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Value-at-risk and asset allocation with stable return distributions*

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SUMMARY: The paper reviews recent empirical evidence on the implications distributional assumptions can have on financial decision making. Specifically, we compare the empirical validity of decisions on risk assessment and asset allocation that are based on the — commonly adopted — normal assumption to those based on the heavy-tailed stable Paretian assumption.

KEYWORDS: Financial modeling, value at risk, portfolio selection, heavy tails. JEL C10, G10, G11.

1. INTRODUCTION

There is overwhelming empirical evidence that the returns on speculative assets follow heavy-tailed and, sometimes, skewed distributions and thus violate the assumption of normality. Among the various distributions that have been proposed as alternatives to the commonly used normal model, the *stable Paretian distribution* (in short, *stable distribution*) represents a particularly attractive candidate. The use of the stable distribution in the context of asset returns was put forth by Mandelbrot (1963) and Fama (1965), who were the first to seriously question the normal assumption for asset returns. The stable distribution is capable of capturing heavy tails and skewness, while preserving many of the desirable analytical properties of the normal. An important property is that stable distributions have domains of attraction. The Central Limit Theorem for normalized sums of identically and independently distributed (i.i.d.) random variables determines the domain of attraction of each stable law. Therefore, any distribution in the domain of attraction of a specified stable distribution will have properties that are close to those of the stable distribution. Another attractive feature is the stability property — that is, stable distributions are stable with respect to the summation of i.i.d. random stable variables. Finally, the stable distribution includes the normal as a special case. Thus, it does not rule out the normal assumption, but rather represents a generalization of the normal model.[1]

This paper illustrates some of the consequences when replacing the normal assumption by the more general stable Paretian assumption. Specifi-

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[1] Detailed accounts of the properties of stable distributed variables can be found in Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994). Overviews of the use of stable distributions in finance are given in McCulloch (1996) and Rachev and Mittnik (2000).

cally, we survey some recent empirical evidence comparing implications of distributional assumptions on risk assessment — in terms of value-at-risk (VaR) measures — and asset allocation.[2] The paper is organized as follows. Section 2 summarizes some of the relevant properties of stable random variables. Section 3 considers implications on risk assessment. The influence of distributional assumptions on asset allocation is addressed in Section 4. Section 5 concludes.

2. Properties of Stable Random Variables

A random variable R is said to be *stable*[3] if for any $a > 0$ and $b > 0$ there exist constants $c > 0$ and $d \in \mathbf{R}$ such that

$$aR_1 + bR_2 \sim cR + d \ ,$$

where R_1 and R_2 are independent copies of R and "$\sim$" denotes equality in distribution. There are no general closed-form expressions for the density and distribution functions of stable distributions. Instead, they can be described by their characteristic function

$$\Phi_R(\theta) = \begin{cases} \exp\left\{i\mu\theta - \sigma^\alpha|\theta|^\alpha\left(1 - i\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right)\right\} & \text{if } \alpha \neq 1 \ , \\ \exp\left\{i\mu\theta - \sigma|\theta|\left(1 + i\beta\frac{2}{\pi}\mathrm{sign}(\theta)\ln\theta\right)i\mu\theta\right\} & \text{if } \alpha = 1 \ , \end{cases}$$

where $\alpha \in (0, 2]$ is the *index* of *stability*, $\beta \in [-1, 1]$ is the *skewness parameter*, $\sigma > 0$ is the *scale parameter*, and $\mu \in \mathbf{R}$ is the *location parameter*. If R is a stable random variable, we write $R \sim S_\alpha(\beta, \sigma, \mu)$. If $\alpha = 2$, the stable distribution specializes to the normal distribution. In financial applications, one typically finds that $1 < \alpha < 2$, implying that the mean $\mathrm{E}(R) = \mu$ is finite but the variance $\mathrm{Var}(R)$ is not (see below). Stable distributions are unimodal; and the smaller α is, the more peaked they are around the center and the heavier are the tails. Thus, the index of stability can be interpreted as a *measure of kurtosis.* If the skewness parameter β is zero, the distribution is symmetric. If $\beta > 0$ ($\beta < 0$), the distribution is skewed to the right (left), with larger magnitudes of β indicating more pronounced skewness. The scale parameter, σ, implies that any stable random variable $R \sim S_\alpha(0, \sigma, 0)$ can be written as $R = \sigma R_1$, where $R_1 \sim S_\alpha(0, 1, 0)$. The scale parameter generalizes the definition of standard deviation; and the analogue of the variance is the *variation* of R, defined by σ^α.

