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FINITE DIFFERENCE METHODS AND JUMP PROCESSES ARISING IN THE PRICING OF CONTINGENT CLAIMS: A SYNTHESIS

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Since the seminal article by Black and Scholes on the pricing of corporate liabilities, the importance in finance of contingent claims has become widely recognized. The key to the valuation of such claims has been found to lie in the solution to certain partial differential equations. The best known of these was derived by Black and Scholes, in their original article, from the assumption that the value of the asset underlying the contingent claim follows a geometric Brownian motion.

Depending on the nature of the boundary conditions which must be satisfied by the value of the contingent claim, the Black-Scholes partial differential equation and its extensions may or may not have an analytic solution. Analytic solutions have been derived under certain conditions for the values of a call option (Black and Scholes [1], Merton [11]), of a risky corporate discount bond (Merton [12]), of European put options (Black and Scholes [1], Merton [11]), of the capital shares of dual funds (Ingersoll [8]), and of convertible bonds (Ingersoll [9]). In many realistic situations, however, analytic solutions do not exist, and the analyst must resort to other methods. These include the finite difference approximation to the differential equation employed extensively by Brennan and Schwartz [3, 4, 5], numerical integration used by Parkinson [13], and Monte Carlo methods advocated by Boyle [2].

Complementing the above work, Cox and Ross [6, 7] have analyzed the pricing of contingent claims when the value of the underlying asset follows a jump process rather than a diffusion process, and have shown that in the limit the jump process approaches a pure diffusion process. The major purpose of this paper is to demonstrate that approximation of the Black-Scholes partial differential equation by use of the finite difference method is equivalent to approximating the diffusion process by a jump process and that therefore the finite difference approximation is a type of numerical integration. In particular, we establish

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that the simpler explicit finite difference approximation is equivalent to approximating the diffusion process by one of the jump processes described by Cox and Ross, while the implicit finite difference approximation amounts to approximating the diffusion process by a more general type of jump process. As a preliminary to this, we show that certain simplifications of the numerical procedure are made possible by taking a log transform of the Black-Scholes equation. In the subsequent sections we discuss the explicit and implicit finite difference approximations, respectively.

I. The Log Transform of the Black-Scholes Equation

The basic Black-Scholes equation is

$$1/2\sigma^2 S^2 H_{SS} + rS H_S + H_t - rH = 0 \tag{1}$$

where S is the value of the underlying asset, t is time, H(S,t) is the value of the contingent claim, r is the riskless rate of interest, σ^2 is the instantaneous variance rate of the return on the underlying asset, and subscripts denote partial differentiation.

To obtain the log transform of (1) we define

$$y \equiv \ln S \tag{2}$$

$$W(y,t) \equiv H(S,t) \tag{3}$$

so that

$$H_S = W_y e^{-y} \tag{4}$$

$$H_{SS} = (W_{yy} - W_y)e^{-2y} \tag{5}$$

$$H_t = W_t. \tag{6}$$

Then, making the appropriate substitutions in (1), we obtain the transformed equation:

$$\frac{1}{2}\sigma^2 W_{yy} + (r - \frac{1}{2}\sigma^2)W_y + W_t - rW = 0. \tag{7}$$

Notice that (7) unlike (1) is a partial differential equation with constant coefficients. This simplifies the numerical analysis, and, as we shall see below, makes it possible to employ an explicit finite difference approximation to (7), whereas the explicit finite difference approximation to (1) is in general unstable.

II. The Explicit Finite Difference Approximation

To obtain a finite difference approximation to (7), we replace the partial derivatives by finite differences, and to this end define

$$W(y,t) = W(ih,\ jk) = W_{i,j}$$

where h and k are the discrete increments in the value of the underlying asset and the time dimension, respectively. For the explicit approximation, the partial derivatives are approximated by

$$W_y = (W_{i+1,j+1} - W_{i-1,j+1})/2h$$

$$W_{yy} = (W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1})/h^2$$

$$W_t = (W_{i,j+1} - W_{i,j})/k$$

so that the corresponding difference equation is

$$(8)\quad W_{i,j}(1+rk) = aW_{i-1,j+1} + b\,W_{i,j+1} + c\,W_{i+1,j+1} \qquad \begin{array}{l} i = 1,\ \ldots,\ (n-1) \\ j = 1,\ \ldots,\ m \end{array}$$

