A TWO-FACTOR INTEREST RATE MODEL AND CONTINGENT CLAIMS VALUATION

FRANCIS A. LONGSTAFF AND EDUARDO S. SCHWARTZ

In Longstaff and Schwartz [1992] we develop a two-factor general equilibrium model of the term structure of interest rates using the framework pioneered by Cox, Ingersoll, and Ross [1985a, 1985b]. In it we derive closed form expressions for the prices of discount bonds and other interest rate-sensitive contingent claims. The two factors are the instantaneous riskless (short-term) interest rate and the instantaneous variance of changes in this short-term interest rate.

The model is attractive not only because it provides for closed form expressions in a two-factor world, but also because it explicitly allows for a stochastic volatility factor. As the parameters of the model can be estimated using both time series and cross sections of prices (or yields), the framework provides reasonable dynamics of the factors or state variables.

In this article we extend the Longstaff-Schwartz model to allow it to fit the initial discount function exactly. We do this by simply deriving a partial differential equation for forward prices and using the actual discount factors to obtain the present value of the forward prices. This method can then be used for valuing all European interest rate-contingent claims. This extension of the model cannot be used to price discount bonds because by construction all discount bonds used exactly fit the model,[1] but it has the advantage of using all the information in the current term structure to price interest rate-contingent claims within the framework of the two-factor model.

We also show that this version of the Longstaff-Schwartz model can be fitted into the Heath, Jarrow, and Morton [1992] framework and thus their numerical procedures can be used to value American claims.

We briefly summarize the results of the

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Longstaff-Schwartz model needed for deriving the interest rate-contingent claims valuation model, then derive the fundamental partial differential equation and the risk-adjusted processes for forward prices, and the forward density for the state variables. We show applications of the model in valuation of interest rate caps, floating-rate swaptions, and volatility caps. We also show the relation between the model and the framework recently developed by Heath, Jarrow, and Morton [1992]; in particular we derive the implicit assumptions made in our model with respect to the volatilities of the forward rate process.

THE LONGSTAFF-SCHWARTZ MODEL

Starting from assumptions about the stochastic evolution of two exogenous state variables that affect the returns on physical investment and the preferences of a representative investor, Longstaff and Schwartz [1992] follow the general equilibrium framework of Cox, Ingersoll, and Ross [1985a] to derive the fundamental partial differential equation for all default-free interest rate-contingent claims, H(x,y,):

$$\frac{x}{2}H_{xx} + \frac{y}{2}H_{yy} + (\gamma - \delta x)H_x + (\eta - \nu y)H_y - (\alpha x + \beta y)H = H_\tau \qquad (1)$$

where γ, δ , η, and υ are the parameters of the risk-adjusted process for the unspecified, uncorrelated (modified) state variables x and y:

$$dx = (\gamma - \delta x)dt + \sqrt{x}\, dZ_1 \qquad (2)$$

$$dy = (\eta - \nu y)dt + \sqrt{y}\, dZ_2 \qquad (3)$$

and α and β are parameters of the return process for physical investment.

In this framework the equilibrium instantaneous interest rate and the variance of the changes in this rate are given respectively by:

$$r = \alpha x + \beta y \qquad (4)$$

$$V = \alpha^2 x + \beta^2 y \qquad (5)$$

These last two equations provide the empirical link of the model. As both r and V are linear functions of the state variables x and y, it is easy to perform a change of variables and express all the results of the model in terms of r and V as state variables. This is the approach taken in Longstaff and Schwartz [1992]. For computational convenience here, we opt for expressing all our results in terms of the original state variables x and y, although at any point in the analysis a change of variables is possible to express the results in terms of the "observable" state variables r and V.