VaR calculations involve the tails of the return distribution. The tails of (non-Gaussian) stable distributions have a power decay characterized by

$$\lim_{\lambda \to +\infty} \lambda^\alpha \mathrm{Prob}(R > \lambda) = k_\alpha \frac{1 + \beta}{2} \sigma^\alpha$$

[2] The survey draws heavily on Khindanova *et al.* (2001), Ortobelli *et al.* (2001), and Doganoglu and Mittnik (2001).

[3] The terms *stable Paretian*, *α-stable*, and *Pareto-Lévy-stable* are also being used in the literature.

and

$$\lim_{\lambda \to +\infty} \lambda^\alpha \text{Prob}(R < -\lambda) = k_\alpha \frac{1-\beta}{2} \sigma^\alpha ,$$

where

$$k_\alpha = \begin{cases} \dfrac{2}{\pi} & \text{if } \alpha = 1 , \\ \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} & \text{if } \alpha \neq 1 . \end{cases}$$

For $\alpha = 2$ the p-th absolute moment, $\mathrm{E}|R|^p = \int_0^\infty \mathrm{P}(|R|^p > x)dx$, is always *finite*, while for $\alpha < 2$ we require $p < \alpha$. Thus, the second moment of any non-Gaussian stable distribution is *infinite*.

Stable distributions are closed under summation. More specifically, linear combinations of independent stable random variables with stability index α are again stable with the same index α. Only stable random variables possess this property, which is an important advantage in portfolio analysis. If $R_1, R_2, \ldots, R_N$ are independent stable random variables with stability index α, that is, $R_i \sim S_\alpha(\beta_i, \sigma_i, \mu_i)$, then $R_P = \sum_{i=1}^N w_i R_i \sim S_\alpha(\beta_P, \sigma_P, \mu_P)$ with

$$\sigma_P = \begin{cases} ((|w_1|\sigma_1)^\alpha + \cdots + (|w_N|\sigma_N)^\alpha)^{\frac{1}{\alpha}} & \text{if} \quad \alpha \neq 1 , \\ |w_1|\sigma_1 + \cdots + |w_N|\sigma_N & \text{if} \quad \alpha = 1 , \end{cases}$$

$$\beta_P = \begin{cases} \dfrac{\text{sign}(w_1)\beta_1(|w_1|\sigma_1)^\alpha + \cdots + \text{sign}(w_N)\beta_N(|w_N|\sigma_N)^\alpha}{(|w_1|\sigma_1)^\alpha + \cdots + (|w_N|\sigma_N)^\alpha} & \text{if} \quad \alpha \neq 1 , \\ \dfrac{\text{sign}(w_1)\beta_1|w_1|\sigma_1 + \cdots + \text{sign}(w_N)\beta_N|w_N|\sigma_N}{|w_1|\sigma_1 + \cdots + |w_N|\sigma_N} & \text{if} \quad \alpha = 1 , \end{cases}$$

and

$$\mu_P = \begin{cases} w_1\mu_1 + \cdots + w_N\mu_N & \text{if} \quad \alpha \neq 1 , \\ -\frac{2}{\pi}(w_1 \ln|w_1|\sigma_1\beta_1 + \cdots + w_N \ln|w_N|\sigma_N\beta_N) & \text{if} \quad \alpha = 1 . \end{cases}$$

For $\alpha < 2$, the variance of a stable random variable is infinite, so that one cannot express risk and dependence in terms of variances or correlations. However, there are analogues for stable laws. The scale parameter can play the role of a risk measure — as the standard deviation does for the normal. The dependence between stable random variables depends on the structure of the underlying multivariate distribution. To elaborate on this, let R be a random *vector* of dimension N. R is *stable* if for any $a > 0$ and $b > 0$ there exist $c > 0$ and an N-dimensional vector D, such that $aR_1 + bR_2 \sim cR + D$, where R_1 and R_2 are independent copies of vector R. The characteristic function of an N-dimensional vector is

$$\Phi_R(\theta) = \begin{cases} \exp\left\{i\theta'\mu - \int_{S_N} |\theta' s|^\alpha \left(1 - i\,\text{sign}(\theta' s)\tan\frac{\pi\alpha}{2}\right)\Gamma(ds)\right\} \\ \hfill \text{if} \quad \alpha \neq 1 , \\ \exp\left\{i\theta'\mu - \int_{S_N} |\theta' s| \left(1 + i\frac{2}{\pi}\,\text{sign}(\theta' s)\ln(\theta' s)\right)\Gamma(ds)\right\} \\ \hfill \text{if} \quad \alpha = 1 , \end{cases}$$