where

$$a = [\tfrac{1}{2}(\sigma/h)^2 - \tfrac{1}{2}(r - \tfrac{1}{2}\sigma^2)/h]k,$$

$$b = [1 - (\sigma/h)^2\ k],\ \text{and}$$

$$c = [\tfrac{1}{2}\sigma/h)^2 + \tfrac{1}{2}(r - 1/2\sigma^2)/h]k\ .$$

For any given value of j, (8) allows us to solve for $W_{i,j}$ $(i = 1,\ \ldots,\ n-1)$ in terms of $W_{i,j+1}$. The extreme values of $W_{i,j}$, $W_{o,j}$, and $W_{n,j}$, must be given by the boundary conditions to the problem.[1] Then, given the values of $W_{i,j}$ corresponding to the maturity of the contingent claim, we may solve (8)

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[1]Note that we are implicitly assuming that the lower boundary condition is of the form $W(0,t) = Z_t$. More generally the boundary condtion may be $W(i_j h,t) = Z_t$; this will simply change the range of i in (8) without changing anything essential.

recursively for all values of $W_{i,j}$.

Notice that the coefficients of (8) are independent of i and that $a + b + c = 1$. For the stability of the explicit solution, it is necessary that the coefficients of (8) be nonnegative (McCracken and Dorn [10]). While appropriate choice of h and k may guarantee this for (8), the corresponding coefficients of the explicit approximation to (1) depend on i, and will be negative for sufficiently large values of i, so that this explicit finite difference approximation may not be applied to the untransformed equation (1).

For the nonnegativity condition to be satisfied, it is necessary that h and k be chosen so that

$$h \leq \sigma^2/|(r - \tfrac{1}{2}\sigma^2)|$$

(9) and

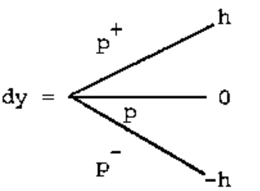
$$k \leq \sigma^2/(r - \tfrac{1}{2}\sigma^2)^2 \ .$$

If the conditions (9) are satisfied, the coefficients of the RHS of (8) may be interpreted as probabilities since they are nonnegative. Writing p^- for a, p for b and p^+ for c, (8) becomes

$$W_{i,j} = \frac{1}{(1+rk)} \quad p^- \, W_{i-1,j+1} + p \, W_{i,j+1} + p^+ \, W_{i+1,j+1} \tag{10}$$

Thus the value of the contingent claim at time instant j may be regarded as given by its expected value at (j+1) discounted at the riskless rate, r. The expected value of the claim at the next instant is obtained by assuming that y, the logarithm of the stock price follows the jump process

(11)



which is formally identical to a jump process discussed by Cox and Ross [6, equation (8)], where $\mu = 0$. The local mean and variance of (11) are

$$E[dy] = h[p^+ - p^-] \tag{12}$$

$$= (r - \tfrac{1}{2}\sigma^2)k \ .$$

$$V[dy] = h^2[p^+ + p^-] - (E[dy])^2 \tag{13}$$

$$= \sigma^2 k - (r - \tfrac{1}{2}\sigma^2)^2 k^2 \ .$$

Thus the diffusion limit of (11) is

$$dy = (r - 1/2\sigma^2)dt + \sigma\ dz \tag{14}$$

where dz is a Gauss-Wiener process with $E[dz] = 0$, $E[dz^2] = dt$; this implies that the diffusion limit of dS is

$$\frac{dS}{S} = r\ dt + \sigma\ dz. \tag{15}$$

Now as Cox and Ross [6] have pointed out, if a riskless arbitrage portfolio can be established between the contingent claim and the underlying asset, the resulting valuation equation is preference free. Therefore we may value the contingent claim under any convenient assumption about preferences, in particular under the assumption of risk neutrality, which implies that the diffusion process for the underlying asset is (15) and that the value of the contingent claim is obtained by discounting its expected future value at the riskless rate of interest as is done in (10).