The solution to Equation (1) subject to the appropriate boundary conditions determines the value of any interest rate-contingent claim. In the case of a discount bond in which the appropriate boundary condition at maturity is given by:

$$F(x,y,0) = 1.0 \qquad (6)$$

the equation has a simple closed form solution:

$$F(x,y,\tau) = E_1(\tau)\exp[E_2(\tau)\, x + E_3(\tau)y]$$

where

$$E_1(\tau) = A^{2\gamma}(\tau) \bullet B^{2\eta}(\tau)\exp(\kappa\tau)$$
$$E_2(\tau) = (\delta - \phi)(1 - A(\tau))$$
$$E_3(\tau) = (\upsilon - \psi)(1 - B(\tau))$$

$$A(\tau) = \frac{2\phi}{(\delta + \phi)(\exp(\phi\tau) - 1) + 2\phi}$$

$$B(\tau) = \frac{2\psi}{(\nu + \psi)(\exp(\psi\tau) - 1) + 2\psi} \qquad (7)$$

$$\phi = \sqrt{2\alpha + \delta^2}$$

$$\psi = \sqrt{2\beta + \nu^2}$$

$$\kappa = \gamma(\delta + \phi) + \eta(\nu + \psi)$$

PRICING FORWARD CONTRACTS

The value of any European interest rate-contingent claim with no intermediate payout is equal to its forward price multiplied by the value of a unit dis-

count bond with the same maturity as the contingent claim. We can obtain the present value of the contingent claim by discounting its forward price using the current discount function. This method assures that model prices will exactly fit the term structure used in the computations.

Consider an interest rate-contingent claim with the general payoff function at maturity:

$$H(x,y,0) = G(x,y) \tag{8}$$

Note that this type of payoff function will accommodate any function of r, V, yield, etc. At any time before maturity this claim satisfies Equation (1).

We try the separation of variables:

$$H(x,y,\tau) = F(x,y,\tau) \times M(x,y,\tau) \tag{9}$$

where $F(x,y,\tau)$ is the unit discount bond.

Taking the appropriate partial derivatives in (9), substituting in (1), collecting terms and recalling that the value of a unit discount bond, $F(x,y,\tau)$, also satisfies (1), it can be shown that forward prices satisfy the partial differential equation:

$$\frac{x}{2}M_{xx} + \frac{y}{2}M_{yy} + (\gamma - \delta x + \frac{F_x}{F}x)M_x + (\eta - \nu y + \frac{F_y}{F}y)M_y = M_\tau \tag{10}$$

with the same terminal boundary condition as the contingent claim, because at maturity the value of the discount bond is one:

$$M(x,y,0) = G(x,y) \tag{11}$$

The value of the two terms in (10) that depend on the value of the unit discount bond can be obtained from Equation (7):

$$\frac{F_x}{F} = E_2(\tau) \tag{12}$$

$$\frac{F_y}{F} = E_3(\tau) \tag{13}$$

This analysis shows that in the framework of the two-factor model we can obtain the forward price of any European interest rate-contingent claim by solving Equation (10), subject to the appropriate terminal condition. The present value of the claim can then be obtained by discounting this forward price using the current unit riskless discount bond to the maturity of the claim.

Another interpretation of Equation (10) is that the forward price of the claim is given by:

$$M(x,y,\tau) = E[G(x,y)] \tag{14}$$

where the expectation is taken with respect to the joint density for x and y implied by the dynamics:

$$dx = (\gamma - \delta x + E_2(\tau)x)dt + \sqrt{x}\, dZ_1 \tag{15}$$

$$dy = (\eta - \nu y + E_3(\tau)y)dt + \sqrt{y}\, dZ_2 \tag{16}$$

This density is a bivariate non-central chi-square density with closed form, given the current values of the state variables x_0 and y_0:[2]

$$q(x,y \mid x_0,y_0) = \frac{4}{a(\tau)c(\tau)}\left(\frac{x}{b(\tau)x_0}\right)^{\gamma-1/2}\left(\frac{y}{d(\tau)y_0}\right)^{\eta-1/2} \exp\left(\frac{-2}{a(\tau)}(x + b(\tau)x_0)\right)\exp\left(\frac{-2}{c(\tau)}(y + d(\tau)y_0)\right) I_{2\gamma-1}\left(\frac{4}{a(\tau)}\sqrt{b(\tau)xx_0}\right)I_{2\eta-1}\left(\frac{4}{c(\tau)}\sqrt{d(\tau)yy_0}\right) \tag{17}$$