where the *spectral measure* Γ is a bounded nonnegative measure on the unit sphere S_N, unit vector $s \in S_N$ is the integrand, and μ is the location vector. If $\alpha > 1$ then $\mu = \mathrm{E}(R)$ is the mean vector. The scale parameter or *variation* of a linear combination of the components $w'R = w_1R_1 + \ldots + w_NR_N$ satisfies

$$\sigma^\alpha(w'R) = \int_{S_N} |w's|^\alpha \Gamma(ds) \ . \tag{1}$$

If $R = (R_1, \ldots, R_N)'$ represents a vector of the individual returns in a portfolio with weights $w = (w_1, \ldots, w_N)'$, then the variation $\sigma^\alpha(w'R)$ is a measure of the portfolio risk; and the *covariation* is a measure of the dependence between two (symmetric) stable random variables with $1 < \alpha < 2$. The covariation is defined by

$$[R_i; R_j]_\alpha = \frac{1}{\alpha} \frac{\partial \sigma^\alpha(w_1R_1 + w_2R_2)}{\partial w_i} \Bigg|_{w_i=0; w_j=1} = \int_{S_N} s_i s_j^{<\alpha-1>} \Gamma(ds) \ ,$$

where $x^{<k>} = |x|^k \mathrm{sign}(x)$. The covariation matrix $([R_i, R_j]_\alpha)_{i,j}$, $i, j = 1, \ldots, d$, reflects the dependence structure among the individual returns in the portfolio.

3. Value at Risk

3.1. The Concept. Financial institutions have to evaluate their exposure to market risks which arise from variations in asset prices. A commonly used measure for assessing market risks is the *Value at Risk* (VaR). The VaR reflects the loss a financial position is expected not to exceed at a given probability level. Formally, the VaR is defined as the upper bound of the one-sided confidence interval

$$\mathrm{Prob}(R_{t,\tau} < -\mathrm{VaR}_c) = 1 - c \ , \tag{2}$$

where $R_{t,\tau}$ denotes the return of the portfolio over the period $(t, t+\tau]$; and c is the confidence level or target probability. Typical values for c are 0.95, 0.975 or 0.99. Thus, empirical VaR calculations involve the estimation of lower-order quantiles — for example 5%, 2.5%, or 1% — of the portfolio-return distributions.

Different methods for constructing the portfolio-return distributions have been suggested. Common methods are the delta method, historical simulation and Monte-Carlo simulation. The delta method is based on the normal assumption for the return distribution; the historical approach does not impose distributional assumptions, but is rather unreliable in estimating low quantiles due to the small number of available observations in the tails; and the performance of the Monte-Carlo method depends on the quality of distributional assumptions for the underlying risk factors.

From definition (2), VaRs are implied by

$$1 - c = F_R(-\mathrm{VaR}_c) = \int_{-\infty}^{-\mathrm{VaR}_c} f_R(x)dx \ ,$$

where $F_R(x) = \mathrm{Prob}(R \leq x)$ is the cumulative distribution function and $f_R(x)$ the probability density function of R. If returns can be modeled by a parametric distribution, VaRs can be derived from the distributional parameters. Consider a portfolio that consists of a single asset. Assuming — as is common — that the returns of the single-asset portfolio are normal, VaRs are fully determined by two parameters: the mean, μ, and the standard deviation, σ. Then, the VaR derivation reduces to finding the $(1-c)$-quantile, z_{1-c}, of the standard normal distribution; that is

$$1 - c = \int_{-\infty}^{-VaR_c} \phi_{\mu,\sigma}(x)dx = \int_{-\infty}^{z_{1-c}} \phi_{0,1}(z)dz$$

with $-\mathrm{VaR}_c = z_{1-c}\sigma + \mu$, where $\phi_{\mu,\sigma}(z)$ is the normal density function with mean μ and standard deviation σ.

If the returns are assumed to follow a stable distribution, the procedure for calculating VaRs remains unchanged. The only modification is that the quantile, z_{1-c}, has to be derived from the standardized *stable* distribution $S_\alpha(\beta, 1, 0)$ and that σ represents the scale parameter.

For short investment horizons one can ignore the expected return and set $\mu = 0$, because — under either distributional assumption — the magnitude of μ is negligible relative to σ, so that $\mathrm{VAR}_c \approx -z_{1-c}\sigma$.

