THE MARKET PORTFOLIO MAY BE MEAN-VARIANCE EFFICIENT AFTER ALL

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January 14, 2009

ABSTRACT

Testing the CAPM boils down to testing the mean/variance efficiency of the market portfolio. Numerous studies have examined the mean/variance efficiency of various market proxies by employing sample parameters, and have concluded that these proxies are inefficient. Shrinkage methods do not seem to help. These findings cast doubt about one of the cornerstones of modern finance. This study adopts a reverse-engineering approach: given a particular market proxy, we find the minimal variations in sample parameters required to ensure that the proxy is mean/variance efficient. Surprisingly, slight variations in parameters, well within estimation error bounds, suffice to make the proxy efficient. Thus, many conventional market proxies could be perfectly consistent with the CAPM and useful for estimating expected returns.

Keywords: Mean-Variance efficiency, CAPM, portfolio optimization, beta.

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INTRODUCTION

Testing the Capital Asset Pricing Model (CAPM) is equivalent to testing the mean/variance efficiency of the market portfolio (see Roll [1977] and Ross [1977]). The efficiency of the market portfolio has very important implications regarding the debate over passive versus active investing, and regarding the use of betas for pricing risky assets. Many studies that have examined the mean/variance efficiency of various market proxies have found that these proxies are inefficient, and typically far from the efficient frontier[1]. Moreover, it is well known that portfolios on the efficient frontier typically involve many short positions[2], which implies, of course, that the positive-by-definition market portfolio cannot be efficient. These results hold both when the sample return parameters are employed, and when the return parameters are adjusted by various shrinkage methods.[3] This constitutes a very dark cloud hanging over one of the most fundamental models of modern finance. In light of this evidence, should the CAPM be taken seriously by financial economists, or is it just a pedagogical tool for MBA classes, grossly inconsistent with the empirical evidence?

This paper shows that a small variation of the sample parameters, well within their estimation error bounds, can make a typical market proxy efficient. Thus, the empirically measured return parameters and the market portfolio weights are perfectly consistent with the CAPM using a typical proxy. How is this possible, and how can it be reconciled with the many previous studies that have shown that the market proxy is

[1] See, for example, Ross [1980], Gibbons [1982], Jobson and Korkie [1982], Shanken [1985], Kandel and Stambaugh [1987], Gibbons, Ross, and Shanken [1989], Zhou [1991], and MacKinlay and Richardson [1991].

[2] As shown, for example, by Levy [1983], Green and Hollifield [1992], and Jagannathan and Ma [2003].

[3] Jagannathan and Ma [2003] show that constraining the weights of the minimum-variance portfolio to be non-negative is equivalent to modifying the covariance matrix in a way which typically shrinks the large elements of the covariance matrix. When this shrinkage is employed, however, only a small number of assets are held in positive proportions (and the rest have weights of zero). This is, again, not an encouraging result for the hope of finding an efficient market portfolio by employing shrinkage techniques.

*in*efficient? While most studies suggest various variations of the return parameters relative to the sample parameters and check whether these variations lead to an efficient market proxy, we take a reverse approach: we first require that the return parameters ensure that the market proxy is efficient. Given this requirement, we look for parameters that are as "close" as possible to their sample counterparts. Surprisingly, parameters that make the market proxy efficient can be found very close to the sample parameters. Hence, minor changes in estimation error reverse previous negative and disappointing finding for the CAPM.

We hasten to add that the efficiency, or lack thereof, for a market proxy can never be a definitive test of the macro CAPM, which requires the market portfolio of *all* assets, including real-estate, human capital, etc. Nonetheless, it would be reassuring if typical proxies could be shown to be less <u>in</u>efficient than previously believed.

This paper is organized as follows. The next section introduces the methods employed. Section II describes the data and the results. Section III discusses implications for asset pricing. Section IV concludes.

I. METHODS

Given a market proxy, *m*, we look for the "minimal" variation of sample parameters that would make it mean/variance efficient. Denote the vector of market proxy portfolio weights by x_m, and denote the vector of sample average returns and the vector of sample standard deviations by μ^{sam} and σ^{sam}, respectively. C^{sam} denotes the sample covariance matrix, and ρ^{sam} denotes the sample correlation matrix.

The objectives being sought are an expected return vector μ and a covariance matrix *C* that on the one hand make portfolio *m* mean/variance efficient, and on the other hand are as close as possible to their sample counterparts. For simplicity, when considering the covariance matrix *C* we allow variation only in the standard deviations, while retaining the same sample correlations:

$$\begin{bmatrix} \\ & C & \\ \\ \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \begin{bmatrix} \\ & \rho^{sam} & \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \quad (1)$$

Allowing the correlations to vary as well introduces technical difficulties, but can only make the results stronger, as it allows more degrees of freedom in the optimization procedure described below.

