decreasing in τ. On the other hand, Cornell (1999) reports that "the common view at the company (Amgen, a pharmaceutical company) is that risk declines as the project moves out of basic research and toward commercial sale. Consequently, projects still in basic research (and that therefore have the longest cash flow maturity) are assigned the highest discount rates, whereas those that have completed all clinical test are assigned the lowest." This would imply that σ_Y increases with τ. Thus, the dependence of discount rates on the maturity of the underlying cash flow may be highly sensitive to the assumption about the time path of the rate of information arrival. This issue has received very little attention to date, and we turn to it next.

To examine the effect of time variation in the rate of cash flow information arrival on the term structure of discount rates, consider a situation in which Y, the expectation of a real cash flow, x, to be received at time $T = t + \tau$, follows a driftless *arithmetic* process with time dependent volatility:[14]

$$dY = \sigma_x e^{-\kappa_x \tau} dz_x \equiv \sigma_Y(\tau) dz_x. \tag{35}$$

Assume further that innovations in the cash flow expectation are related to innovations in the market return by:

$$dY = e^{-\kappa_x \tau} \beta_{xM} \sigma_M dz_M + \sigma_e dz_e \equiv \beta_{YM}(\tau) \sigma_m dz_M + \sigma_e dz_e$$

where β_{xM} is constant, and the Brownian motion dz_e is orthogonal to all other variables in the economy. A constant β_{xM} implies a constant ρ_{xm} since $\sigma_x \rho_{xm} = \beta_{xM} \sigma_M (\zeta_M + \zeta_r \rho_{Mr} + \zeta_\eta \rho_{M\eta})$. On the other hand, both $\sigma_Y(\tau)$ and $\beta_{YM}(\tau)$ decrease with τ for $\kappa_x > 0$, while they increase with τ for $\kappa_x < 0$.

Then, extending Theorem 1 in Brennan *et al.* (2003) to the case of an arithmetic process for Y, the value of the claim to the cash flow is given in the following theorem:

Theorem 1 *(Arithmetic Process for Y)*
In an economy in which the investment opportunity set is described by (1) and (3-4), the value at time t of a claim to a real cash flow x at time T, whose expectation, Y, follows the stochastic process (35), is given by:

$$\begin{aligned} V(Y, \tau, r, \eta) &= \mathrm{E}_t^Q \left[Y_T \exp^{-\int_t^T r(s)ds} \right] \\ &= [Y_t + F(\tau)\eta_t + E(\tau)] \exp \{A^*(\tau) - B(\tau) r_t - D^*(\tau)\eta_t\} \end{aligned} \tag{36}$$

where Q denotes the risk neutral probability measure, and $A^*(\tau)$, $B(\tau)$, $D^*(\tau)$, $E(\tau)$ and $F(\tau)$, are given in equations (B10 - B14) in Appendix B.

The (continuously compounded) risk-adjusted discount rate, $\phi(\beta_{xM}, \tau)$, is then given by:

$$\phi(\beta_{xM}, \tau) \equiv -\frac{\ln v}{\tau} = -\frac{\ln [Y_t + E(\tau) + F(\tau)\eta_t]}{\tau} - \frac{A^*(\tau)}{\tau} + \frac{B(\tau)}{\tau} r + \frac{D^*(\tau)}{\tau} \eta. \tag{37}$$

[14]If a continuous real cash flow rate, $x(t)$, follows an Ornstein-Uhlenbeck process $dx = \kappa_x(\bar{x} - x)dt + \sigma_x dz_x$, then $Y(t) \equiv E_t[x(T)]$, the expectation at time t of the cash flow rate at time T, will follow the process (35).

Under this modified information structure, the risk adjusted discount rate continues to be increasing in the real interest rate, r, and also in η provided, as before, that $\rho_{mr} < 0$ and $\rho_{xm} > 0$. However, when $\beta_{xM} \neq 0$, the term structure of discount rates now depends also on the parameter κ_x, which captures the relation between the rate of information-arrival and the time to maturity of the cash flow. A positive value of κ_x implies that the rate of information arrival, $\sigma_Y(\tau)$, increases as t approaches T, the date of the cash flow realization. As a result, the ratio of the conditional variance of the cash flow x at time t to the remaining time to maturity also increases with time, tending to increase the discount rate for cash flow claims near maturity and to decrease them for cash flows far from maturity. The effect is most pronounced for maturities in the range of 5-15 years as seen in Figure 6, which plots term structures of discount rates for different values of κ_x when $r = \bar{r}, \eta = \bar{\eta}$. When κ_x is negative, the reverse effect is observed: now the rate of information arrival is highest at long maturities and this tends to raise discount rates for long maturity cash flows. As a result, it is possible for the term structure of discount rates to take on a U-shape, as shown in the Figure. In summary, the relation between discount rates and cash flow maturity is highly sensitive to the time pattern of σ_Y, which is captured in this example by the parameter κ_x.

5.3 Valuation Factors and Volatility

Cochrane (1991) has pointed out that time variation in discount rates can substantially increase the volatility of stock prices relative to a constant discount rate model and, consistent with this, Dechow et. al. (2001) report that stock return volatility increases with a measure of the duration of the stock's income stream. To assess the effect on stock prices of the variation in discount rates induced by variation in the state variables, η and r, the present value factor $v(r, \eta, \tau)$ was calculated for each month from January 1983 to December 2000 using the Kalman-filter estimates of r and η, and the parameters of their joint stochastic process reported in Table 1. The cash flow risk parameter, β_{YM}, was set to unity so that, as seen in Figure 3, the corresponding security beta is about 1.4 (1.25) when the cash flow maturity is twenty (five) years. The present value factor is plotted in Figure 7a for cash flow maturities of one, five, and twenty years, and summary statistics are reported in Table 2. The present value factor for a cash flow maturity of twenty years[15] exhibits the strongest variation over time, ranging from a low of 0.12 in February 1985 to a high of 0.76 in January 1999. Figure 7b, which plots the corresponding discount rates, shows that the implied discount rates on these dates were 10.6% and 1.4%, respectively. A striking feature of Figure 7b is the decline in the twenty-year discount rate from over 8.5% in 1992 to 2-4% at the end of the decade. This was the period in which the S&P 500 index rose from around 400 to 1300. The figures suggest that time-varying discount rates have major effects on the level of stock prices and on market volatility. This is confirmed by the last line of Table 2, which shows that the annualized "return" volatility, $\sigma\left(dv/v\right)$, ranges from 4% for $\tau = 1$ to 27% for $\tau = 20$. Similar patterns can be found for $\beta_{YM} = 0.5$ which corresponds to a security beta of about 0.8 (0.7) when the cash flow maturity is twenty (five) years. β_{YM} has only a small effect on present values and discount rates at the twenty-year maturity, but its effect is more significant for shorter maturities, especially when η is large.

To provide a visual assessment of the relative importance of r and η for the variability of the present value factor, Figure 8 plots time series of: the pure maturity element, $\ln A(\tau)$; the element that captures the effect of variation in interest rates, $\ln v^r \equiv A(\tau) - B(\tau)r_t$; and the present value factor itself, $\ln v$, for $\beta_{YM} = 1.0$ and $\tau = 20$ years. Figure 8 shows that $\ln v$ is much more variable than $\ln v^r$: $\ln v$ ranges from -0.3 to -2.1 while $\ln v^r$ ranges only between -0.3 and -1.05, with the incremental variation driven by the variation in η. In the late 1990's $\ln v$ is very close to $\ln v^r$, even exceeding it at times. This is because the estimated value of η is very small and even negative at these times. Since η measures the risk-return trade-off that is available in the capital market, these

[15]Note that a simple Gordon model implies that the duration of a stock ranges from 20 to 50 years as the dividend yield changes from 5% to 2%.

very low and even negative values are consistent with claims that the market was seized by "irrational exuberance" at this time.

Table 2 reports summary statistics on v, $v^r \equiv e^{A(\tau)-B(\tau)r}$ and $v^\eta \equiv e^{A(\tau)-D(\tau)\eta}$ for three different maturities when $\beta_{YM} = 1$. For all maturities, v^η is more influential than v^r. When $\tau = 20$, the variability of v^η, as well as the volatility of its "rate of return", $\sigma(dv^\eta/v^\eta)$, is almost twice that of v^r. For the shorter maturities, the ratio is around three times. Although the variability in v due to η is more important than that due to r, the annualized "return" volatility of v^r is still 13% for $\tau = 20$. This contrasts with the finding of Campbell and Ammer (1993) that variation in real interest rates accounts for only a minor component of the variation in stock returns.