We have established therefore that the explicit finite difference approximation to the Black-Scholes differential equation is equivalent to making the permissible assumption of risk-neutrality and approximating the diffusion process (15) by the jump process (11). Notice however that the variance of the approximating jump process given by (13) is a downward biased estimate of the variance of the approximated diffusion process (14). The bias is equal to the square of the expected jump, $(r - \frac{1}{2}\sigma^2)k$. Using the stability condition (9), the upper bound on this bias is σ^4.

The recursive valuation equation (10) may be regarded as a type of numerical integration where the probabilities are taken, not from the normal density function, but from a jump process, (11), approximating the Gauss-Wiener process (14). This approach is almost identical to the numerical integration procedure employed by Parkinson [13], who also approximated the normal distribution by a related but different three-point distribution.

III. The Implicit Finite Difference Approximation

The implicit finite difference approximation to (7) is obtained by approximating the partial derivatives by the finite differences

$$W_{yy} = (W_{i+1,j} - 2W_{i,j} + W_{i-1,j})/h^2 \tag{16}$$

$$W_y = (W_{i+1,j} - W_{i-1,j})/2h \tag{17}$$

(18) $$W_t = (W_{i,j+1} - W_{i,j})/k$$

so that the differential equation is written in finite difference form as:

(19) $$a\, W_{i-1,j} + b\, W_{i,j} + c\, W_{i+1,j} = (1-rk) W_{i,j+1} \qquad i = 1, \ldots n \quad j = 1, \ldots m$$

where

(20) $$a = [\, -\tfrac{1}{2}(\sigma/h)^2 + \tfrac{1}{2}(r - \tfrac{1}{2}\sigma^2)/h]k$$

(21) $$b = 1 + (\sigma/h)^2 k$$

(22) $$c = [- \tfrac{1}{2}(\sigma/h)^2 - \tfrac{1}{2}(r - \tfrac{1}{2}\sigma^2)/h]k$$

For any value of j, (19) constitutes a system of n equations in the (n+2) unknowns $W_{i,j}$ $(i = 0,1, \ldots, n+1)$. To complete the system, it is necessary to introduce two boundary conditions. Assume that these are given by knowing $W_{0,j}$ and $W_{n+1,j}$:

(23) $$W_{0,j} = \alpha_j$$

(24) $$W_{n+1,j} = \beta_j$$

Then we may eliminate $W_{0,j}$ and $W_{n+1,j}$ from the first and last equations of (19) to obtain:

(25)
$$\begin{aligned}
b\, W_{1,j} + c\, W_{2,j} &= (1-rk) W_{1,j+1} - a\alpha_j = f_1 \\
&\quad - - \circ \circ \circ - - \\
a\, W_{i-1,j} + b\, W_{i,j} + c\, W_{i+1,j} &= (1-rk) W_{i,j+1} = f_i \\
&\quad - - \circ \circ \circ - - \\
a\, W_{n-1,j} + b\, W_{n,j} &= (1-rk) W_{n,j+1} - c\,\beta_j = f_n
\end{aligned}$$

This system of equations may be written in matrix form as

(26) $$\underline{\underline{A}}\ \underline{W} = \underline{f}\ .$$

And by recursive solution of (26), knowing the values of $W_{i,j}$ at maturity, we generate the whole set of $W_{i,j}$ values. Note that since $\underline{\underline{A}}$ is independent of j, the matrix must be inverted only once, so that each time step simply involves the multiplication of a vector by this matrix inverse. This is admittedly a more complex calculation than was required for the explicit solution: on the other hand, the implicit solution procedure is potentially more accurate.

Our objective is to demonstrate that the elements of this matrix inverse may be viewed as discounted probabilities, and that therefore the implicit solution procedure generates successively earlier values of $W_{i,j}$ by discounting the expected value at the end of the next time increment assuming risk neutral preferences.