In order to obtain the parameters (μ,σ) that are "closest" to their sample counterparts, (μ^{sam},σ^{sam}), we define the following distance measure *D* between any parameter set (μ,σ) and the sample parameter set:

$$D\left((\mu,\sigma),(\mu,\sigma)^{sam}\right) \equiv \sqrt{\alpha \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\mu_i - \mu_i^{sam}}{\sigma_i^{sam}}\right)^2 + (1-\alpha)\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\sigma_i - \sigma_i^{sam}}{\sigma_i^{sam}}\right)^2}\,, \quad (2)$$

where *N* is the number of assets, and $0 \le \alpha \le 1$ is a parameter determining the relative weight assigned to deviations of the means relative to deviations of the standard deviations. Recall that the larger the standard deviation of a given asset's returns, the larger the statistical errors involved in estimating this asset's parameters, and the larger the confidence intervals for these parameters. This is the rationale for dividing the deviations in (2) by σ_i^{sam} - the resulting distance measure "punishes" deviations in the parameters of assets with low standard deviations more heavily than similar deviations in assets with higher standard deviations. The ultimate test of whether a set of parameters (μ,σ) can be considered as "reasonably close" to the sample parameters is the proportion of parameters that deviate from the standard estimation error bounds around their sample counterparts, and the size of those deviations. Intuitively, a parameter set is "reasonably close" when 95% or more of the parameters are within the 95% confidence intervals of the sample parameters (Below we also employ the more formal Bonferroni [1935] multiple-comparison test). The choice of the distance measure D in eq.(2), and its minimization in the optimization problem described below, are designed to minimize the statistical significance of the deviations between μ and σ and their sample counterparts.

To find the set of parameters (μ,σ) that make the proxy m mean/variance efficient and are closest to the sample parameters, we solve the following optimization problem:

<u>Optimization Problem 1:</u>

Minimize $D\left((\mu,\sigma)\ ,(\mu,\sigma)^{sam}\right)$

Subject to:

$$\text{(i)}\quad \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \Bigg[\quad \rho^{sam} \quad \Bigg] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \begin{bmatrix} x_{m1} \\ x_{m2} \\ \vdots \\ \\ x_{mN} \end{bmatrix} = q \cdot \begin{bmatrix} \mu_1 - r_z \\ \mu_2 - r_z \\ \vdots \\ \\ \mu_N - r_z \end{bmatrix}$$

where q>0 is the constant of proportionality, and r_z is the zero-beta rate. Both q and r_z are free variables in the optimization. Thus, there are 2N+2 variables in the optimization: N μ's, N σ's, q and r_z. Any set of these 2N+2 parameters satisfying (i) makes the proxy portfolio mean/variance efficient (see, for example Roll [1977]). We are looking for the set of parameter vectors (μ^*, σ^*) that satisfy this mean/variance efficiency condition and are closest to the sample parameters[4].

Our approach is different from the approach employed in previous studies, such as Black, Jensen, and Scholes [1972] and Gibbons, Ross, and Shanken [1989], for example, in two main regards. First, we are not required to assume the existence of a risk-free asset. Second, and more importantly, the standard approach looks at the adjustment to the empirical average returns required to make the market proxy efficient (i.e. the stocks' alphas), and asks whether these adjustments are statistically plausible. In contrast, we are looking at *simultaneous* adjustments to the average returns *and the standard deviations* (and could, in principle, include adjustments to the correlations as well). Thus, while the standard approach examines the statistical plausibility of a single vector of alphas, we examine a multitude of vectors of average return and standard deviation adjustments. This allows us many more degrees of freedom relative to the standard approach, and explains why we find that only small adjustments are required to make the market proxy efficient.

In some situation one may have beliefs about the proxy portfolio's ex-ante mean and standard deviation, and would like to find the set of parameters that are closest to the sample parameters, and at the same time ensure that the proxy portfolio is mean/variance efficient *with the pre-specified mean and standard deviation*. Denoting the pre-specified mean and standard deviation by μ_0 and σ_0, respectively, the optimization problem solved in this case is:

Optimization Problem 2:

Minimize $D\left((\mu,\sigma)\ ,(\mu,\sigma)^{sam}\right)$

[4] This optimization problem is similar in spirit to Sharpe's [2007] "reverse optimization" problem. Levy [2007] employs an analogous technique to find mean/variance efficient portfolios that have all-positive weights. This approach was first used in a very innovative paper by Best and Grauer [1985].

Subject to:

$$\text{(i)} \quad \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \rho^{sam} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} x_m = q \cdot \begin{bmatrix} \mu_1 - r_z \\ \mu_2 - r_z \\ \vdots \\ \\ \mu_N - r_z \end{bmatrix}$$

$$\text{(ii)} \quad x'_m \, \mu = \mu_0$$

$$\text{(iii)} \quad x_m' \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} \rho^{sam} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & & \\ & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix} x_m = \sigma_0^2,$$

where, again, x_m is the vector of a given proxy's portfolio weights.

The next section presents solutions to these optimization problems with empirical equity data in order to ascertain how large the deviations from the sample parameters must be in order to ensure mean/variance efficiency.

II. DATA AND RESULTS

Our demonstration sample consists of the 100 largest stocks in the U.S. market (according to December 2006 market capitalizations), which have a complete monthly return records over the period January 1997 - December 2006 (120 return observations). Columns (2) and (4) in Table I report the sample average returns and standard deviations for 30 of these stocks (the complete information for all 100 stocks is given in Table AI in the Appendix). The average sample correlation is 0.24.

Following previous research (e.g., Stambaugh [1982]), we examine a market proxy whose weights are market capitalizations, in this case of the 100 stocks as of December 2006,

$$x_{mi} = \frac{\textit{market cap of firm } i}{\sum_{j=1}^{100} \textit{market cap of firm } j}.$$

The proxy portfolio and the sample mean/variance frontier are shown in Figure 1 by the triangle and thin line, respectively. As the figure illustrates, the proxy portfolio is far from the efficient frontier when the sample parameters are employed. This is consistent with previous studies.

To solve Optimization Problem 1 numerically, we implement Matlab's *fmincon* function, which is based on the interior-reflective Newton method and the sequential quadratic programming method.