6 Conclusion

In this paper we have used a simple ICAPM with a time-varying interest rate and market Sharpe ratio to analyze the dependence of security market betas and risk-adjusted discount rates on the maturity and risk characteristics of the underlying cash flow. Applying a Kalman filter to zero-coupon Treasury bond yields and inflation data, estimates of the model parameters and time series of the state variables r and η were obtained. The parameterized model was then used first to examine the relation between security betas and the maturities of the underlying cash flows. Security betas were shown to be increasing in maturity, which is consistent with empirical findings of Cornell (1999) and Dechow et. al. (2001). Expected returns, on the other hand, can be increasing, decreasing or non-monotone functions of maturity, depending on the risk characteristics of the cash flow and the values of the state variables. The term structure of risk-adjusted discount rates was shown to depend on the risk characteristics of the underlying cash flow and the values of the state variables, r and η. The estimated time series of r and η was used to construct time series of present value factors and risk adjusted discount rates. The estimates show that the level of the state variables is likely to be an important determinant of the level of stock prices, and that the resulting time variation in discount rates is likely to be an important component of stock return volatility.

Given the empirical failures the simple CAPM, the evidence that we have presented here, both of time-variation in investment opportunities and of the major effect of this on valuation, suggests that, for purposes of valuation, there are major gains to moving beyond the simple CAPM to models such as the one we have presented which takes account, both of the time variation in investment opportunities, and of the risk premia that are associated with this time-variation.

Appendix

A. Parameters of the Valuation Equation

In equation (9), $B(\tau)$, $D(\tau)$ and $A(\tau)$ are given by:

$$
\begin{aligned}
B(\tau) &= {\kappa_r}^{-1}(1 - e^{-\kappa_r \tau}) && \text{(A1)}\\
D(\tau) &= d_1 + d_2 e^{-\kappa_\eta^* \tau} + d_3 e^{-\kappa_r \tau} && \text{(A2)}\\
A(\tau) &= a_1 \tau + a_2 \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} + a_4 \frac{1 - e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} + a_5 \frac{1 - e^{-2\kappa_r \tau}}{2\kappa_r}\\
&+ a_7 \frac{1 - e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} + a_8 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r}. && \text{(A3)}
\end{aligned}
$$

where $\kappa_\eta^* \equiv \kappa_\eta + \sigma_\eta \rho_{m\eta}$, and $d_1, \cdots, d_3$ are defined in the following equations:

$$
\begin{aligned}
d_1 &= \frac{\sigma_Y \rho_{mY}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{\kappa_r \ \kappa_\eta^*}\\
d_2 &= -\frac{\sigma_Y \rho_{mY}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_\eta^*} = -d_1 - d_3\\
d_3 &= \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_r}
\end{aligned}
$$

Define $a_0 \equiv \frac{\sigma_{r\eta}}{\kappa_r} - \sigma_{Y\eta} - \kappa_\eta^* \bar{\eta}^*$, $\bar{\eta}^* \equiv \frac{\kappa_\eta \bar{\eta}}{\kappa_\eta^*}$, and $\bar{r}^* \equiv \bar{r} + \frac{\sigma_{Yr}}{\kappa_r}$, then $a_1, \cdots, a_8$ are expressed as:

$$
\begin{aligned}
a_1 &= \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\eta^2}{2} d_1^2 - \bar{r}^* + a_0 d_1\\
a_2 &= \bar{r}^* - \frac{\sigma_r^2}{\kappa_r^2} - \frac{\sigma_{r\eta}}{\kappa_r} d_1 + a_0 d_3 + \sigma_\eta^2 d_1 d_3\\
a_4 &= a_0 d_2 + \sigma_\eta^2 d_1 d_2\\
a_5 &= \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\eta^2}{2} d_3^2 - \frac{\sigma_{r\eta}}{\kappa_r} d_3\\
a_7 &= \frac{\sigma_\eta^2}{2} d_2^2\\
a_8 &= -\frac{\sigma_{r\eta}}{\kappa_r} d_2 + \sigma_\eta^2 d_2 d_3
\end{aligned}
$$

The partial derivative of $D(\tau)$ with respect to τ is given by:

$$D_\tau = \sigma_Y \rho_{mY} e^{-\kappa_\eta^* \tau} + \frac{\sigma_r \rho_{mr}}{\kappa_\eta^* - \kappa_r}(e^{-\kappa_\eta^* \tau} - e^{-\kappa_r \tau}). \tag{A4}$$

The assumptions $\rho_{mY} > 0$ and $\rho_{mr} < 0$ imply that $D_\tau \geq 0$ and $D(\tau) \geq 0 \ \forall \tau$.

In equation (12), $B(\tau)$, $\widehat{C}(\tau)$, $\widehat{D}(\tau)$, and $\widehat{A}(\tau)$ are defined as:

$$B(\tau) = \kappa_r^{-1}\left(1 - e^{-\kappa_r \tau}\right) \tag{A5}$$

$$\widehat{C}(\tau) = \kappa_\pi^{-1}\left(1 - e^{-\kappa_\pi \tau}\right) \tag{A6}$$

$$\widehat{D}(\tau) = \hat{d}_1 + \hat{d}_2 e^{-\kappa_\eta^* \tau} + \hat{d}_3 e^{-\kappa_r \tau} + \hat{d}_4 e^{-\kappa_\pi \tau} \tag{A7}$$

$$\begin{aligned} \widehat{A}(\tau) = {} & \hat{a}_1 \tau + \hat{a}_2 \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} + \hat{a}_3 \frac{1 - e^{-\kappa_\pi \tau}}{\kappa_\pi} + \hat{a}_4 \frac{1 - e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} \\ & + \hat{a}_5 \frac{1 - e^{-2\kappa_r \tau}}{2\kappa_r} + \hat{a}_6 \frac{1 - e^{-2\kappa_\pi \tau}}{2\kappa_\pi} + \hat{a}_7 \frac{1 - e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} \\ & + \hat{a}_8 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r} + \hat{a}_9 \frac{1 - e^{-(\kappa_\eta^* + \kappa_\pi)\tau}}{\kappa_\eta^* + \kappa_\pi} + \hat{a}_{10} \frac{1 - e^{-(\kappa_r + \kappa_\pi)\tau}}{\kappa_r + \kappa_\pi}. \end{aligned} \tag{A8}$$

where κ_η^* is defined above. Expressions for $\hat{d}_1, \cdots, \hat{d}_4$ and $\hat{a}_1, \cdots, \hat{a}_{10}$ are omitted for brevity.

In equation (16), the expressions for $\tilde{A}(\tau)$,$\cdots$,$F(\tau)$ are given by:

$$\begin{aligned}
B(s-t) &= \kappa_r^{-1}\left(1-e^{-\kappa_r(s-t)}\right) && \text{(A9)}\\
C(s-t) &= \kappa_\pi^{-1}\left(1-e^{-\kappa_\pi(s-t)}\right) && \text{(A10)}\\
F(s-t) &= \kappa_g^{-1}\left(1-e^{-\kappa_g(s-t)}\right) && \text{(A11)}\\
\tilde{D}((s-t)) &= \tilde{d}_1+\tilde{d}_2 e^{-\kappa_\eta^*(s-t)}+\tilde{d}_3 e^{-\kappa_r(s-t)}+\tilde{d}_4 e^{-\kappa_\pi(s-t)}+\tilde{d}_5 e^{-\kappa_g(s-t)} && \text{(A12)}\\
\tilde{A}((s-t)) &= \tilde{a}_1(s-t)+\tilde{a}_2\frac{1-e^{-\kappa_r(s-t)}}{\kappa_r}+\tilde{a}_3\frac{1-e^{-\kappa_\pi(s-t)}}{\kappa_\pi}+\tilde{a}_4\frac{1-e^{-\kappa_\eta^*(s-t)}}{\kappa_\eta^*}\\
&\quad+\tilde{a}_5\frac{1-e^{-2\kappa_r(s-t)}}{2\kappa_r}+\tilde{a}_6\frac{1-e^{-2\kappa_\pi(s-t)}}{2\kappa_\pi}+\tilde{a}_7\frac{1-e^{-2\kappa_\eta^*(s-t)}}{2\kappa_\eta^*}\\
&\quad+\tilde{a}_8\frac{1-e^{-(\kappa_\eta^*+\kappa_r)(s-t)}}{\kappa_\eta^*+\kappa_r}+\tilde{a}_9\frac{1-e^{-(\kappa_\eta^*+\kappa_\pi)(s-t)}}{\kappa_\eta^*+\kappa_\pi}+\tilde{a}_{10}\frac{1-e^{-(\kappa_r+\kappa_\pi)(s-t)}}{\kappa_r+\kappa_\pi}\\
&\quad+\tilde{a}_{11}\frac{1-e^{-(\kappa_\eta^*+\kappa_g)(s-t)}}{\kappa_\eta^*+\kappa_g}+\tilde{a}_{12}\frac{1-e^{-(\kappa_g+\kappa_\pi)(s-t)}}{\kappa_g+\kappa_\pi}+\tilde{a}_{13}\frac{1-e^{-(\kappa_r+\kappa_g)(s-t)}}{\kappa_r+\kappa_g}\\
&\quad+\tilde{a}_{14}\frac{1-e^{-\kappa_g(s-t)}}{\kappa_g}+\tilde{a}_{15}\frac{1-e^{-2\kappa_g(s-t)}}{2\kappa_g} && \text{(A13)}
\end{aligned}$$

where $\kappa_\eta^* \equiv \kappa_\eta+\sigma_\eta\rho_{m\eta}$, and $\tilde{d}_1,\ldots,\tilde{d}_5$, $\tilde{a}_1,\ldots,\tilde{a}_{15}$ are constants whose values are available upon request.

B. Proof of Theorem 1

The real part of the economy is described by the processes for the real pricing kernel, the real interest rate, and the maximum Sharpe ratio (1), (3), and (4). Under the risk neutral probability measure Q, we can write these processes as:

$$\begin{aligned}
dr &= \kappa_r(\bar{r}-r)dt-\sigma_r\rho_{mr}\eta dt+\sigma_r dz_r^Q && \text{(B1)}\\
d\eta &= \kappa_\eta^*(\bar{\eta}^*-\eta)dt+\sigma_\eta dz_\eta^Q && \text{(B2)}
\end{aligned}$$

where $\kappa_\eta^*=\kappa_\eta+\sigma_\eta\rho_{m\eta}$ and $\bar{\eta}^*=\frac{\kappa_\eta\bar{\eta}}{\kappa_\eta^*}$.

Let Y, whose stochastic process is given by (35), denote the expectation of a real cash flow which is realizable at a future date T, x_T. The process for Y under the risk neutral probability measure can be written as:

$$dY_t = -\eta\sigma_x\rho_{xm}e^{-\kappa_x(T-t)}dt + \sigma_x e^{-\kappa_x(T-t)}dz_x^Q. \tag{B3}$$

The real value at time t of the claim to the real cash flow at time T, x_T, is given by expected discounted value of the real cash flow under Q:

$$V(\xi, r, \pi, \eta, T-t) \;=\; \mathrm{E}_t^Q\left[x_T \exp^{-\int_t^T r(s)ds}\right] = \mathrm{E}_t^Q\left[Y_T \exp^{-\int_t^T r(s)ds}\right] \tag{B4}$$

Using equation (B3), we have

$$Y_T \;=\; Y_t - \sigma_x\rho_{xm}\int_t^T e^{-\kappa_x(T-s)}\eta_s ds + \sigma_x\int_t^T e^{-\kappa_x(T-s)}dz_x^Q(s). \tag{B5}$$

Thus, equation (B4) can be decomposed as:

$$\begin{aligned} V(\xi, r, \pi, \eta, T-t) &= \mathrm{E}_t^Q\left[e^{-\int_t^T r_s ds}\right]\left\{Y_t + \sigma_x\rho_{xm}\frac{e^{-\kappa_\eta^*(T-t)} - e^{-\kappa_x(T-t)}}{\kappa_\eta^* - \kappa_x}\eta_t\right. \\ &+ \left.\sigma_x\rho_{xm}\bar{\eta}^*\left[\frac{1-e^{-\kappa_x(T-t)}}{\kappa_x} - \frac{e^{-\kappa_\eta^*(T-t)} - e^{-\kappa_x(T-t)}}{\kappa_\eta^* - \kappa_x}\right]\right\} \\ &+ \mathrm{E}_t^Q\left\{\left[\int_t^T \frac{\sigma_x\rho_{xm}\sigma_\eta\left(e^{\kappa_\eta^*(s-T)} - e^{\kappa_x(s-T)}\right)}{\kappa_\eta^* - \kappa_x}dz_\eta^Q(s) + \sigma_x\int_t^T e^{-\kappa_x(T-s)}dz_x^Q(s)\right]e^{-\int_t^T r_s ds}\right\}. \end{aligned} \tag{B6}$$

The last term in the previous equation can be simplified by the following calculation:

$$\begin{aligned} G_1 &\equiv \mathrm{E}_t^Q\left\{\left[\int_t^T \frac{\sigma_x\rho_{xm}\sigma_\eta\left(e^{\kappa_\eta^*(s-T)} - e^{\kappa_x(s-T)}\right)}{\kappa_\eta^* - \kappa_x}dz_\eta^Q(s)\right]e^{-\int_t^T r_s ds}\right\} \\ &= \mathrm{E}_t^Q\left[\int_t^T \frac{\sigma_x\rho_{xm}\sigma_\eta\left(e^{\kappa_\eta^*(s-T)} - e^{\kappa_x(s-T)}\right)}{\kappa_\eta^* - \kappa_x}dz_\eta^Q(s)\right]\mathrm{E}_t^Q\left[e^{-\int_t^T r_s ds}\right] \\ &+ \mathrm{Cov}\left(\int_t^T \frac{\sigma_x\rho_{xm}\sigma_\eta\left(e^{\kappa_\eta^*(s-T)} - e^{\kappa_x(s-T)}\right)}{\kappa_\eta^* - \kappa_x}dz_\eta^Q(s), e^{-\int_t^T r_s ds}\right) \\ &= \mathrm{Cov}\left(\int_t^T \frac{\sigma_x\rho_{xm}\sigma_\eta\left(e^{\kappa_\eta^*(s-T)} - e^{\kappa_x(s-T)}\right)}{\kappa_\eta^* - \kappa_x}dz_\eta^Q(s), -\int_t^T r_s ds\right)\mathrm{E}_t^Q\left[e^{-\int_t^T r_s ds}\right], \end{aligned} \tag{B7}$$

where the last equality follows from Stein's Lemma.

In a similar way,

$$
\begin{aligned}
G_2 &\equiv \mathrm{E}_t^Q \left\{ \left[\sigma_x \int_t^T e^{-\kappa_x (T-s)} dz_x^Q(s) \right] e^{-\int_t^T r_s ds} \right\} \\
&= -\mathrm{E}_t^Q \left[e^{-\int_t^T r_s ds} \right] \left\{ \left[-\frac{\sigma_{x\eta}\sigma_r\rho_{mr}}{\kappa_\eta^* \kappa_r} + \frac{\sigma_{xr}}{\kappa_r} \right] \frac{1 - e^{-\kappa_x (T-t)}}{\kappa_x} \right. \\
&- \frac{\sigma_{x\eta}\sigma_r\rho_{mr}}{\kappa_\eta^*(\kappa_\eta^* - \kappa_r)} \frac{1 - e^{-(\kappa_\eta^* + \kappa_x)(T-t)}}{\kappa_\eta^* + \kappa_x} \\
&+ \left. \left[-\frac{\sigma_{x\eta}\sigma_r\rho_{mr}}{\kappa_r(\kappa_\eta^* - \kappa_r)} + \frac{\sigma_{xr}}{\kappa_r} \right] \frac{1 - e^{-(\kappa_x + \kappa_r)(T-t)}}{\kappa_x + \kappa_r} \right\}.
\end{aligned}
\tag{B8}
$$