where

$$a(\tau) = A(\tau)(\exp(\phi\tau) - 1)/\phi,$$

$$b(\tau) = A^2(\tau)\exp(\phi\tau),$$

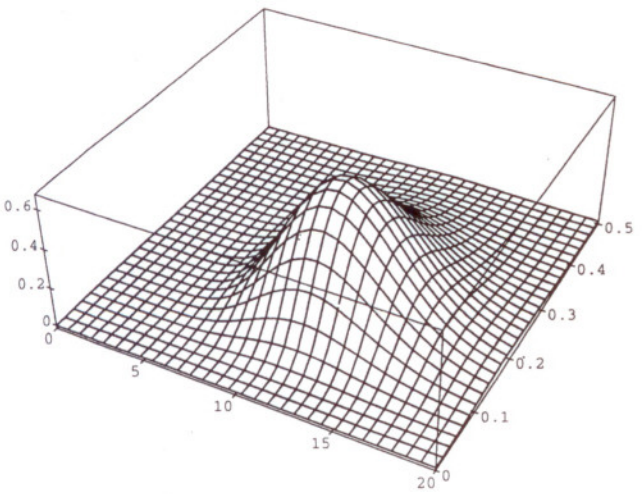
$$c(\tau) = B(\tau)(\exp(\psi\tau) - 1)/\psi,$$

$$d(\tau) = B^2(\tau)\exp(\psi\tau),$$

and $I_p(.)$ is the modified Bessel function of order p.

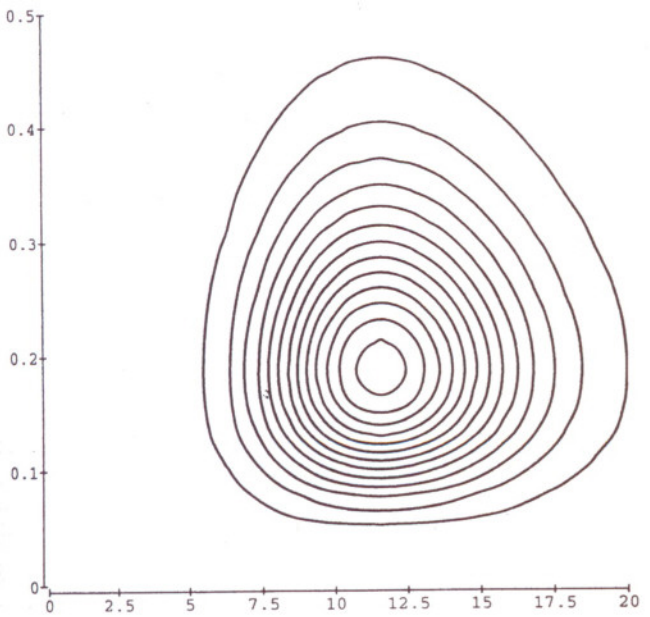
The value of the forward price for the claim can then be obtained directly from (14) by integration, or by using Monte Carlo simulation with the risk-

EXHIBIT 1 ■ Density Function in Terms of x and y



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = One Year
Current Value of r = 0.06717
Current Value of V = 0.00081

EXHIBIT 2 ■ Contour of Density Function in Terms of x and y



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = One Year
Current Value of r = 0.06717
Current Value of V = 0.00081

adjusted processes for the uncorrelated state variables (15) and (16).

Exhibits 1 and 2 illustrate the density function and its contour as a function of the state variables x and y. Exhibits 3 and 4 illustrate the same density and its contour as a function of the transformed state variables r and V. These figures show that r and V are correlated and that the value of V is bounded by αr and βr, i.e., $\alpha r < V < \beta r$.

CAP PRICING

Consider a simple cap that at time τ from now pays the maximum between the difference of the short rate and the cap rate, c, and zero. From (4) and (14), the forward price of this cap is given by:

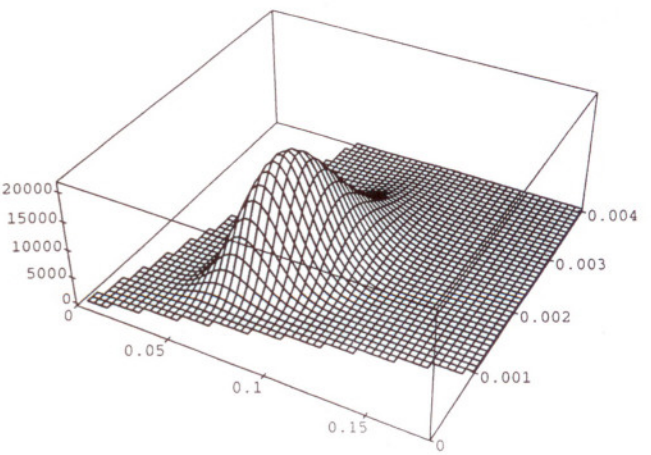
$$M(x,y,\tau) = E[\text{Max}(\alpha x + \beta y - c, 0)] \qquad (18)$$

where the expectation is taken with respect to the density function (17).