The solution $\left(\mu^*, \sigma^*\right)$ is given in Columns (3) and (5) of Table I. The t-values for the expected returns μ^* are given in Column (6). These t-values reveal that the difference between the sample average return, μ_i^{sam}, and μ_i^* is non-significant at the 95% level for all stocks (this is true not only for the 30 stocks shown in the table, but for the other 70 stocks as well). Column (7) provides the ratio $\left(\sigma_i^*\right)^2 / \left(\sigma_i^{sam}\right)^2$ for each stock. The 95% confidence interval for this ratio is the range [0.790-1.319]. [5]

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[5] The ratio $\frac{(n-1)s^2}{\sigma^2}$ is distributed according to the χ^2_{n-1} distribution, where σ^2 is the population variance, s^2 is the sample variance (or $\left(\sigma^{sam}\right)^2$ in the notation used in this paper), and *n* is the number of observations. We have 120 monthly return observations, hence *n*=120. As we are looking for the 95% confidence interval for s^2/σ^2, we need to find the critical values c_1 and c_2 for which $P\left(\chi^2_{119} > c_1\right) = 0.025$, and $P\left(\chi^2_{119} < c_2\right) = 0.025$. For large *n*, $\sqrt{2\chi^2_n} - \sqrt{2n-1}$ can be approximated by the standard normal distribution. Thus, the critical values c_1 and c_2 satisfy $\sqrt{2c_1} - \sqrt{2 \cdot 119 - 1} = 1.96$ and $\sqrt{2c_2} - \sqrt{2 \cdot 119 - 1} = -1.96$, which yield: $c_1 = 150.6$ and $c_2 = 90.2$. Thus, the 95% confidence interval for

The values in Column (7) reveal that for all stocks the ratio $\left(\sigma_i^*\right)^2 \big/ \left(\sigma_i^{sam}\right)^2$ is well within this range (and this is also true for the 70 stocks not shown in the table). Thus, the solution $\left(\mu^*, \sigma^*\right)$ to the optimization problem is very close to the sample parameter set in the sense that none of the parameters is significantly different from its sample counterpart.

More formally, as we have 2N=200 parameters, we are simultaneously testing 200 hypotheses (each stating that the given parameter is not different than its sample counterpart at the 5% significance level). The Bonferroni [1935] test states that we should reject the multiple-comparison hypothesis at the 5% level if *any one* of the parameters is significantly different than its sample counterpart at the (5/200)% level (see also Miller [1991]). As none of our parameters is significantly different at the 5% level, of course none is significant at the much lower (5/200)% level, and we cannot reject the multiple comparison hypothesis.

(Please insert Table I and Figure 1 about here)

To confirm that the parameters $\left(\mu^*, \sigma^*\right)$ make the proxy portfolio mean/variance efficient, one can examine the efficient frontier and the location of the proxy portfolio in the mean-standard-deviation plane with these parameters. These are illustrated by the bold line and the star in Figure 1. The figure shows that with the parameters $\left(\mu^*, \sigma^*\right)$ the proxy portfolio lies on the efficient frontier. It is interesting to note that while the modified parameters $\left(\mu^*, \sigma^*\right)$ do not have a big impact on the expected return or the standard deviation of the proxy portfolio (the star is located very close to the triangle), they do have a big effect on the shape of the frontier. Why is the modified frontier much flatter than the sample frontier?

$s^2 \big/ \sigma^2$ is given by $90.2 < 119 \cdot s^2 \big/ \sigma^2 < 150.6$ or: $0.758 < s^2 \big/ \sigma^2 < 1.266$. Alternatively, this range can be also stated as $0.790 < \sigma^2 \big/ s^2 < 1.319$.

The explanation can be found in Figure 2, which shows the adjustment to the expected return, $\mu_i^* - \mu_i^{sam}$, as a function of the sample average return, μ_i^{sam}. The figure reveals that high sample returns tend to get negative corrections $\left(\mu_i^* < \mu_i^{sam}\right)$, while the opposite holds for low sample returns. Thus, the cross sectional variation of μ_i^* is smaller than the cross sectional variation of μ_i^{sam}, which explains why the frontier is flatter (recall that in the limiting case where all expected returns are identical, the frontier becomes completely flat – it is a horizontal line). Figure 2 shows that the corrections to the sample means implied by the optimization are reminiscent of standard statistical shrinkage methods. However, unlike the standard shrinkage methods, the method employed here ensures that the proxy is mean/variance efficient.[6]

(Please insert Figure 2 about here)

There is excellent intuition behind such a result when one recalls two facts (a) the efficient frontier itself is the result of an optimization; it gives the minimum variance for each level of mean return, and (b) sample parameter estimates are equal to true population parameters plus estimation errors. An efficient frontier computed using sample estimates optimizes with respect to sampling errors in addition to true parameters, so assets with over-estimated means are likely to be weighted too heavily in frontier portfolios and vice versa for assets with under-estimated means. This suggests that an efficient frontier computed using population parameters, if they were only known, would fall well inside the frontier computed using sample estimates, at

[6] One may wonder whether the adjustment $\mu^* - \mu^{sam}$ is similar for stocks that are relatively highly correlated with one another. In order to check this, we calculate the sample return correlation for each pair of stocks (i,j), and examine the relation across pairs between this sample correlation and the difference between the adjustments of the two stocks, i.e. $\left(\mu_i^* - \mu_i^{sam}\right) - \left(\mu_j^* - \mu_j^{sam}\right)$. We find no such relation (R^2=0.009), i.e. pairs that are more highly correlated are not more likely to have similar adjustments.

least at most points. The main exception would be near the global minimum variance portfolio, whose weights do not depend on mean returns; indeed, such a relation is exactly what we see depicted in Figure 1.