To consider the multi-asset case, let R_P denote the portfolio return and R_i, $i = 1, \ldots, N$, the return on asset i over the investment horizon. Moreover, let w_i, $i = 1, \ldots, N$, with $\sum_{i=1}^{N} w_i = 1$, represent the i-th asset's weight in the portfolio. If the N asset returns are jointly normal the portfolio return, a linear combination of normal random variables, is also normally distributed with portfolio return, R_P, and portfolio variance, σ_P^2, given by

$$R_P = \sum_{i=1}^{N} w_i R_i$$

and

$$\sigma_P^2 = w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \ ,$$

respectively. Here, σ_i denotes the standard deviation of returns on the i-th asset; and ρ_{ij} is the correlation between the returns on assets i and j. Then, the portfolio VaR is computed by $\mathrm{VaR}_c \approx -z_{1-c}\sigma_P$. In this setting, portfolio risk is represented by a combination of linear exposures to normally distributed factors; and it suffices to evaluate the covariance matrix of the risk factors.

Again, the necessary modifications are — in principle — straightforward, when assuming that the returns follow a multivariate stable rather than a multivariate normal distribution. The portfolio scale σ_P is obtained by evaluating (1); and, as in the single-asset case, z_{1-c} is the quantile of the underlying standardized (univariate) stable distribution.

3.2. UNIVARIATE EMPIRICAL COMPARISONS. In this section we illustrate the empirical validity of the normal and stable assumptions for VaR calculations in a single-asset setting. To do so, we consider three stock indices, namely the S&P500, the DAX30, and the CAC40 (see Table 1 for sample information). To obtain the VaR estimates, we fit normal and stable distributions to the entire samples and calculate the VaR as (the negative of) the $(1-c)$-th quantile of a fitted distribution. The maximum likelihood estimates (see Mittnik *et al.* (1999)) of the distributional parameters are also reported in Table 1. The VaR_c estimates (upper entries) for probability

	S&P500	DAX30	CAC40
Sample Size	7327	8630	2756
Time Period	1/1/70–1/30/98	1/4/65–1/30/98	7/10/87–1/30/98
Normal:			
μ	0.032	0.026	0.028
σ	0.930	1.002	1.198
Stable:			
α	1.71	1.82	1.78
β	0.004	-0.084	-0.153
μ	0.036	0.027	0.027
σ	0.512	0.392	0.698

TABLE 1. Sample information and parameter estimates.

levels $c = 0.99$ and $c = 0.95$, implied by the fitted distributions, and their deviations from the empirical quantiles (lower entries) are reported in Table 2. The results indicate that the stable models provide, in general, a better fit for both the empirical $\text{VaR}_{.99}$ and $\text{VaR}_{.95}$ values. In fact, the stable model yields a better fit in five of the six cases. The sole exception is the $\text{VaR}_{.99}$ estimate for the S&P500. The overall fit, measured in terms of 100 times the mean squared deviation (MSD), drops from 6.256 to 3.230 for $\text{VaR}_{.99}$, and from 1.379 to 0.436 for $\text{VaR}_{.95}$, when the α parameter is freely estimated and not — as under the normal assumption — restricted to $\alpha = 2$. Similarly, the mean deviations (MD) are improved, namely from -0.243 to 0.098 ($\text{VaR}_{.99}$) and from 0.117 to -0.026 ($\text{VaR}_{.95}$). One of the patterns that emerge is that the normal model *under*estimates the 99% and *over*estimates

	VaR$_{.99}$			VaR$_{.95}$		
	Emp.	Normal	Stable	Emp.	Normal	Stable
S&P500:						
Estimate	2.293	2.131	2.559	1.384	1.497	1.309
Deviation		-0.162	0.266		0.113	-0.075
DAX30:						
Estimate	2.564	2.306	2.464	1.508	1.623	1.449
Deviation		-0.258	-0.100		0.115	0.059
CAC40:						
Estimate	3.068	2.760	3.195	1.819	1.943	1.756
Deviation		-0.308	0.127		0.124	-0.063
MSD$\times$100		6.256	3.230		1.379	0.436
MD		-0.243	0.098		0.117	-0.026

TABLE 2. Empirical and estimated VaRs.

the 95% VaR levels. The stable model has a systematic (negative) bias for the 95% level, but to lesser extent than the normal; and it is, on average, more on the conservative side for the 99% level. Overall, for the series considered, the stable assumption leads to a considerable improvement of the VaR estimates.

4. Optimal Asset Allocation

We now turn to the problem of determining the optimal asset allocation. First, we consider the most trivial case where an investor is faced with the situation of allocating funds between one risky and one risk-free asset. Subsequently, we consider the problem of choosing an optimal portfolio subject to specified VaR constraints. Here, we focus on two types of portfolios. First, the two-asset case with one risky and one risk-free asset is presented. Then, the problem of selecting from a set of several risky assets is illustrated.