The simple form of the matrix, $\underline{\underline{A}}$, suggests the use of Gaussian elimination to solve the equation system. We proceed by multiplying the second equation of (25) by (b/a) and subtracting from it the first equation to obtain a transformed second equation from which $W_{1,j}$ has been eliminated: we proceed in this way, multiplying each equation by (b/a) and subtracting from it its transformed predecessor, obtaining the transformed system of equations:

$$
\begin{aligned}
b_1^* W_{1,j} + c_1^* W_{2,j} &= f_1^* \\
b_2^* W_{2,j} + c_2^* W_{3,j} &= f_2^* \\
- - \circ \circ \circ - - & \\
b_{n-1}^* W_{n-1,j} + c_{n-1}^* W_{n,j} &= f_{n-1}^* \\
b_n^* W_{n,j} &= f_n^*
\end{aligned}
\tag{27}
$$

In the first equation

$$b_1^* = b,\ c_1^* = c,\ f_1^* = f_1$$

and in general

$$b_i^* = (b/a)\ b_{i-1}^* - c_{i-1}^* \tag{28}$$

$$c_i^* = (c/a)\ b_{i-1}^* \tag{29}$$

$$f_i^* = (f_i/a)\ b_{i-1}^* - f_{i-1}^*. \tag{30}$$

Substituting for c_{i-1}^* in (28) from (29), we obtain the difference equation for b_i^*:

$$b_i^* = (b/a)\ b_{i-1}^* - (c/a)\ b_{i-2}^*. \tag{31}$$

The solution to this difference equation, given the initial conditions $b_1^* = b$, $c_1^* = c$ is:

$$b_i^* = (a^2/\sqrt{b^2-4ac})(\lambda_1^{i+1} - \lambda_2^{i+1}) \tag{32}$$

where

$$\lambda_1 = (b + \sqrt{b^2-4ac})/2a \tag{33}$$

$$\lambda_2 = (b - \sqrt{b^2-4ac})/2a. \tag{34}$$

Then, substituting for b_{i-1}^* from (32) in (29), c_i^* may be written as:

$$c_i^* = (ac/\sqrt{b^2-4ac})(\lambda_1^i - \lambda_2^i). \tag{35}$$

The expression for f_i^* is obtained by substituting for b_i^* in (30) and solving recursively for f_2^*, f_3^* ... This yields

$$f_i^* = (a/\sqrt{b^2-4ac}) \sum_{j=1}^{i} L_j \; f_j (-1)^{(i-j)} \tag{36}$$

where

$$L_j = \lambda_1^j - \lambda_2^j.$$

The matrix inversion is completed by solving the system of equations (27) starting with the last equation. Define $Z_i = \Sigma_{j=1}^{i} L_j \; f_j (-1)^{(i-j)}$. Then

$$W_{n,j} = f_n^*/b_n^* = Z_n/aL_n$$

$$W_{n-1,j} = \frac{Z_{n-1}}{aL_n} - \frac{c}{a^2}\frac{L_{n-1}\,Z_n}{L_n\,L_{n+1}}$$

$$- - \circ \circ \circ - - .$$

$$W_{n-q,j} = \frac{L_{n-q}}{a}\left[\frac{Z_{n-q}}{L_{n-q}\,L_{n-q+1}} - \frac{c}{a}\frac{Z_{n-q+1}}{L_{n-q+1}\,L_{n-q+2}} + \frac{c^2}{a^2}\frac{Z_{n-q+2}}{L_{n-q+2}L_{n-q+3}} + \ldots - + \left(\frac{-c}{a}\right)^q \frac{Z_n}{L_n\,L_{n+1}}\right] \tag{37}$$

$$- - \circ \circ \circ - - .$$

Set $(n-q) = i$ and collect coefficients of $W_{i,j+1}$ in (37), recalling that

$f_j = (1-rk)W_{i,j+1}$. Denoting the coefficient of $W_{i,j+1}$ by $(1-rk)p_i$, we have:

(38) $$p_i = \frac{L_i^2}{a} \Sigma_{j=i}^{n} (c/a)^{j-i} (1/L_j\ L_{j+1})$$

(39) $$p_{i-q} = (-1)^q \frac{L_i\ L_{i-q}}{a} \Sigma_{j=i}^{n} (c/a)^{j-i} (1/L_j\ L_{j+1})$$

(40) $$p_{i+q} = (-1)^q \frac{L_i\ L_{i+q}}{a} \Sigma_{j=i+q}^{n} (c/a)^{j-(i+q)} (1/L_j\ L_{j+1}).$$

The values of p_{i+q} $(q = 1-i, \ldots, -1, 0, +1, \ldots n-i)$ are the elements of the i^{th} row of $\underline{\underline{A}}^{-1}$. We shall now show that as the boundaries become indefinitely remote p_{i+q} may be interpreted as the probability that the logarithm of the stock price will jump by qh. As the lower boundary becomes remote $i \to \infty$, while $(n-i) \to \infty$ as the upper boundary becomes remote.