Substituting equations (B7) and (B8) into equation (B6), the value is then given by

$$
V(Y, r, \eta, \tau) = [Y_t + F(\tau)\eta_t + E(\tau)] \exp\{A^*(\tau) - B(\tau) r_t - D^*(\tau)\eta_t\} \tag{B9}
$$

where $\tau \equiv T - t$ and

$$
B(\tau) = \kappa_r^{-1} \left(1 - e^{-\kappa_r \tau}\right) \tag{B10}
$$

$$
F(\tau) = \sigma_x \rho_{xm} \frac{e^{-\kappa_\eta^*(T-t)} - e^{-\kappa_x (T-t)}}{\kappa_\eta^* - \kappa_x} \tag{B11}
$$

$$
D^*(\tau) = \frac{\sigma_r \rho_{mr}}{\kappa_\eta^* - \kappa_r} \left[\frac{1 - e^{-\kappa_\eta^*(T-t)}}{\kappa_\eta^*} - \frac{1 - e^{-\kappa_r (T-t)}}{\kappa_r} \right] \tag{B12}
$$

$$
\begin{aligned}
A^*(\tau) &= a_1 \tau + a_2 \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} + a_3 \frac{1 - e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} + a_4 \frac{1 - e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} \\
&+ a_5 \frac{1 - e^{-2\kappa_r \tau}}{2\kappa_r} + a_6 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r}
\end{aligned}
\tag{B13}
$$

$$
\begin{aligned}
E(\tau) &= e_1 F(\tau) + e_2 \frac{1 - e^{-\kappa_x \tau}}{\kappa_x} + e_3 \frac{1 - e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} + e_4 \frac{1 - e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} \\
&+ e_5 \frac{1 - e^{-(\kappa_\eta^* + \kappa_x)\tau}}{\kappa_\eta^* + \kappa_x} + e_6 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r} + e_7 \frac{1 - e^{-(\kappa_x + \kappa_r)\tau}}{\kappa_x + \kappa_r}.
\end{aligned}
\tag{B14}
$$

The constant coefficients, $a_1, \ldots, a_6$, are given by:

$$a_1 = -\bar{r} + \frac{\sigma_r \rho_{mr} \bar{\eta}^*}{\kappa_r} - \frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{\kappa_r^2 \kappa_\eta^*} + \frac{1}{2}\left(\frac{\sigma_r \sigma_\eta \rho_{mr}}{\kappa_\eta^* \kappa_r}\right)^2 + \frac{1}{2}\left(\frac{\sigma_r}{\kappa_r}\right)^2 \tag{B15}$$

$$\begin{aligned} a_2 &= \bar{r} - \frac{\sigma_r \rho_{mr}}{\kappa_r (\kappa_\eta^* - \kappa_r)} \kappa_\eta^* \bar{\eta}^* - \left(\frac{\sigma_r}{\kappa_r}\right)^2 \\ &- \frac{\sigma_r^2 \rho_{mr}^2 \sigma_\eta^2}{(\kappa_\eta^* - \kappa_r)\kappa_r^2 \kappa_\eta^*} - \frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{(\kappa_\eta^* - \kappa_r)\kappa_\eta^* \kappa_r} + 2\frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{(\kappa_\eta^* - \kappa_r)\kappa_r^2} \end{aligned} \tag{B16}$$

$$a_3 = \frac{\sigma_r \rho_{mr}}{\kappa_\eta^* - \kappa_r} \bar{\eta}^* + \frac{\sigma_r^2 \rho_{mr}^2 \sigma_\eta^2}{(\kappa_\eta^* - \kappa_r)\kappa_r \left(\kappa_\eta^*\right)^2} - \frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{(\kappa_\eta^* - \kappa_r)\kappa_\eta^* \kappa_r} \tag{B17}$$

$$a_4 = \frac{1}{2}\left(\frac{\sigma_r \sigma_\eta \rho_{mr}}{\kappa_\eta^* (\kappa_\eta^* - \kappa_r)}\right)^2 \tag{B18}$$

$$a_5 = \frac{1}{2}\left(\frac{\sigma_r \sigma_\eta \rho_{mr}}{\kappa_r (\kappa_\eta^* - \kappa_r)}\right)^2 + \frac{1}{2}\left(\frac{\sigma_r}{\kappa_r}\right)^2 - \frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{(\kappa_\eta^* - \kappa_r)\kappa_r^2} \tag{B19}$$

$$a_6 = -\left(\frac{\sigma_r \sigma_\eta \rho_{mr}}{\kappa_\eta^* - \kappa_r}\right)^2 \frac{1}{\kappa_\eta^* \kappa_r} + \frac{\sigma_r^2 \rho_{mr} \sigma_\eta \rho_{r\eta}}{(\kappa_\eta^* - \kappa_r)\kappa_\eta^* \kappa_r} \tag{B20}$$

and the constant coefficients, $e_1, \ldots, e_7$, are given by:

$$e_1 = -\bar{\eta}^* \tag{B21}$$

$$e_2 = -\sigma_x \rho_{xm} \bar{\eta}^* + \frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x}\left[-\frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_\eta^* \kappa_r} + \frac{\sigma_r \rho_{r\eta}}{\kappa_r}\right] + \frac{\sigma_{x\eta} \sigma_r \rho_{mr}}{\kappa_\eta^* \kappa_r} - \frac{\sigma_{xr}}{\kappa_r} \tag{B22}$$

$$e_3 = \frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x}\left[\frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_\eta^* \kappa_r} - \frac{\sigma_r \rho_{r\eta}}{\kappa_r}\right] \tag{B23}$$

$$e_4 = \frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x} \frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_\eta^* (\kappa_\eta^* - \kappa_r)} \tag{B24}$$

$$e_5 = \frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x}\left[-\frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_\eta^* (\kappa_\eta^* - \kappa_r)}\right] + \frac{\sigma_{x\eta} \sigma_r \rho_{mr}}{\kappa_\eta^* (\kappa_\eta^* - \kappa_r)} \tag{B25}$$

$$e_6 = \frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x}\left[-\frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_r (\kappa_\eta^* - \kappa_r)} + \frac{\sigma_r \rho_{r\eta}}{\kappa_r}\right] \tag{B26}$$

$$e_7 = -\frac{\sigma_x \rho_{xm} \sigma_\eta}{\kappa_\eta^* - \kappa_x}\left[-\frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_r (\kappa_\eta^* - \kappa_r)} + \frac{\sigma_r \rho_{r\eta}}{\kappa_r}\right] - \frac{\sigma_{x\eta} \sigma_r \rho_{mr}}{\kappa_r (\kappa_\eta^* - \kappa_r)} + \frac{\sigma_{xr}}{\kappa_r}. \tag{B27}$$

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Table 1
Summary Statistics and Model Parameter Estimates

Bond yields are monthly constant maturity zero coupon U.S. Treasury yields for maturities of three and six months, and one, two, three, four, five, ten and fifteen years for the period of January 1983 to December 2000. Inflation is calculated from the CPI data for the same sample period. The mean and standard deviation are in percent per year. Panel B reports parameter estimates for the stochastic process of the investment opportunity set based on the CPI and bond yield data. The long run mean of r, π and η are exogenously set: $\bar{r} = 2.82\%$, $\bar{\pi} = 3.22\%$ and $\bar{\eta} = 0.7$. The variance of the measurement error in the inflation equation is also set exogenously at $\sigma_P = 0.72\%$. Parameter estimates were obtained using a Kalman filter algorithm. t-ratios are in parenthesis. The correlation coefficients of the market return with the innovations in the state variables are sample correlations using the filtered time series of r and η. n.a.: stands for not available.