Rewriting this expectation in terms of the transformed state variables r and V, the forward price of the cap can be expressed as:

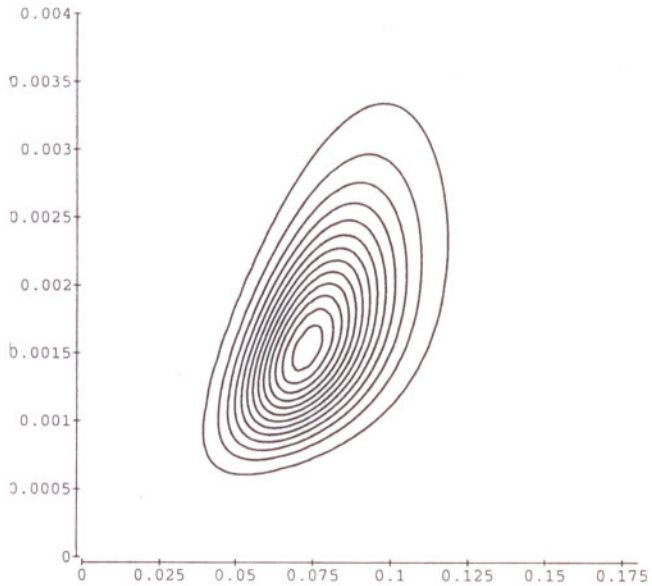
$$M(r_0, V_0, \tau) = \int_c^{\infty} \int_{\alpha r}^{\beta r} (r - c) q(r, V | r_0, V_0) dV dr \qquad (19)$$

EXHIBIT 3 ■ Density Function in Terms of r and V



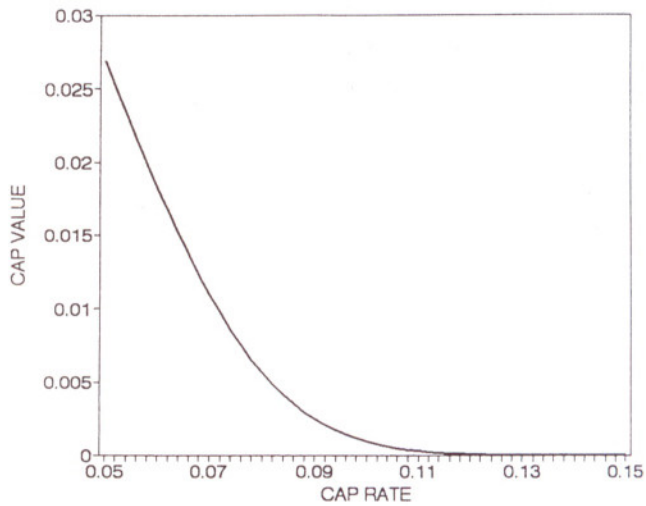
Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = One Year
Current Value of r = 0.06717
Current Value of V = 0.00081

EXHIBIT 4 ■ Contour of Density Function in Terms of r and V



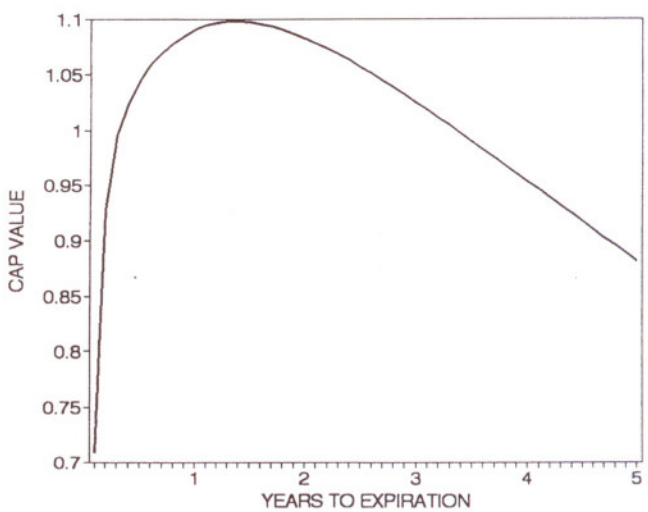
Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = One Year
Current Value of r = 0.06717
Current Value of V = 0.00081