First note that

(41) $$\frac{L_i}{L_{i+q}} = \frac{\lambda_1^i - \lambda_2^i}{\lambda_1^{i+q} - \lambda_2^{i+q}} = \frac{1}{\lambda_1^q} \frac{1 - (\lambda_2/\lambda_1)^i}{1 - (\lambda_2/\lambda_1)^{i+q}}$$

and that since $|\lambda_2/\lambda_1| < 1$

(42) $$\lim_{i\to\infty} \frac{L_i}{L_{i+q}} = \frac{1}{\lambda_1^q}.$$

Hence as $(n-i)$, $i \to \infty$,[2]

(43) $$\lim_{\substack{i\to\infty \\ n-i\to\infty}} p_i = p_o^* = \frac{1}{a} \left[\frac{1}{\lambda_1} + \frac{c}{a} \frac{1}{\lambda_1^3} + \frac{c^2}{a^2} \frac{1}{\lambda_1^5} \cdots \right]$$

$$= \lambda_1/(a\lambda_1^2 - c)$$

and from (38) and (39)

(44) $$\lim_{\substack{i\to\infty \\ n-i\to\infty}} p_{i-q} = p_{-q}^* = (-\frac{1}{\lambda_1})^q\ p_i^*, \text{ for } q = 1, \ldots, \infty$$

---

[2] Since λ_1 and λ_2 are the roots of the auxiliary equation of the difference equation (31), $\lambda_1\lambda_2 = c/a$. Therefore, $|c/a\lambda_1^2| = |\lambda_2/\lambda_1| < 1$.

$$(45) \qquad \lim_{i\to\infty} p_{i+q} = p_q^* = \left(\frac{-c}{a\lambda_1}\right)^q p_i^*, \text{ for } q = 1, \ldots, \infty.$$

Consider the sum of the p_q^* $(q = -\infty, \ldots, +\infty)$, S:

$$(46) \qquad S = p_o^* \left[\left(1 - \frac{c}{a\lambda_1} + \left(\frac{c}{a\lambda_1}\right)^2 - \left(\frac{c}{a\lambda_1}\right)^3 \ldots\right) - \frac{1}{\lambda_1}\left(1 - \frac{1}{\lambda_1} + \left(\frac{1}{\lambda_1}\right)^2 \ldots\right)\right]$$

$$= p_o^* \left[\frac{a\lambda_1}{a\lambda_1+c} - \frac{1}{1-\lambda_1}\right] = p_o^* \left[\frac{a\lambda_1{}^2-c}{(1+\lambda_1)(a\lambda_1+c)}\right]$$

and, substituting for p_o^* from (43)

$$(47) \qquad S = \frac{\lambda_1}{(1+\lambda_1)(a\lambda_1+c)} .$$

But since λ_1 is a root of the auxiliary equation of (31) and $b = 1-(a+c)$, $(1+\lambda_1)(a\lambda_1+c) = \lambda_1$ so that S=1. Thus the sum of the weights p_q^* $(q=-\infty, \ldots, +\infty)$ equals 1.

Moreover each element p_q^* is nonnegative so long as[3]

$$(48) \qquad h^2 \leq \sigma^4/(r - \tfrac{1}{2}\sigma^2)^2 .$$

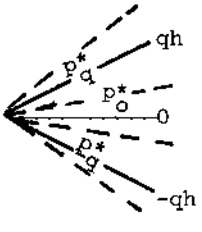
Thus since the p_q^* are nonnegative and sum to unity, they may be interpreted as probabilities and we have

$$(49) \qquad W_{i,j} = (1-rk) \sum_{q=-\infty}^{\infty} p_q^* \quad W_{i+q,j+1}$$

$$\approx \frac{1}{1+rk} \sum_{q=-\infty}^{\infty} p_q^* \quad W_{i+q,j+1}.$$

Again, the value of the contingent claim at time instant j may be regarded as given by the expected value of its value at (j+1) discounted at the riskless rate, r. In this case the expected value of the claim at the next instant is obtained by assuming that y, the logarithm of the stock price, follows the generalized jump process

dy =



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[3]For a proof see Appendix.