Panel A: Summary Statistics for Bond Yields and Inflation (% per year)

Bond Maturity (years)	0.25	0.5	1	2	3	4	5	10	15	Infl.
Mean	6.02	6.21	6.48	6.93	7.14	7.47	7.64	7.80	8.10	3.22
Std. Dev.	1.83	1.87	1.92	1.96	1.96	1.98	1.99	1.92	1.87	0.72

Panel B: Model Parameters

Parameter	σ_b	κ_r	σ_r	κ_η	σ_η	κ_π	σ_π
Estimates	0.225%	0.069	0.966%	0.103	0.424	0.000	0.554%
	(56.65)	(57.65)	(22.76)	(5.81)	(18.48)	(0.72)	(6.12)
Parameter	$\rho_{r\eta}$	$\rho_{r\pi}$	ρ_{rm}	$\rho_{\eta\pi}$	$\rho_{\eta m}$	$\rho_{\pi m}$	ρ_{Pm}
Estimates	-0.307	-0.141	-0.652	-0.269	0.518	-0.128	0.0
	(2.78)	(2.26)	(10.02)	(2.30)	(7.64)	(1.50)	n.a.
Other Statistics	$\bar{r}$	$\bar{\pi}$	$\bar{\eta}$	σ_P	ρ_{Mr}	$\rho_{M\eta}$	ML
	2.82%	3.22%	0.70	0.72%	-0.140	-0.311	12,242,6

Table 2

Summary Statistics on the Valuation Factor

The table reports the variation in the valuation factor, $v \equiv e^{A(\tau)-B(\tau)r-D(\tau)\eta}$, that is attributable to variation in state variables r: $v^r \equiv e^{A(\tau)-B(\tau)r}$; and η: $v^\eta \equiv e^{A(\tau)-D(\tau)\eta}$. Factors are calculated for maturities of 1, 5, and 20 years when the cash flow beta is unity. Under the ICAPM assumption the pricing kernel is $dm/m = -rdt - \eta(\zeta_M dz_M - \zeta_\eta dz_\eta - \zeta_r dz_r)$. The correlations between the innovations in r and η and the market return, ρ_{Mr} and $\rho_{M\eta}$, are the sample estimates, -0.14 and -0.31 respectively. The coefficients of the pricing kernel match the yield-based estimates of the parameters $\rho_{mr}(-0.65)$ and $\rho_{m\eta}(0.52)$ and satisfy the constraint that Var $(dm/m) = \eta^2$, which implies that $\zeta_M = 0.74$, $\zeta_\eta = 0.64$ and $\zeta_r = -0.35$. The sample period is from January 1983 to December 2000. $\sigma(.)$ is the standard deviation of the time series of values. $\sigma(dv/v)$ is the annualized volatility of the proportional changes in values.

	$\tau = 1$ Year			$\tau = 5$ Years			$\tau = 20$ Years		
	v	v^r	v^η	v	v^r	v^η	v	v^r	v^η
	$A(\tau) = -0.003$			$A(\tau) = -0.048$			$A(\tau) = -0.373$		
	$B(\tau) = 0.967$			$B(\tau) = 4.230$			$B(\tau) = 10.859$		
	$D(\tau) = 0.076$			$D(\tau) = 0.257$			$D(\tau) = 0.459$		
Mean	0.908	0.975	0.929	0.691	0.866	0.762	0.371	0.544	0.476
Min	0.777	0.937	0.815	0.392	0.727	0.484	0.119	0.343	0.205
Max	1.020	1.009	1.033	1.021	1.004	1.073	0.758	0.787	0.851
$\sigma(.)$	0.057	0.015	0.059	0.141	0.057	0.160	0.131	0.091	0.173
Mean(dv/v)	0.011	0.000	0.011	0.045	0.003	0.043	0.100	0.013	0.089
$\sigma(dv/v)$	0.040	0.012	0.039	0.140	0.052	0.134	0.269	0.134	0.242

Figure 1

The One Period Investment Opportunity Set

The figure plots the instantaneous one-period investment opportunity set given by the capital market line. Variable $r(Z)$ is the real interest rate or the intercept of the capital market line, and variable $\eta(Z)$ is the maximum Sharpe ratio or slope of the capital market line.

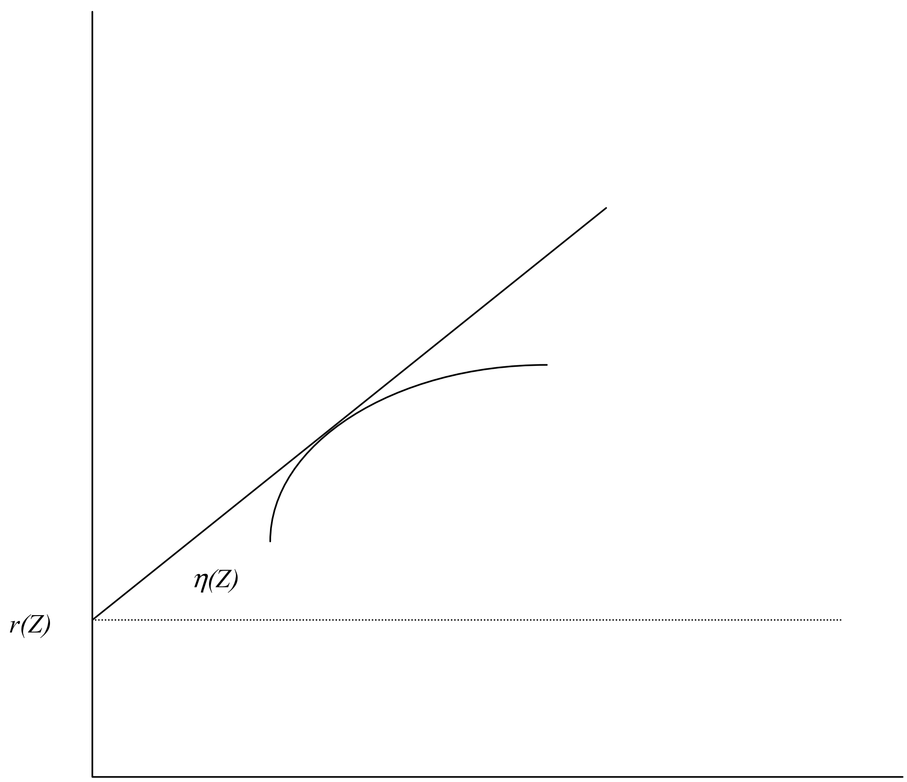
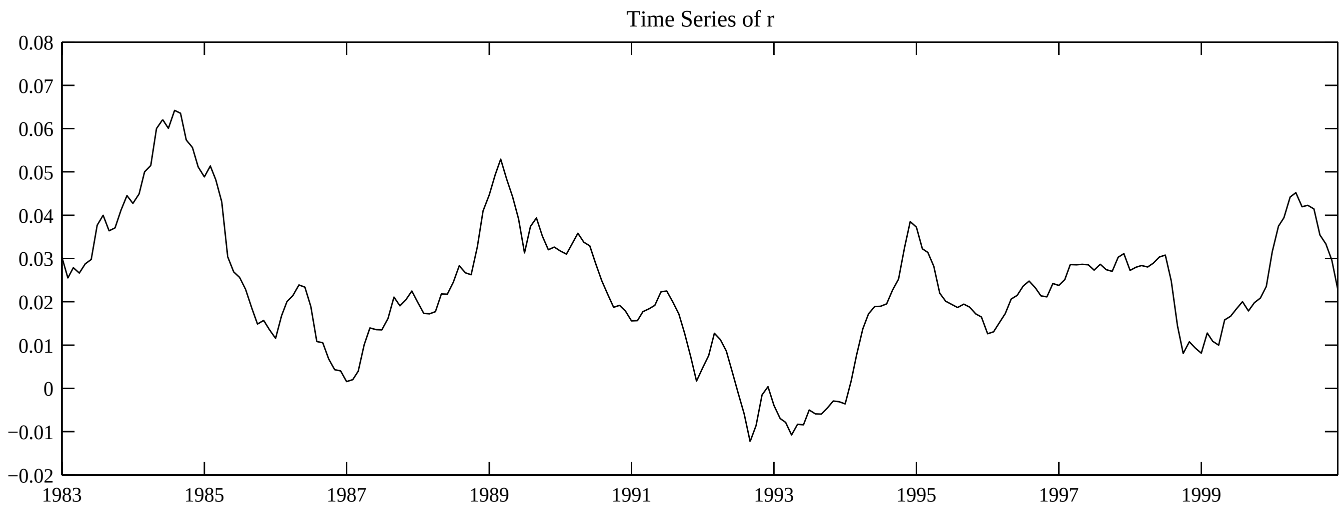


Figure 2
Time Series of State Variables, r and η

The figures plot the time series of r and η estimates, filtered from the constant maturity zero-coupon Treasury yields and the realized inflation rate from January 1983 to December 2000.