The implication of these results is quite striking. In contrast to "common wisdom", they show that the empirical proxy portfolio parameters are perfectly consistent with the CAPM if one allows for only slight estimation errors in the return moments. The reason that most previous studies have found that the market proxy is *in*efficient, even when various standard shrinkage methods have been employed, is that the variation of the parameters necessary to make the proxy portfolio efficient is very specific. While this variation is in the spirit of shrinkage, it is specifically designed to ensure the efficiency of the proxy portfolio, and thus it is fundamentally different than the standard statistical shrinkage methods.

With the solution $\left(\mu^*,\sigma^*\right)$ to Optimization Problem 1 the proxy portfolio has a monthly expected return of 1.4% and a standard deviation of 4.6% (see Figure 1), which are very close to its sample values, 1.5% and 4.6%. These values were produced by the optimization (given the proxy portfolio weights). In some situations one may have beliefs about the proxy portfolio's ex-ante return parameters, and may wish to look for solutions that are consistent with these beliefs. For example, suppose one would like to find vectors μ and σ such that the proxy portfolio is efficient and has an expected return and a standard deviation of $\mu_0 = 2\%$ and $\sigma_0 = 4\%$, respectively. Are such index values compatible (in a statistical sense) with the sample parameters and with a mean/variance efficient index? To answer this question, Optimization Problem 2 can be solved with $\mu_0 = 2\%$ and $\sigma_0 = 4\%$. We will consider the solution $\left(\mu^*,\sigma^*\right)$ compatible with the sample parameters if 95% or more of the parameters are within the 95% confidence intervals of their sample counterparts.

Of course, $\mu_0 = 2\%$ and $\sigma_0 = 4\%$ is just one example. A more complete picture would scan the mean/variance plane and map the range of proxy portfolios' return parameters, μ_0 and σ_0, that are compatible with the CAPM and the sample returns and market proxy weights.

Figure 3 shows the results of this analysis. For each combination of pre-specified proxy portfolio parameters (μ_0, σ_0) we solve Optimization Problem 2. The points scanned are shown by the circles in the mean/variance plain. If the resulting optimal parameter set (μ^*, σ^*) is found to be statistically compatible with the sample parameters, $(\mu^{sam}, \sigma^{sam})$, and with the CAPM (mean/variance efficiency of the index), the point is marked as a filled circle; if not, it is left transparent. For example, the point $(\mu_0 = 2\%,\ \sigma_0 = 4\%)$, (indicated by a down arrow in Figure 3), is indeed consistent with the sample parameters and the proxy being efficient. In contrast, the point to the left, $(\mu_0 = 2\%, \sigma_0 = 3.95\%)$, is not consistent. The figure shows that the range of possible proxy portfolio return parameters that can be made simultaneously consistent with the CAPM and the sample parameters is in fact quite large. The proxy portfolio expected return can be as small as 0.5% or as large as 2%, and the standard deviation can be as small as 4% or as large as 5%. These are percent per month.

(Please insert Figure 3 about here)

It would be interesting to redo this analysis using indexes with even more individual assets, but there are technical difficulties. When the number of assets exceeds the number of time series observations, the correlation matrix is singular, which produces some instabilities in the optimization problem. We can, however, partially investigate this issue by varying the number of assets for N<100 and looking for any trend in the range of proxy portfolio return parameters consistent with the CAPM.

For example, repeating the analysis for the 50 largest stocks (instead of the 100) yields the results shown in Figure 4. These results are comparable to those obtained with 100 stocks in the range of proxy portfolio return parameters consistent with the CAPM is roughly similar. To investigate in more detail possible systematic effects of the number of assets, we repeat this analysis for N=10, 20, …, 100 stocks. For each value of N we measure the area of admissible proxy portfolio parameters (estimated by the polygon containing the admissible points, see, for example the polygon in Figure 4). The results are shown in Figure 5. Although the area is an approximation of the precise area of admissible points, because of the discreteness of the points, (and as indicated by the error bars in Figure 5), Figure 5 shows that the area does not seem to change systematically with the number of assets. Thus, the results seem robust to the identity of stocks and to the number of stocks contained in any market index proxy.

(Please insert Figures 4 and 5 about here)

III. IMPLICATIONS FOR ASSET PRICING AND PRACTICAL USE OF THE CAPM

The Security Market Line (SML) formula is probably the most widespread method for estimating the cost of capital and for pricing risky assets. Using beta and the SML formula for estimating the expected return, rather than employing the sample average return directly, is usually justified on the basis that the statistical estimation of beta is more stable than that of the average return. However, when there are questions about how well the SML relationship holds empirically, there are serious doubts about employing betas for pricing.[7] While we cannot prove that the SML relationship holds empirically with the ex-ante parameters, our analysis does provide another reason for employing betas for estimating the cost of capital.