The locan mean and variance of this process are shown in the Appendix to be given by

(50) $$E[dy] = (r - \frac{1}{2}\sigma^2)k$$

(51) $$V[dy] = \sigma^2 k + (r - \frac{1}{2}\sigma^2)^2 k^2 .$$

Taking the diffusion limit as $k \to 0$, y again follows the stochastic differential equation (14) which again implies that the stochastic process for S is (15). Notice that for finite k the variance of the jump process approximation to the diffusion process is biased upwards by the square of the expected size of the jump. This suggests that the accuracy of the implicit method could be improved by adjusting the variance used in (19) by subtracting from the true variance the square of the expected change in the logarithm of the underlying asset value obtained under the assumption of risk neutrality.

Thus the implicit finite difference approximation to the log transform of the Black-Scholes differential equation (7) is also equivalent to approximating the diffusion process by a jump process. In this case the jump process is a generalized one which allows for the possibility that the stock price will jump to an infinity of possible future values rather than just three. It would appear that this "more realistic" approximation would result in more accurate determination of the value of the contingent claim, but this conjecture must wait upon detailed numerical analysis.

IV. Summary

In this paper we have established that the coefficients of the difference equation approximation to the Black-Scholes partial differential equation correspond to the probabilities of a jump process approximation to the underlying diffusion process. The simpler explicit finite difference approximation corresponds to a three-point jump process of the type discussed by Cox and Ross [6], while the more complex implicit finite difference approximation corresponds to a generalized jump process to an infinity of possible points.

APPENDIX

1. Condition for nonnegativity of weights in implicit solution.

(43) can also be written as

$$p_o^* = (b + \sqrt{b^2-4ac})/(b^2-4ac + b\sqrt{b^2-4ac})$$

but from (21) $b > 0$, and from (20), (21) and (22) $b^2-4ac > 0$. Therefore $p_o^* > 0$.

Then from (44) $p_{-q}^* > 0$, iff $\lambda_1 < 0$ which from (33) requires that $a < 0$. Then from (45), $p_q^* > 0$, also iff $c/a > 0$, so that c must also be negative. From (20) and (22), c and a are negative if and only if (48) is satisfied.

2. Mean and variance of the generalized Jump Process.

$$E(dy) = h\left[\sum_{q=1}^{\infty} q\, p_q^* - \sum_{q=1}^{\infty} q\, p_{-q}^*\right].$$

Substituting for p_q^* and p_{-q}^* from (44) and (45),

$$E(dy) = h\left[-\frac{c}{a\lambda_1}\left(1 - 2\frac{c}{a\lambda_1} + 3\left(\frac{c}{a\lambda_1}\right)^2 - \dots\right)\right.$$
$$\left. + \frac{1}{\lambda_1}\left(1 - \frac{2}{\lambda_1} + \frac{3}{\lambda_1 2} - \dots\right)\right] p_i^* .$$

Summing and using (43),

$$E(dy) = h\left[-\frac{ac\lambda_1}{(a\lambda_1+c)}2 + \frac{\lambda_1}{(1+\lambda_1)}2\right]\frac{\lambda_1}{a\lambda_1{}^2-c}$$

$$E(dy) = (a-c)h = (r - \tfrac{1}{2}\sigma^2)k .$$

Q.E.D.

$$V(dy) = \sum_{q=0}^{\infty} p_{-q}^*(-qh-(a-c)h)^2 + \sum_{q=1}^{\infty} p_q^*\,(qh-(a-c)h)^2$$
$$= h^2\left[\sum_{q=0}^{\infty} q^2\, p_{-q}^* + \sum_{q=1}^{\infty} q^2\, p_q^* - (a-c)^2\right].$$

Summing the series and substituting for p_o^* as above we obtain:

$$V(dy) = h^2\left[\left\{\frac{\lambda_1(1-\lambda_1)}{(1+\lambda_1)^3} - \frac{ac\lambda_1(a\lambda_1-c)}{(a\lambda_1+c)^3}\right\}\frac{\lambda_1}{a\lambda_1{}^2-c} - (a-c)^2\right].$$

Simplifying yields:

$$V(dy) = -h^2[(a+c)b + 4ac]$$

and, substituting for a, b, and c, we obtain (51).

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