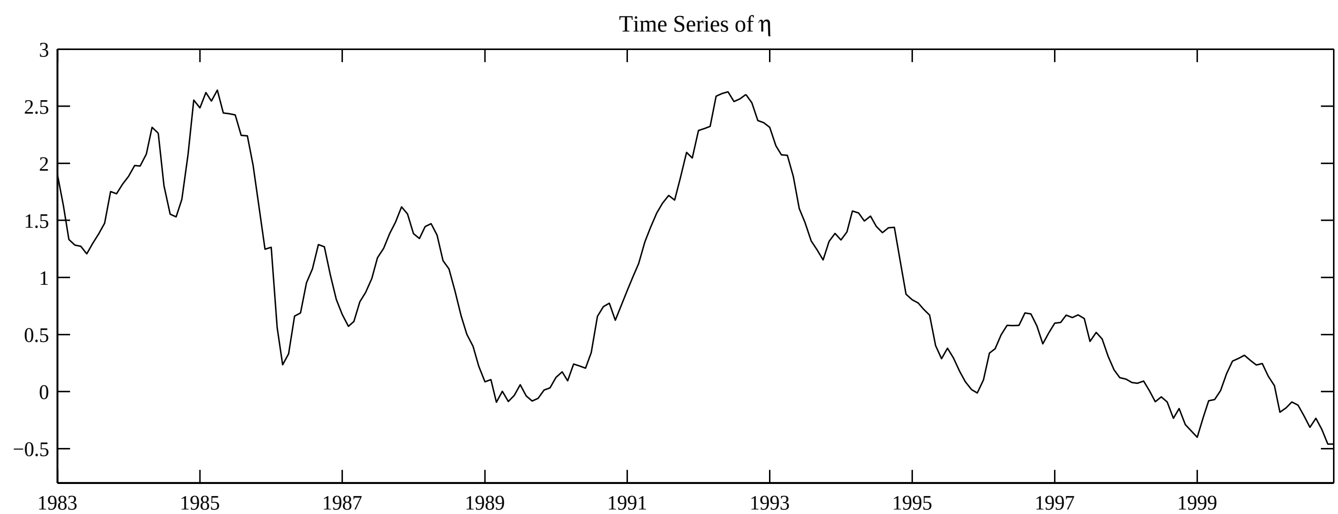


Figure 3
Beta, Expected Excess Return, and the Maturity of the Underlying Cash Flow
$(\eta = \bar{\eta} = 0.7)$

Panel a of the figure plots the beta of a security and Panel b of the figure plots the unconditional instantaneous expected excess on the security as the maturity of the underlying cash flow changes from zero to thirty years under $\beta_{YM} = 0.0$, 0.05, and 1.0. The security betas are calculated from equation (19) and the expected excess returns are calculated from equation (23), both using the parameter estimates reported in Table 1.

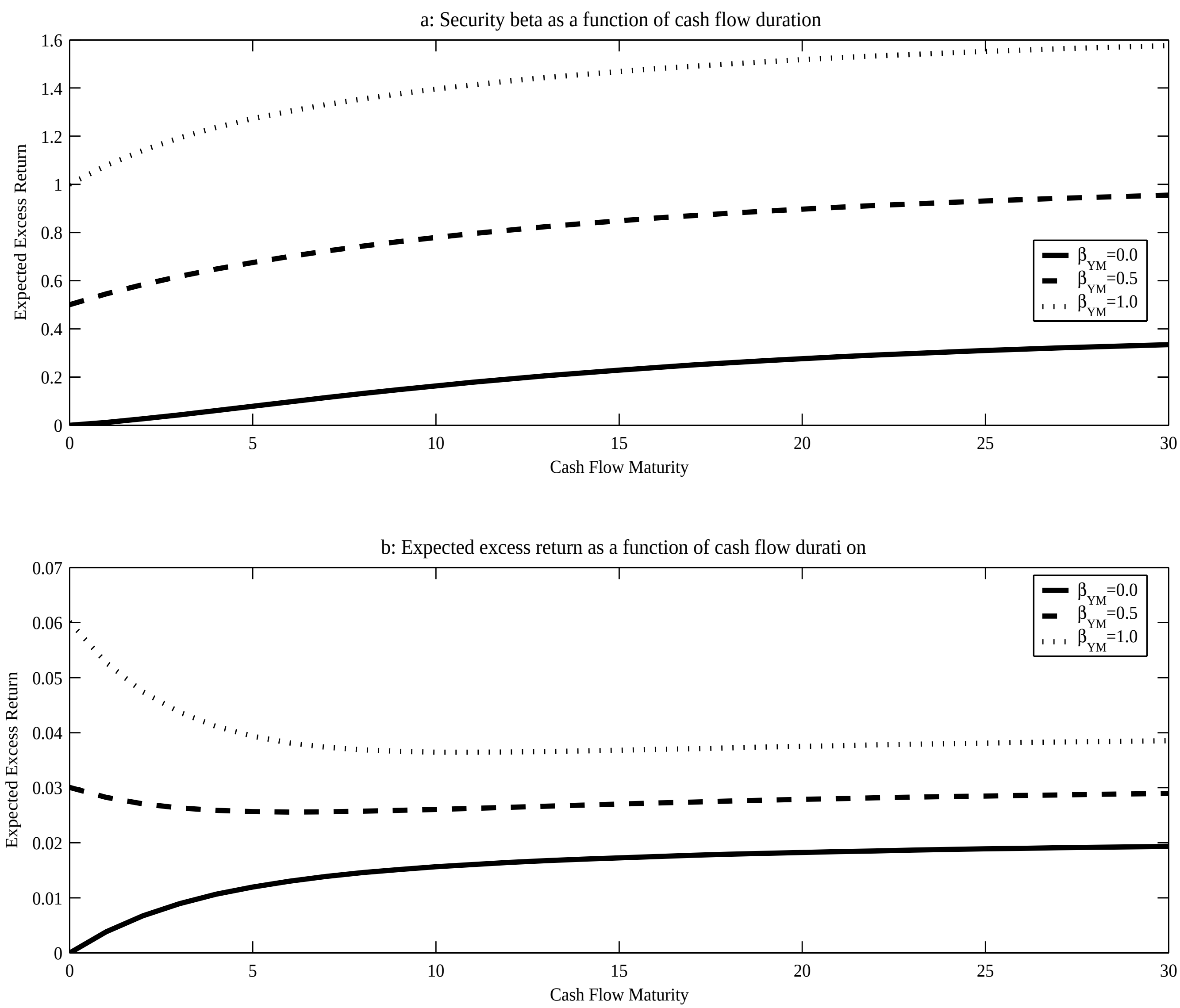


Figure 4
Term Structure of Discount Rates
$(r = \bar{r} = 2.8\%,\ \eta = \bar{\eta} = 0.7)$

This figure plots the discount rates defined in equation (33) as a function of the cash flow maturity τ, as τ varies from one to thirty years.