Suppose that the CAPM holds with the true ex-ante parameters (μ^*, σ^*), and that the empirically measured parameters are $(\mu^{sam}, \sigma^{sam})$. The true and sample betas of stock *i* are given respectively by:

$$\beta_i^* = \frac{\sum_{j=1}^{N} x_{mj}\sigma_i^*\sigma_j^*\rho_{ij}}{x_m' C x_m} \quad (3a) \qquad \beta_i^{sam} = \frac{\sum_{j=1}^{N} x_{mj}\sigma_i^{sam}\sigma_j^{sam}\rho_{ij}}{x_m' C^{sam} x_m}, \quad (3b)$$

where x_m denotes the market portfolio weights. The true cost of equity of firm *i* is μ_i^*. If one employs the observable β_i^{sam} in the SML formula instead of the correct β_i^*, how accurate will be the resulting cost of capital estimate? In other words, how close are β_i^{sam} and β_i^*? The answer is shown in Figure 6, where the parameter set (μ^*, σ^*) employed is the solution to Optimization Problem 1. The figure reveals that the difference between β_i^{sam} and β_i^* is very small. The reason is that both the denominators and the numerators of (3a) and (3b) are very similar. The variance of the market proxy is quite close whether the optimized parameters or the sample parameters are employed (compare the horizontal location of star and the triangle in Figures 3 and 4). As for the covariances in the numerator, note that $\sigma_j^* \approx \sigma_j^{sam}$, and in addition, the deviations tend to cancel each other out in the summation, as in some cases $\sigma_j^* > \sigma_j^{sam}$, while in others $\sigma_j^* < \sigma_j^{sam}$ (see Column 7 in Table I).[8]

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[7] This is, of course, one of the major debates in finance. See, for example, Reinganum [1981], Levy [1981], Lakonishok and Shapiro [1986], Chen, Roll, and Ross [1986], Fama and French [1992], and Roll and Ross [1994].

[8] Figure 6 shows the relation between the β_i^{sam}'s and the β_i^*'s when we use a value of $\alpha = 0.75$ in the distance measure D (see eq.(2)). When a higher value of α is employed, the μ_i^*'s are closer to their

Since the market proxy is efficient with the true parameters $\left(\mu^*, \sigma^*\right)$, the following relationship holds exactly:

$$\mu_i^* = r_z + \beta_i^*(\mu_m - r_z), \tag{4}$$

where r_z is the expected return on the zero-beta portfolio for index m. Common practice substitutes a "riskless" rate, r_f, for r_z, but this is appropriate only when f and z have the same mean return. Since $\beta_i^{sam} \approx \beta_i^*$, employing the SML with the sample beta, as is commonly done in practice, provides an excellent estimate for the true expected return (assuming $r_f = r_z$):

$$\mu_i^* - \left[r_f + \beta_i^{sam}(\mu_m^{sam} - r_f)\right] = \beta_i^*(\mu_m^* - r_f) - \beta_i^{sam}(\mu_m^{sam} - r_f) \approx 0. \tag{5}$$

The above argument is based on taking the true ex-ante parameters as the $\left(\mu^*, \sigma^*\right)$ vectors solving Optimization Problem 1, i.e. the parameters ensuring the CAPM that are closest to the sample parameters. What if, instead, we take another set of parameters that ensures the efficiency of the proxy and is consistent with the sample parameters? For example, suppose that we take as the true parameters those that solve Optimization Problem 2 with $\mu_0 = 2\%$ and $\sigma_0 = 4.25\%$ (see point A in Figure 3). It turns out that with these parameters the β_i^*'s and the β_i^{sam}'s are still very close – see Panel A in Figure 7. This is also true for other points with very different proxy portfolio expected returns and standard deviations – see Panels B,C, and D in Figure 7, corresponding to the points B,C, and D in Figure 3.

This is a strong result: if the CAPM holds in a way that is consistent with the sample parameters, the differences between sample betas and true betas are going to be small. Thus, if the SML formula for pricing, which implies that the CAPM holds

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sample counterparts, and the σ_i^*'s are more distant from their sample counterparts. As a result, the differences between the β_i^{sam}'s and the β_i^*'s also increase. Yet, even with a very high value of $\alpha = 0.97$ the β_i^*'s are still very close to the β_i^{sam}'s, with a correlation of 0.96.

with the ex-ante parameters, one can be confident about using the sample betas, and should not worry about estimation errors in the betas. This conclusion is reached because we are not just looking at the statistical estimation error of a single asset's beta in isolation, as is typically done, but rather at the error in beta *given that the CAPM holds in a way that is consistent with the sample parameters* $(\mu^{sam}, \sigma^{sam})$.

(Please insert Figures 6 and 7 about here)

From a practical perspective, since sample betas are quite close to betas that have been adjusted to render the market proxy mean/variance efficient, improved estimates of expected returns can be obtained from sample betas alone. Sample mean returns should be ignored! To illustrate, Figure 8, Panel A shows the cross-sectional relation between sample mean returns and sample betas for our 100 stocks while Figure 8, Panel B shows the analogous relation for adjusted means and betas. Clearly, the sample means in Panel A are not closely related at all to sample betas but the adjusted means in Panel B are perfectly related to adjusted betas.[9]

(Please insert Figure 8 about here)

Consequently, to obtain an improved expected return for any stock, first calculate the adjusted mean return for the market index proxy and for its corresponding zero-beta portfolio.[10] Plugging these numbers along with the sample beta (because it's close to the adjusted beta) into the usual CAPM formula delivers the improved estimate of expected return. Making the market index proxy mean/variance efficiency produces useful betas for many practical purposes such as estimation of the cost of equity capital for a firm or of the discount rate for a risky project.

[9] The slight deviations from linearity in Figure 8, Panel B are caused by rounding error.

[10] For most proxies, the sample means will be close to the adjusted means.

IV. CONCLUSION

Market proxy portfolios are typically very far from the sample efficient frontier. Many studies have tried various adjustments to the sample parameters to make the market proxy mean/variance efficient, without success. Thus, the "common wisdom" is that the empirical return parameters and market portfolio weights are incompatible with the CAPM theory.