4.1. Optimal Allocation Between One Risky and One Risk-free Asset.

4.1.1. The Optimization Problem. Consider the problem of determining the optimal allocation between a risk-free asset, guaranteeing a return of r_f, and a risky asset. Let the (uncertain) return of the risky asset be denoted by R and the expected return by μ_R, so that

$$R_P = \lambda r_f + (1 - \lambda)R$$

is the return of an investment of one unit with λ denoting the share invested in the risk-free asset. We assume that the investor's objective is to maximize

$$U(R_P) = \mathrm{E}(R_P) - c\mathrm{E}(|R_P - \mathrm{E}(R_P)|^\rho) , \tag{3}$$

where c and ρ are positive constants.

If the risky asset follows a stable distribution, that is, $R \sim S_\alpha(\beta, \sigma_R, \mu_R)$ with $\alpha > 1$, then

$$R_P \sim S_\alpha(\mathrm{sign}(1-\lambda)\beta_R, |1-\lambda|\sigma_R, \lambda r_f + (1-\lambda)\mu_R), \quad \lambda \neq 1 ;$$

and, if there is no short selling, that is, $\lambda \in [0, 1)$, then

$$R_P \sim S_\alpha(\beta_R, (1-\lambda)\sigma_R, \lambda r_f + (1-\lambda)\mu_R), \quad \lambda \in [0,1) ; \tag{4}$$

and $R_P = r_f$, for $\lambda = 1$. Relationship (4) implies that the portfolio mean is given by $\mu_P = \lambda\mu_f + (1-\lambda)\mu_R$, the portfolio scale parameter by $\sigma_P = (1-\lambda)\sigma_R$, and that the portfolio skewness remains unchanged.

The solutions to the maximization of (3) can be represented as the half-line in the mean-dispersion plane defined by

$$\mu_P = r_f + \frac{\mu_R - r_f}{\sigma_R}\sigma_P, \quad \sigma_P \geq 0 .$$

The investor should choose a portfolio $\mu_P = \lambda\mu_f + (1-\lambda)\mu_R$ that maximizes (3) for some $\rho \in [1, \alpha)$ and $\lambda \in [0, 1]$. To solve the asset allocation problem

$$\max_\lambda \mathrm{E}(R_P) - c\mathrm{E}\left(|R_P - \mathrm{E}(R_P)|^\rho\right) , \tag{5}$$

note first that, for all $\rho \in [1, \alpha)$ and $1 < \alpha < 2$, we have

$$U(R_P) = \lambda r_f + (1-\lambda)\mu_R - c(H(\alpha,\beta,\rho))^\rho (1-\lambda)^\rho \sigma_R^\rho ,$$

where

$$(H(\alpha,\beta,\rho))^\rho = \frac{2^{\rho-1}\Gamma\left(1-\frac{\rho}{\alpha}\right)}{\rho\int_0^\infty u^{-\rho-1}\sin^2 u du}\left(1+\beta^2\left(\tan^2\left(\frac{\alpha\pi}{2}\right)\right)\right)^{\rho 2\alpha} \\ \times \cos\left(\frac{r}{\alpha}\arctan\left(\beta\tan\left(\frac{\alpha\pi}{2}\right)\right)\right)$$

(see Samorodnitsky and Taqqu (1994), Hardin (1984)). When R is normally distributed (i.e., $\alpha = 2$), then for all $\rho > 0$,

$$U(R_P) = \lambda r_f + (1-\lambda)\mu_R - c\sqrt{\frac{2^\rho}{\pi}}\Gamma\left(\frac{\rho+1}{2}\right)(1-\lambda)^\rho\sigma_R^\rho .$$

Assuming $\rho \in (1, \alpha)$ and $\mu_R > r_f$, the first-order condition of (5) yields

$$\lambda^* = 1 - \left(\frac{\mu_R - r_f}{\rho\, c\, \sigma_R^\rho V(\alpha,\beta,\rho)}\right)^{\frac{1}{\rho-1}} , \tag{6}$$

where

$$V(\alpha,\beta,\rho) = \begin{cases} (H(\alpha,\beta,\rho))^\rho & \text{if } 1 < \alpha < 2 \ , \\ \sqrt{\frac{2^\rho}{\pi}}\Gamma\left(\frac{\rho+1}{2}\right) & \text{if } \alpha = 2 \ . \end{cases}$$

In the empirical application presented next, we will use (6) to derive the optimal allocation when investing in a portfolio consisting of one risky and one risk-free asset.