EXHIBIT 5 ■ Cap Value as a Function of the Cap Rate



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = One Year
Current Value of r = 0.06717
Current Value of V = 0.00081

EXHIBIT 6 ■ Cap Value as a Function of the Time to Expiration



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$,
$\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Cap Rate = 0.07
Current Value of r = 0.06717
Current Value of V = 0.00081

The present value of the cap is obtained by multiplying the forward price by the τ-maturity discount factor from the current discount function.

Exhibit 5 shows the value of a one-year cap for different values of the cap rate c. Note that the value of the cap drops off rapidly as the cap rate increases because of the mean reversion of interest rates in the model. Specifically, mean reversion implies that the probability of a cap with a high value of c being in the money at expiration is much less than it would be in the absence of mean reversion.

Exhibit 6 plots the value of a cap as a function of its time to expiration, where the cap rate is held fixed at 0.07. The time decay of a cap is very different from what we would expect in models such as the Black-Scholes model. For example, the cap value is an increasing function for values of τ up to 1.3 years, and then becomes a decreasing function for longer maturities. This hump-shaped pattern of time decay in the value of the cap occurs because the variance of future

values of r does not grow linearly with time as it does in the Black-Scholes model. Instead, it grows at a slower-than-linear rate because the model implies that r has a steady state distribution. This effect, in conjunction with the higher discount factor as τ increases, results in the hump-shaped pattern.

These two examples show that the implications of the Longstaff-Schwartz model for valuing and hedging interest rate caps can differ significantly from those of the Black-Scholes model, which is often applied to interest rate options despite its assumption of constant interest rates.

FLOATING-RATE SWAPTIONS

Floating-rate swaptions are simple options that at time τ from now pay the maximum of the difference between the floating rate and zero. For example, a swaption that pays the maximum of the difference between the T-year yield and the short rate and zero has a payoff function given by

$$H(x,y,0) = \text{Max}(0, -(\ln E_1(T) + E_2(T)x + E_3(T)y)/T - r)] \quad (20)$$

where E_1, E_2, and E_3 are defined in (7). From (7) and (14), the forward price of the swaption is

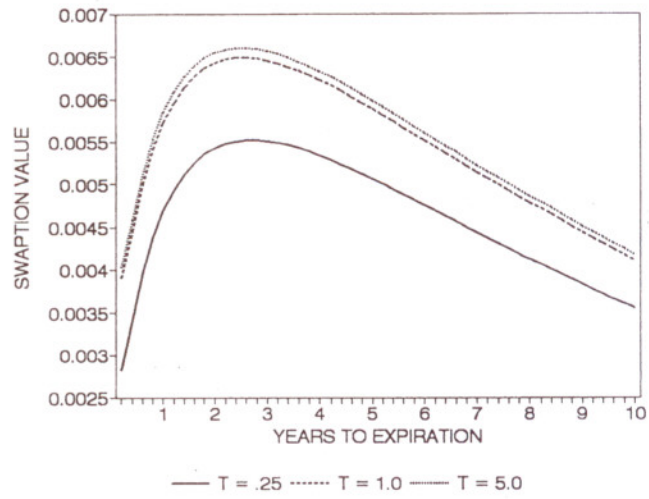
$$M(x,y,\tau) = E[\text{Max}(0, -(\ln E_1(T) + E_2(T)x + E_3(T)y)/T - r)] \quad (21)$$

The present value of the swaption is given by multiplying the forward price by the τ-maturity discount factor from the current discount function.

Exhibit 7 shows the value of the swaption for different values of the time until expiration, τ, and the maturity of the swap yield T. In general, the swaption is an increasing function of the maturity of the swap yield. This is because the smaller the value of T, the more correlated the swap yield and the short-term rate, and the less valuable the option to swap. This last follows because an option to swap for like kind is worth less than an option to swap for something different. Note that for large T, however, the swaption can become a decreasing function of T because the convergence of yields to their steady state distribution makes the swap yield and the short rate more correlated.