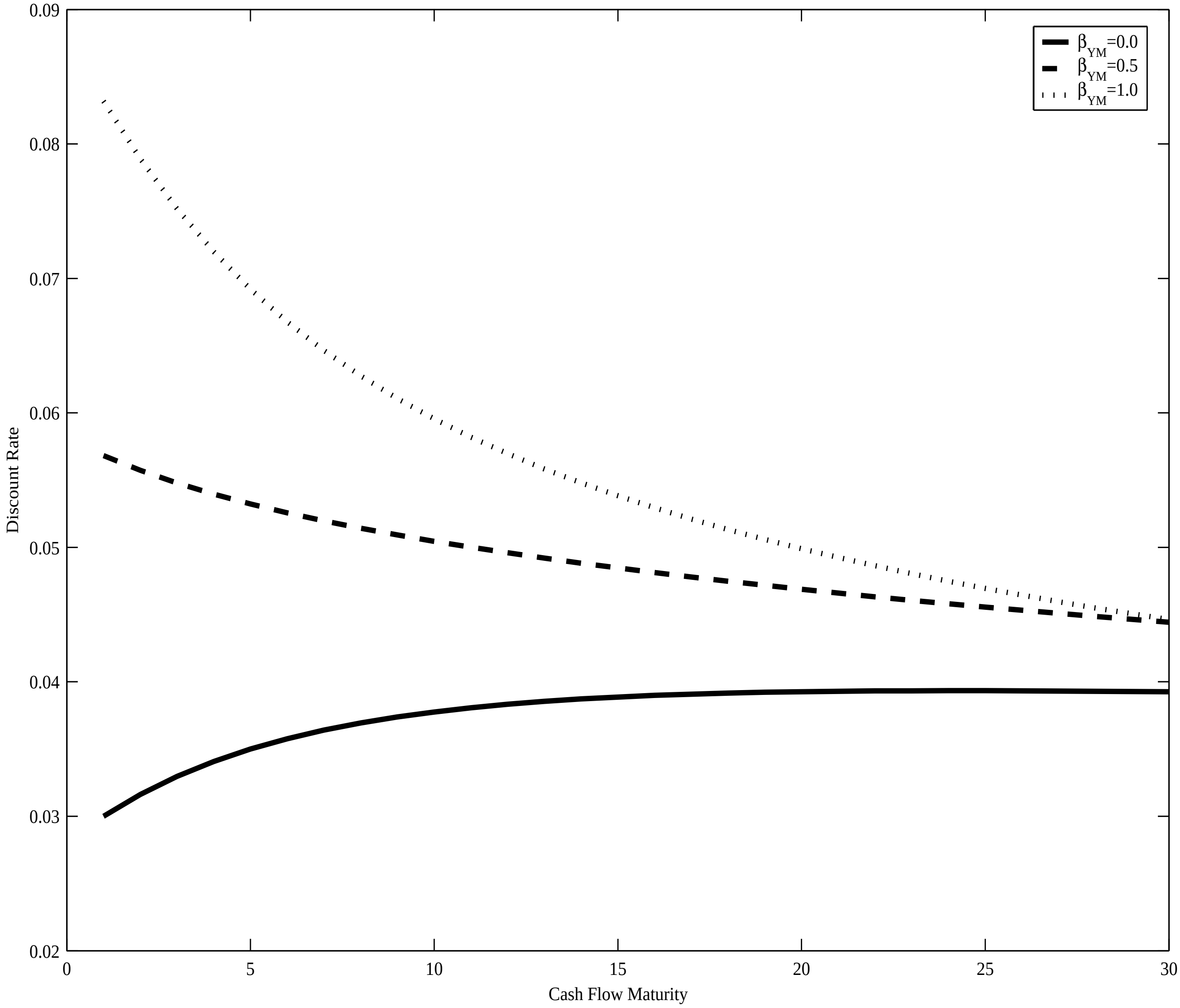


Figure 5

Term Structures of Discount Rates for Different r and η

The figure plots the term structure of discount rates for cash flow maturities from one to thirty years. The calculation of the discount rate is given by equation (33). The middle line in each plot is for $r = \bar{r} = 2.82\%$ and $\eta = \bar{\eta} = 0.7$. In the left panel, the dotted lines are calculated for $\eta = \bar{\eta}$ and r at one standard error (1.55%) above $\bar{r}$ and the dashed lines are for r at one standard error below $\bar{r}$. In the right panel, the dotted lines are calculated for $r = \bar{r}$ and η at one standard error (0.85) above $\bar{\eta}$ and the dashed lines are for η at one standard error below $\bar{\eta}$.

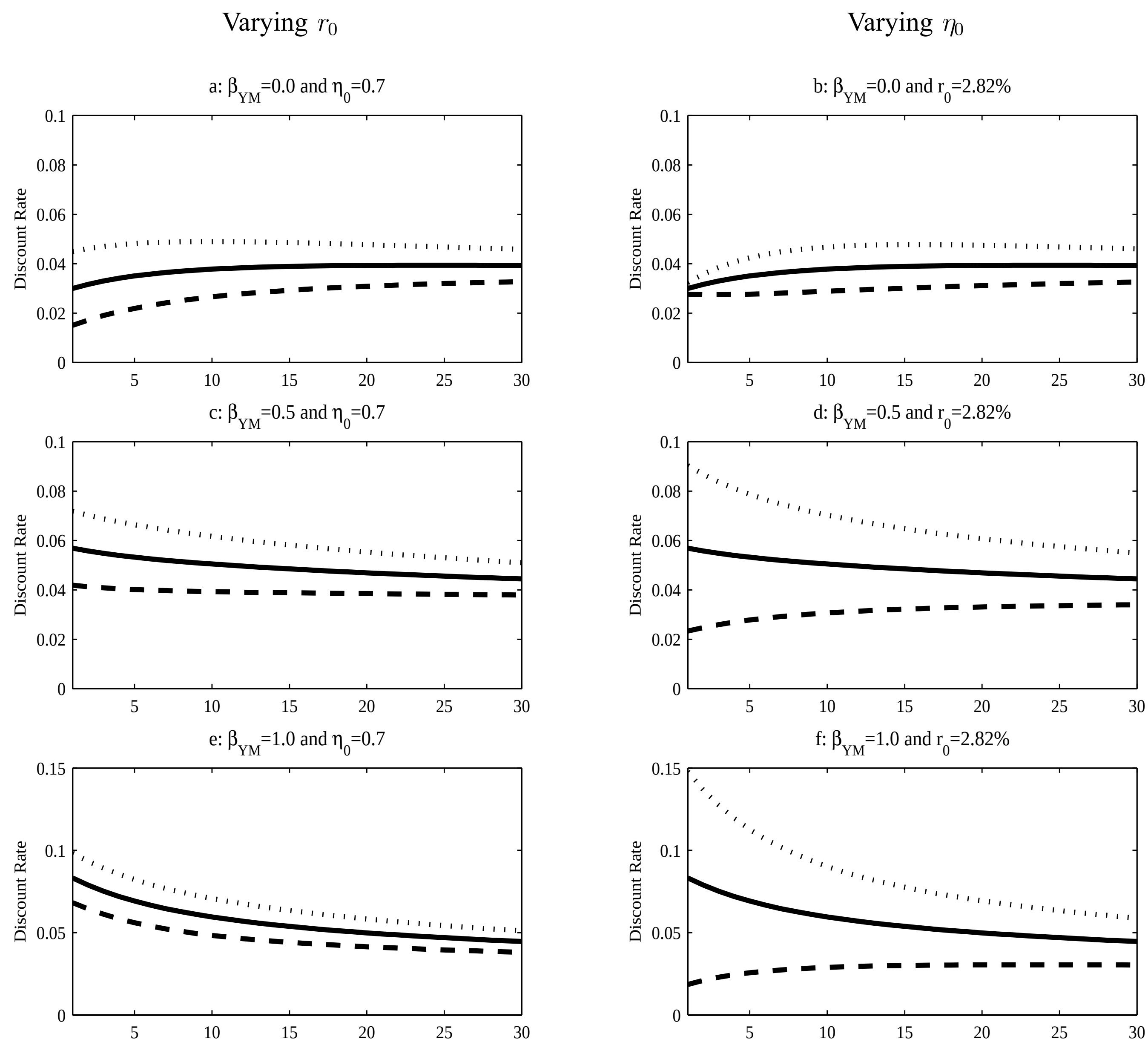


Figure 6
Term Structure of Discount Rates
($r = \bar{r} = 2.8\%$, $\eta = \bar{\eta} = 0.7$, and $\beta_{xM} = 1.0$)

This figure plots the discount rates defined in equation (37) as a function of the cash flow maturity τ, as τ varies from one to thirty years, for different rates of cash flow information arrival. The different rate of inflation arrival is controlled by the parameter κ_x. When $\kappa_x < 0$, information arrival is faster when cash flow maturity date is further away so that σ_Y increases with τ. When $\kappa_x > 0$, information arrival is faster when cash flow gets closer to its maturity dates so that σ_Y decreases with τ. When $\kappa_x = 0$, σ_Y is a constant and equation (37) degenerates into equation (33).