4.1.2. AN EMPIRICAL COMPARISON. To illustrate the consequences of distributional assumptions for asset allocation, we consider the same three daily stock-index series used above (S&P500, DAX30, and CAC40) and make use of the parameter estimates reported in the previous section. We set the annual risk-free rate to 6% or, on a daily basis, $r_f = 0.06/360$. Moreover, we set $\rho = 1.5$ and specify several values for the risk-aversion coefficient, c, in the objective function $\mathrm{E}(R_P) - c\mathrm{E}(|R_P - \mathrm{E}(R_P)|^\rho)$. The optimal λ^*-values under both the normal and stable assumptions are reported in Table 3.

Series	c	Normal	Stable
S&P500	0.03	0.159	0.505
	0.04	0.527	0.721
	0.05	0.697	0.822
	0.10	0.924	0.955
DAX30	0.03	0.557	0.605
	0.04	0.751	0.778
	0.05	0.841	0.858
	0.10	0.960	0.964
CAC40	0.02	0.323	0.567
	0.04	0.831	0.892
	0.10	0.973	0.983

TABLE 3. Optimal allocations under normal and stable assumptions.

The results show that the optimal allocations can differ considerably under the two distributional assumptions. This holds especially for low values of the risk-aversion parameter, c, in which case the normal assumption leads to less conservative portfolios by putting less funds into the risky asset (i.e., λ^* is smaller). By taking the heavy tails of the empirical distributions into account, the stable assumption causes the investor to allocate more funds to the risk-free asset. To see this, recall that the tail behavior of a (non-Gaussian) stable random variable, $X \sim S_\alpha(\beta,\sigma,\mu)$, is determined by

$$\lim_{x\to+\infty} x^\alpha \mathrm{Prob}(\pm X > x) = C_\alpha \frac{1\pm\beta}{2}\sigma^\alpha, \quad 1 < \alpha < 2 \ ,$$

where $C_\alpha = \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$. Because, for $\alpha \in (1,2]$, C_α increases as $\alpha \downarrow 1$, the risk measure $\mathrm{E}(|R_P - \mathrm{E}(R_P)|^\rho)$, with $\rho \in [1,\alpha)$, increases the more one moves away from normality.

4.2. Optimal Asset Allocation Under VaR Constraints.

4.2.1. One Risky and One Risk-free Asset. We now consider the optimal allocation problem of a non-satiable investor, who takes VaR constraints into account by having an objective function of the form

$$\begin{aligned} V(R_P) &= \mathrm{E}(R_P) - c\,\mathrm{Prob}(R_P \le -VaR) \\ &= \lambda r_f + (1-\lambda)\mu_r - cF_P(-VaR)\,, \end{aligned} \tag{7}$$

where F_P is the cumulative distribution function of R_P; and c is a positive real number. In the non-Gaussian case with $1 < \alpha < 2$ and $\lambda \in [0,1)$, we have

$$R_P \sim S_\alpha(\beta_R, (1-\lambda)\sigma_R, \lambda r_f + (1-\lambda)\mu_R)$$

and $V(R_P)$ is given by

$$V(R_P) = \lambda r_f + (1-\lambda)\mu_R - cF_{\alpha,\beta}\left(\frac{-\mathrm{VaR}}{(1-\lambda)\sigma_R} - \frac{\mu_R}{\sigma_R} - \beta\tan\left(\frac{\pi\alpha}{2}\right)\right). \tag{8}$$

In expression (8), $F_{\alpha,\beta}$ denotes the cumulative distribution function of the stable distribution $S_\alpha(\beta,\ 1,\ \text{-}\ \beta\tan(\pi\alpha/2))$ (see Zolotarev (1986)) and is defined by

$$F_{\alpha,\beta}(x) = \begin{cases} 1 - \frac{1}{\pi}\int_{\theta_0}^{\frac{\pi}{2}} \exp\left\{-(x-\zeta)^{\frac{\alpha}{(\alpha-1)}} K(\vartheta,\alpha,\beta)\right\} d\theta & \text{if } x > \zeta\,, \\ \frac{1}{\pi}\left(\frac{\pi}{2} - \theta_0\right) & \text{if } x = \zeta\,, \\ 1 - F_{\alpha,-\beta}(-x) & \text{if } x < \zeta\,, \end{cases}$$

where

$$\begin{aligned} \zeta &= \zeta(\alpha,\beta) = -\beta\tan\left(\frac{\pi\alpha}{2}\right), \\ \theta_0 &= \theta_0(\alpha,\beta) = \frac{\arctan\left(\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{\alpha}, \end{aligned}$$

and

$$K(\theta,\alpha,\beta) = \cos(\alpha\theta_0)^{\frac{1}{\alpha-1}}\left(\frac{\cos\theta}{\sin(\alpha(\theta+\theta_0))}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\theta_0 + (\alpha-1)\theta)}{\cos\theta}\,.$$

In the normal case ($\alpha = 2$), we have

$$V(R_P) = \lambda r_f + (1-\lambda)\mu_R - c\int_{-\infty}^{\frac{-VaR-\lambda r_f-(1-\lambda)\mu_R}{(1-\lambda)\sigma_R}} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{t^2}{2}}\,dt\,. \tag{9}$$

Numerical methods can be used for maximizing (8) or (9) in order to obtain the optimal allocation λ^*.