Exhibit 7 also shows that the time decay of the swaption is more complex than implied by the Black-Scholes model. In the examples shown, the swaption value increases for values of τ up to approximately 2.5 years, and then declines as τ increases. The reason for this pattern is again related to the mean reversion of interest rates in the Longstaff-Schwartz model. For short horizons, the uncertainty about future payoffs grows more rapidly than the discount factor. For values of τ greater than 2.5 years, however, an increase in the time until expiration has less of an effect on the variance of future swaption payoffs than it does on the discount factor.

EXHIBIT 7 ■ Swaption Value as a Function of the Time to Expiration for Three Different Swap Yield Maturities



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$, $\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Current Value of r = 0.06717
Current Value of V = 0.00081

VOLATILITY CAPS

The pricing of a cap on volatility itself is an example of a more exotic type of option that can be valued using our methodology. From (5), the payoff for this claim would be

$$H(x,y,0) = \text{Max}(0, \alpha^2 x + \beta^2 y - K) \quad (22)$$

where K is the strike volatility. A volatility cap is a security that would be useful in hedging fixed-income portfolios against the price effects of shifts in interest rate uncertainty. Similar types of options on implied

volatilities have been proposed in some U.S. markets.

The price of a volatility cap can be determined as in the previous examples. Using (5) and (14), the forward price of a volatility cap equals

$$M(x,y,\tau) = E[Max(0, \alpha^2 x + \beta^2 y - K)] \qquad (23)$$

The price of the volatility cap is again given by multiplying the forward price by the discount factor from the current discount function.

Exhibit 8 graphs the value of the volatility cap as a function of the strike volatility. Interestingly, the convexity of the volatility cap is much less than for an interest rate cap with the same time until expiration. Exhibit 9 shows the value of the volatility cap as a function of the time until expiration of the cap. As in earlier examples, the time decay of the volatility cap is hump-shaped because of the mean reversion in volatility.

FITTING THE TERM STRUCTURE

Heath, Jarrow, and Morton (HJM [1992]) develop an arbitrage methodology for pricing interest rate-sensitive contingent claims given the prices of all zero-coupon bonds. They extend the Ho and Lee [1986] model by imposing a stochastic structure on evolution of the forward rate curve as opposed to the discount function. As our methodology also prices contingent claims given the prices of zero-coupon bonds, it is of interest to compare it with a two-factor HJM framework.

In HJM the assumed stochastic process for changes in the entire forward rate curve is given by[3]

$$df(t,T) = \alpha(t,T)dt + \sigma_1(t,T)dZ_1 + \sigma_2(t,T)dZ_2 \qquad (24)$$

where $f(t,T)$ is the forward rate at time t for date $T > t$. The α and the σs could also be a function of the stochastic factors.

This process for forward rates implies a process for rates of return on discount bonds:[4]

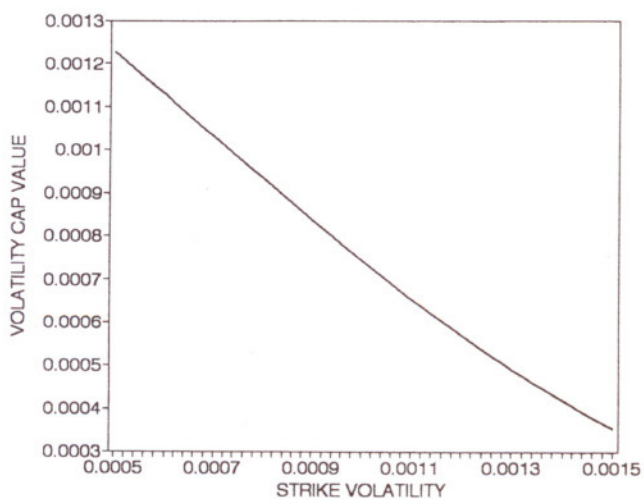
$$\frac{dF}{F} = \mu_F(\cdot)dt + a_1(t,T)dZ_1 + a_2(t,T)dZ_2 \qquad (25)$$