In this paper we hope to change that perception. We show that small variations of the sample parameters, well within the range of estimation error, can make a typical market proxy mean/variance efficient. While such parameter variations are reminiscent of "shrinkage", they differ from those obtained with the standard statistical shrinkage methods: they are the result of "reverse optimization." In this reverse optimization, return parameters are derived to make the market proxy mean/variance efficient while being "close" to their sample counterparts.

The fact that we find many such parameter sets, together with the fact that many previous attempts to vary the return parameters in order to obtain an efficient proxy were unsuccessful, seem to indicate that such parameter sets may be very rare in parameter space – it is very unlikely to "stumble onto one of them" by coincidence. Yet, the reverse optimization problem delivers them simply and directly.

These findings suggest that the CAPM (i.e., ex ante mean/variance efficiency of the market index proxy) is consistent with the empirically observed return parameters and the market proxy portfolio weights. Of course, this does not constitute a proof of the empirical validity of the model, but it shows that the model can not be rejected, in contrast to the widespread belief in our profession. The intuitive idea that shrinkage corrections should increase the empirical validity of the CAPM is shown to be valid - with the right corrections, which are small, the index proxy is perfectly efficient. The analysis also shows that in this framework employing the sample betas provides an excellent estimate of the true expected returns.

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Table I

The Sample Parameters and Closest Parameters Ensuring that the Market Proxy is Mean/variance Efficient

For the sake of brevity, this table reports only 30 of the 100 stocks (the complete table is given in the appendix). The sample parameters are given in the second and fourth columns. The expected returns and standard deviations which are closest to these parameters and ensure that the market proxy is efficient (i.e. the parameters that solve Optimization Problem 1) are given in columns (3) and (5). The t-values for the expected returns are given in column (6), which shows that none of these values are significant at the 95% level (this is also true for the 70 other stocks not shown in the table). Column (7) reports the ratio between the optimized variances $\left(\sigma^*\right)^2$ and the sample variances. The 95% confidence interval for this ratio is [0.790-1.319] (see footnote 5). All of the ratios in the table, as well as the ratios for all other 70 stocks not shown here, fall well within this interval. These results are obtained with a value of $\alpha = 0.75$ in the minimized distance measure D (see eq.(2)). Higher values of α reduce the variation in the expected returns (at the expense of increasing the deviations in the standard deviations).

(1) Stock # (i)	(2) μ_i^{sam}	(3) μ_i^*	(4) σ_i^{sam}	(5) σ_i^*	(6) t-value for μ_i^*	(7) $\left(\sigma_i^*\right)^2 / \left(\sigma_i^{sam}\right)^2$ (the 95% confidence interval for this value is [0.790-1.319])
1	0.024	0.018	0.165	0.167	-0.423	1.019
2	0.021	0.019	0.115	0.115	-0.170	1.003
3	0.011	0.017	0.106	0.104	0.588	0.963
4	0.029	0.023	0.158	0.160	-0.444	1.028
5	0.039	0.022	0.150	0.156	-1.228	1.077
6	0.005	0.011	0.075	0.073	0.952	0.953
7	0.007	0.013	0.072	0.070	0.938	0.942
8	0.012	0.010	0.051	0.052	-0.433	1.028
9	0.013	0.015	0.070	0.069	0.286	0.978
10	0.016	0.018	0.099	0.098	0.185	0.986
11	0.010	0.013	0.067	0.066	0.344	0.977
12	0.016	0.009	0.092	0.093	-0.819	1.025
13	0.015	0.011	0.071	0.072	-0.627	1.035
14	0.019	0.012	0.100	0.102	-0.702	1.034
15	0.011	0.011	0.061	0.061	-0.029	1.006
16	0.032	0.014	0.159	0.162	-1.215	1.044
17	0.023	0.025	0.158	0.157	0.145	0.990
18	0.024	0.021	0.146	0.147	-0.232	1.016
19	0.011	0.012	0.086	0.085	0.199	0.988
20	0.007	0.010	0.067	0.066	0.477	0.979
21	0.011	0.011	0.065	0.065	0.082	0.996
22	0.018	0.016	0.080	0.081	-0.225	1.018
23	0.012	0.008	0.067	0.068	-0.652	1.023
24	0.013	0.004	0.059	0.059	-1.533	0.995
25	0.017	0.014	0.088	0.088	-0.361	1.021
26	0.014	0.013	0.081	0.082	-0.128	1.007

27	0.006	0.012	0.077	0.075	0.810	0.955
28	0.018	0.011	0.077	0.078	-1.058	1.044
29	0.010	0.012	0.087	0.086	0.276	0.989
30	0.010	0.010	0.065	0.064	0.055	0.999