Employing again the three index series used above, we compare the optimal allocations under normal and stable assumptions. Considering the empirical 95% VaR levels of the return series (see Table 2) as the VaR constraints, and assuming different values for the risk-aversion parameter, c, we obtain the optimal allocations reported in Table 4.

Series	c	λ^*	
		Normal	Stable
S&P500	0.3	0.325	0.000
	1.0	0.483	0.496
	2.0	0.530	0.724
	15	0.613	0.981
DAX30	0.2	0.266	0.000
	0.5	0.432	0.355
	1.0	0.495	0.514
	15	0.616	0.967
CAC40	0.2	0.231	0.000
	0.5	0.419	0.375
	1.0	0.484	0.562
	15	0.610	0.978

TABLE 4. Optimal allocations under VaR 95%-constraints.

The results illustrate that the optimal allocations derived under the normal and stable assumptions can be very different. Specifically, we observe that for lower values of the risk-aversion coefficient, c, the "normal investor" invests more in the risk-free asset. This is due to a "kurtosis effect". Stable distributions are more peaked around the center than the normal. Consequently, investors who rely on the stable assumption and who are not too averse to downside risk, give more importance to the mean than "normal investors", who having the same c-value are more willing to sacrifice a higher mean for lower risk.

For higher degrees of risk aversion, it is the "stable investor" who invests more in the risk-free asset due to the "tail effect". The tail probability for normal random variables, Prob($R_P \leq -VaR$), tends to zero exponentially fast, so that the "normal investor" assigns lower probabilities to the tail risk than the "stable investor".

4.2.2. THE MULTI-ASSET CASE. We now turn to asset-allocation problems in the multi-asset case and consider a portfolio of stocks belonging to the DAX30 index. The sample of daily returns consists of 1827 observations which cover the period from January 1991 to April 1998. There were

$N = 26$ stocks which belonged to the DAX30 throughout the sample period considered.

Instead of fitting a full (26-dimensional) multivariate normal or stable distribution to the return series, it is common practice to model the dependencies among the assets by relating them to a — typically small — set of common risk factors. The simplest form of doing so is to fit a single-index model to the return series. Using the broad Composite DAX (CDAX) index as the single risk factor, the single-index model is specified by

$$R_{it} = \mu_i + b_i f_t + \varepsilon_{i,t}, \quad i = 1, \ldots, N ,$$

where f_t denotes the return on the risk-factor (CDAX) during period t; and coefficient b_i reflects the dependence of return R_i on the index. The factor is modeled as a constant plus an error term, i.e., $f_t = \mu_o + \varepsilon_{ot}$. Under the normal assumption we impose, for all $t = 1, \ldots, T$ and $i = 0, 1, \ldots, N$,

$$\varepsilon_{it} \sim N(0, \sigma_i^2) , \tag{10}$$

where subscript 0 refers to the index. Moreover, it is assumed that all ε_{it} are independent over time and with respect to each other. Under the stable assumption, these independence assumptions remain in place but assumption (10) is generalized to

$$\varepsilon_{it} \sim S_\alpha(\beta_i, \sigma_i, 0), \quad i = 0, \ldots, N . \tag{11}$$

Imposing a symmetry restriction for all returns, i.e., $\beta_i = 0$ $(i = 0, \ldots, N)$, the stable single-index model is specified by $3(N+1)$ parameters, namely, $\mu_o, \ldots, \mu_N$, $\sigma_o, \ldots, \sigma_N$, $b_1, \ldots, b_N$, and α. This is one parameter more than the normal single-index model, which imposes $\alpha = 2$. For $N = 26$, a total of 80 (81) parameters need to be estimated for the normal (stable) model, which makes maximum-likelihood estimation practically infeasible. Assuming $1 < \alpha \leq 2$, the location parameters, μ_i, can be estimated by the sample means. Under normality, the factor loadings, b_i, can be estimated by ordinary least squares (OLS) regression. Blattberg and Sargent (1971) show that the OLS estimates of regression parameters are still consistent in the presence of stable disturbances. The variances of the normal errors, σ_i^2, can be estimated from the OLS residuals. For the stable model, given estimates $\hat{\mu}_i$ and $\hat{b}_i$, the scale parameters, σ_i, and the shape parameter, α, can be estimated via maximum likelihood (see Doganoglu and Mittnik (2001) for technical details and estimation results).