with

$$a_i(t,T) \equiv -\int_t^T \sigma_i(t,v)dv, \; i = 1,2 \qquad (26)$$

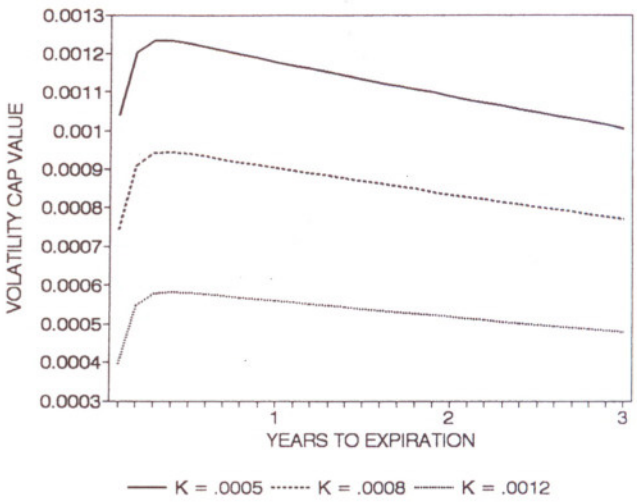
HJM show that in the absence of arbitrage the

EXHIBIT 8 ■ Volatility Cap Value as a Function of the Strike Volatility



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$, $\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Time to Maturity = 0.25 Year
Current Value of r = 0.06717
Current Value of V = 0.00081

EXHIBIT 9 ■ Volatility Cap Value as a Function of the Time to Expiration for Three Different Strike Volatilities



Parameter Values: $\alpha = 0.0050$, $\beta = 0.0814$, $\delta = 0.3299$, $\upsilon = 14.4227$, $\gamma = 4.0224$, $\eta = 3.2033$
Current Value of r = 0.06717
Current Value of V = 0.00081

drift term in (24) for the corresponding risk-adjusted process is a particular function of the volatility terms σ.[5] For the model to have empirical content, it is necessary to specify the functional form of the σs in (24).

To see how the model presented in this article fits into the HJM framework, it is necessary to establish the particular assumptions of the model with respect to the volatilities of forward rates. This can be accomplished by computing the volatilities of bond returns from (7) and comparing them with (25):

$$a_1(\cdot)\frac{F_x}{F}\sqrt{x} = E_2(\tau)\sqrt{x} \qquad (27)$$

$$a_2(\cdot)\frac{F_y}{F}\sqrt{y} = E_3(\tau)\sqrt{y} \qquad (28)$$

Note that $\tau = T - t$.

Finally, from (26), the forward rate volatilities implied by the model are:

$$\sigma_1(\cdot) = -E_2'(\tau)\sqrt{x} \qquad (29)$$

$$\sigma_2(\cdot) = -E_3'(\tau)\sqrt{y} \qquad (30)$$

From (29) and (30) it can be seen that the variance of changes in forward rates implied by the two-factor model is linear in the original factors and therefore also in the transformed factors r and V.

SUMMARY AND CONCLUSIONS

Our extension of the Longstaff-Schwartz two-factor general equilibrium model of the term structure to price forward contracts of interest rate-sensitive contingent claims can be used to determine the present value of any European claim. The procedure uses all the information contained in the current term structure, in addition to the dynamics of the state variables.

We show that the model implies specific assumptions with respect to the stochastic movement of all forward rates and thus can be integrated into the Heath, Jarrow, and Morton [1992] framework.

The approach presented here loses the general equilibrium properties of the Longstaff and Schwartz [1992] model because it does not endogenously determine the price of all discount bonds, but takes them as given. Its merit is that the approach can be cast in an arbitrage framework. The advantage of this is that it applies to nominal interest rates as opposed to real rates and that it can incorporate information embodied in the current term structure.

For the pricing of American option-like contingent claims, numerical procedures involving the stochastic evolution of the discount function or the forward rate curve would be required.

ENDNOTES

[1]This property is not unique to our model, but it is shared by other models such as Heath, Jarrow, and Morton [1992] and Ho and Lee [1986].

[2]The square root processes in (15) and (16) lead to distributions related to non-central chi-squares, and (17) is basically the product of two non-central chi-square densities.

[3]HJM Equation (4) [1992, p. 80].

[4]See HJM Equations (8) and (9), p. 82.

[5]The drift term for the risk-adjusted instantaneous return on discount bonds in (25) is r.

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