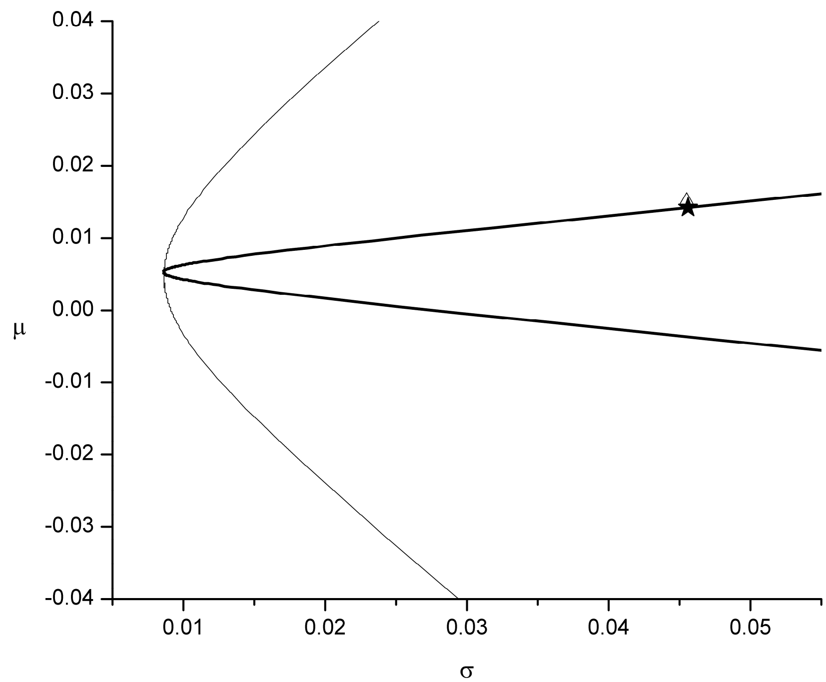

Figure 1: The Efficient Frontier and Market Proxy with the Sample and the Adjusted Return Parameters.

The thin line curve and the triangle (partly hidden behind the star) show the mean/variance frontier and the market proxy with the sample parameters. As typical of other studies, the market proxy is very far from the efficient frontier when the sample parameters are employed. The bold line and the star show the mean/variance frontier and the market proxy with the adjusted parameters (μ^*, σ^*). With these parameters the market proxy is mean/variance efficient.

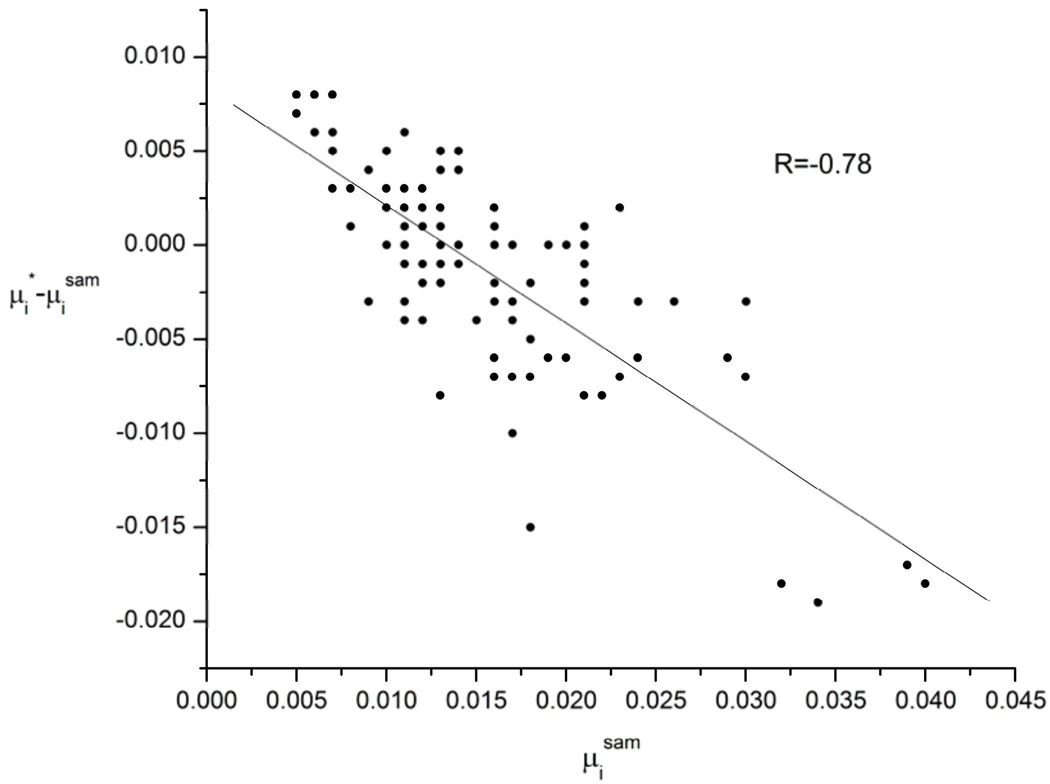

Figure 2: The Correction to the Expected Returns and a Function of the Sample Average Return.

For stocks with high sample average returns, the correction in the expected return tends to be negative. The opposite holds for stocks with low sample average returns. Thus, the corrections produced by the solution to the optimization problem are reminiscent of statistical shrinkage methods.