Portfolio optimization under VaR constrains amounts to maximizing the portfolio return subject to satisfying a particular VaR target, say VaR_c^*. Specifically, that is,

$$\max_w w'\mu$$

subject to

$$\mathrm{VaR}_c \leq \mathrm{VaR}_c^* \,,$$
$$\sum_{i=1}^{N} w_i = 1 \,,$$

and, if there is no short selling, $w_i \geq 0,\ i = 1, \ldots, N$. If the allocation decision relies on estimated quantities, the optimization problem becomes

$$\max_{w} w'\hat{\mu}$$

subject to

$$\widehat{\mathrm{VAR}}_c \leq \mathrm{VaR}_c^* \,,$$
$$\sum_{i=1}^{N} w_i = 1 \,,$$

and, possibly, $w_i \geq 0$, $i = 1, \ldots, N$.

By defining an appropriate grid of VAR_c^* values, one can construct the efficient frontier in the mean-VAR_c plane by deriving the optimal weight vectors associated with each grid point. To conduct empirical comparisons, we can construct three estimated frontiers under the assumption that there is no short selling:

1. The *normal frontier*, which is derived from the estimated single-index model based on the normal assumption.
2. The *stable frontier* assumes a single-index structure with stable errors.
3. The *empirical frontier*, which is derived by computing the sample quantiles of the historical portfolio distributions associated with the optimal weight vectors for each of the grid points.

Figure 1 graphs the three frontiers in the mean-$\mathrm{VaR}_{.95}$ plane. The plots show that the stable frontier (solid line) approximates the empirical frontier (dotted line) more closely than the normal frontier (dash-dot line). Both models fit well in the minimum-risk region. But as one approaches higher VaR^* targets, the fit of the normal deteriorates in that it overestimates the portfolio risk. Summary measures for the goodness of fit confirm the visual impressions of the superior fit of the stable model. Table 5 reports the mean-squared deviation (MSD), mean-absolute derivation (MAD) and mean deviation (MD) of the fitted normal and stable frontiers.

Model	MSD	MAD	MD
Normal	0.0722	0.1960	-0.1956
Stable	0.0034	0.0404	-0.0305

TABLE 5. Summary measures of fitted efficient frontiers.

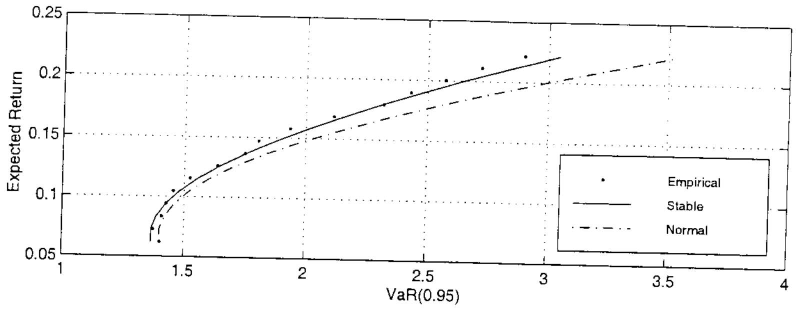

FIGURE 1. Empirical, normal, and stable efficient frontiers in the mean-$\mathrm{VaR}_{.95}$ plane.

The application indicates that the stable assumption provides a more realistic description of the risk/return relationships for portfolios constructed from DAX stocks and, thus, should lead to more efficient allocations of risk. This is particularly important in risk management, where the "cost" of a position is assessed in terms of risk.

5. CONCLUSIONS

We have reviewed some recent empirical evidence on the consequences of the underlying distributional assumption in financial decision making. We have focused on implications for risk assessment, based on the value-at-risk concept, and for asset allocation. Specifically, we have considered the heavy-tailed stable Paretian distribution as an alternative to the commonly adopted normal assumption. The evidence suggests that the stable model leads to more reliable decisions because it can capture the heavy tails — typically encountered in financial data — as well as skewness. The stable assumption does not rule out the normal model, since the latter is a special case of the stable model. In addition to the empirical support, the stable model has attractive theoretical properties, which — in contrast to other alternatives to the normal model — preserve the analytical tractability of financial analyses, such as asset allocation and portfolio management.

We have not addressed questions pertaining to dynamic issues, such as the out-of-sample prediction of risk. Some empirical analyses in this direction can be found in Mittnik *et al.* (2000).

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