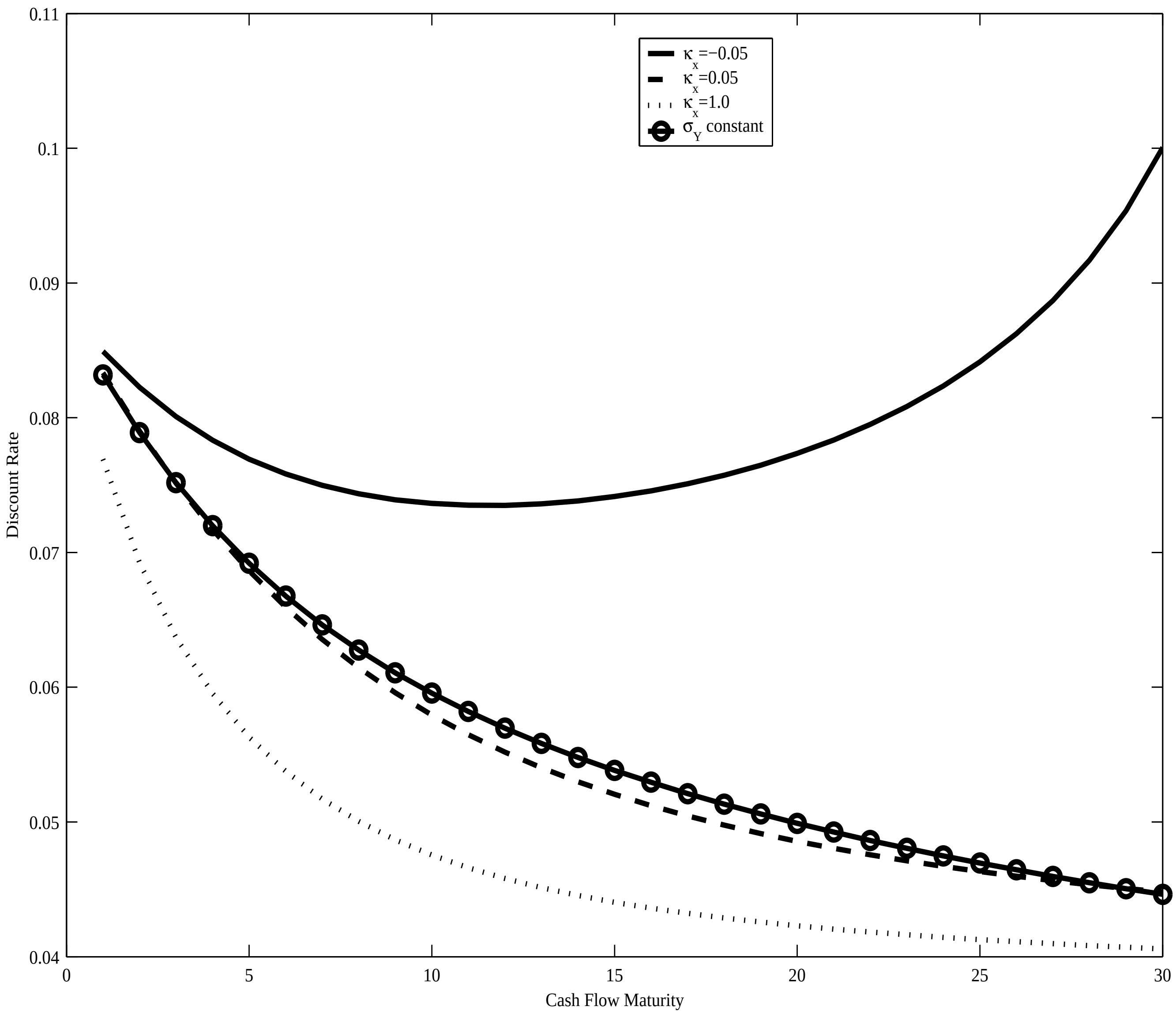


Figure 7
Time Series of the Present Value Factor v and its Corresponding Discount Rate ϕ
$(\beta_{YM} = 1.0)$

Panel a of the figure plots the time series of the present value factor $v = e^{A(\tau)-B(\tau)r_t-D(\tau)\eta_t}$ and Panel b of the figure plots the time series of the discount rate $\phi = -\ln v/\tau$the from January 1983 to December 2000, where r_t and η_t are the estimated values of the state variables at time t. Values are shown for cash flow maturities, τ of 1, 5 and 20 years.

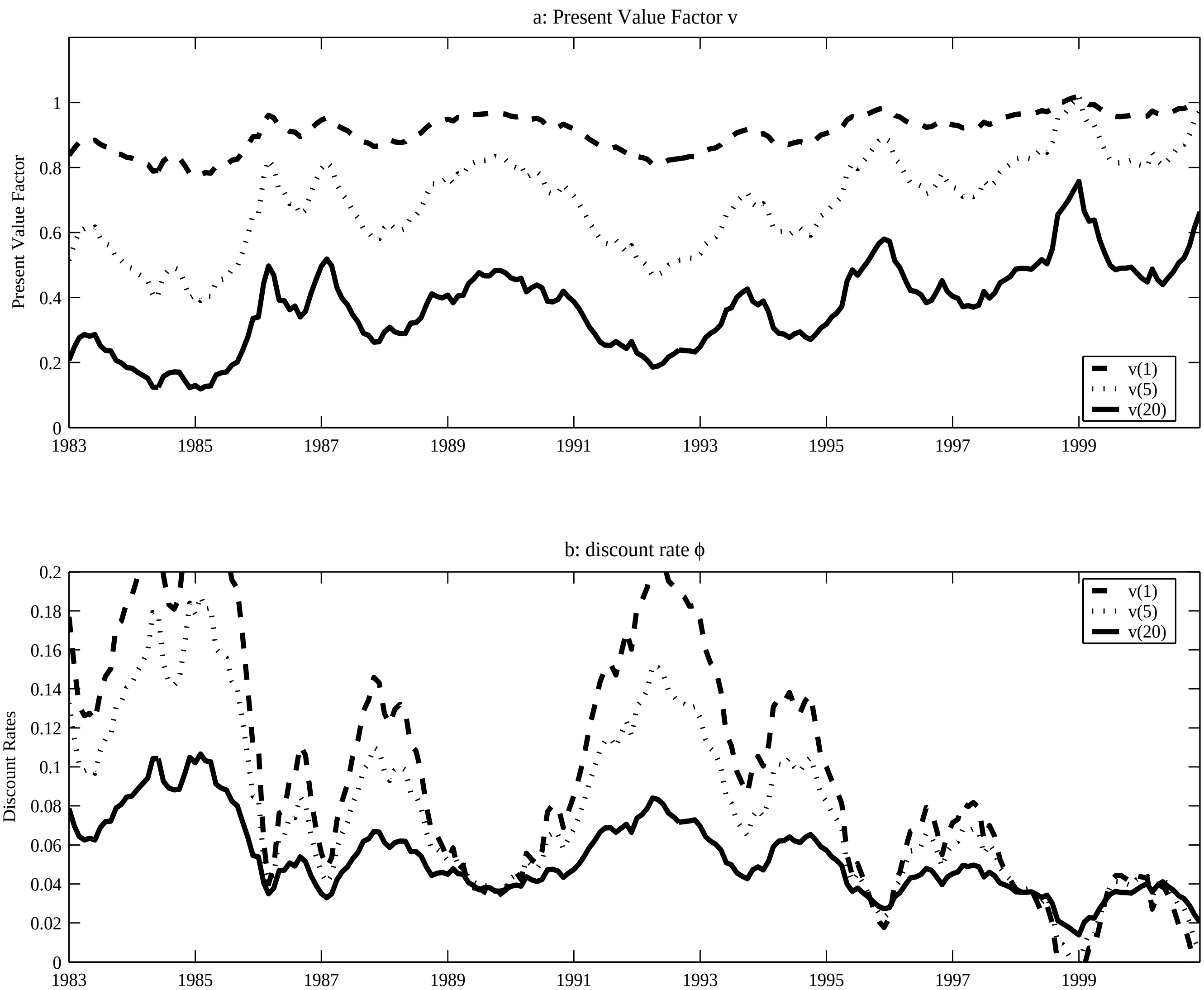


Figure 8

Time Series Decomposition of the Valuation Factor

$(\beta_{YM} = 1.0)$

This figure plots the time series of the valuation factor $v_t = e^{A(\tau)-B(\tau)r_t-D(\tau)\eta_t}$, the part due to r, $v^r = e^{A(\tau)-B(\tau)r_t}$, and the part due to η, $v^\eta = e^{A(\tau)-D(\tau)\eta_t}$ from January 1983 to December 2000, where r_t and η_t are the estimated state variables at time t. Values are shown for cash flow maturities, τ of 1, 5 and 20 years.