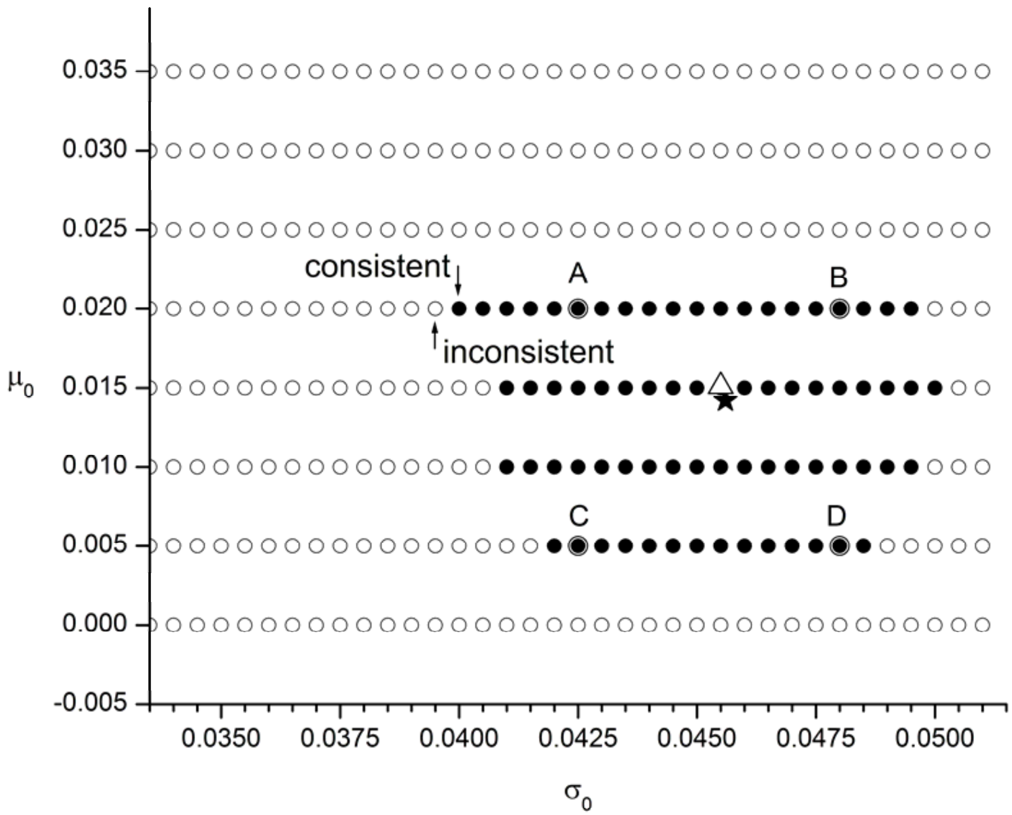

Figure 3: The Set of Proxy Portfolio Parameters Consistent with Mean/variance Efficiency and the Sample Parameters – 100 stocks.

Optimization Problem 2 is solved for each point on the mean-standard deviation plane, (μ_0, σ_0). The resulting parameter set, (μ^*, σ^*), is considered consistent with the sample parameters if 95% or more of the parameters are within the 95% confidence intervals of their sample counterparts. The (μ_0, σ_0) points that are consistent with the mean/variance efficiency of the proxy portfolio and with the sample parameters are indicated by the filled circles. For example, the proxy portfolio can be made mean/variance efficient with a standard deviation of 4% and a mean return of 2%, but not with a standard deviation of 3.95% and a mean return of 2%. The figure shows that given a set of sample parameters and proxy portfolio weights, the proxy portfolio can be made mean/variance efficient with a large range of possible mean and standard deviation combinations. As in Figure 1, the triangle and the star represent the market proxy with the sample parameters and with the parameters solving Optimization Problem 1, respectively.

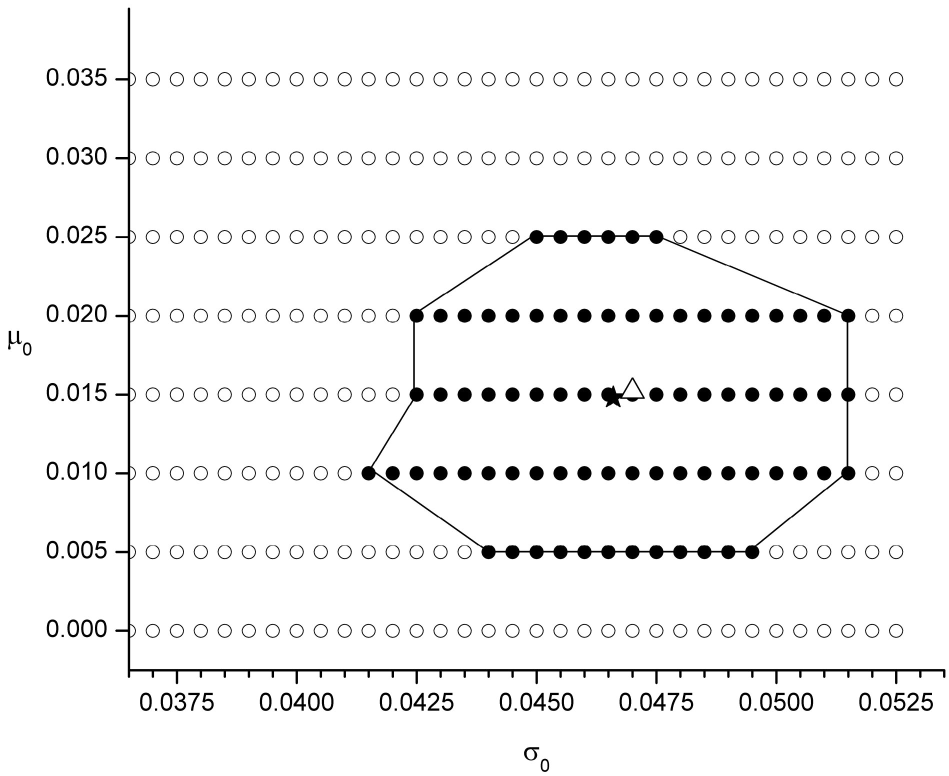

Figure 4: The Set of Proxy Portfolio Parameters Consistent with Mean/variance Efficiency and the Sample Parameters – 50 stocks.

This figure is the same as Figure 3, but it is constructed with only the 50 largest stocks (rather than 100). Again, a wide range of (μ_0, σ_0) is consistent with the efficiency of the market proxy. The area of a polygon drawn through the outer consistent points is an approximation to the range of consistency.

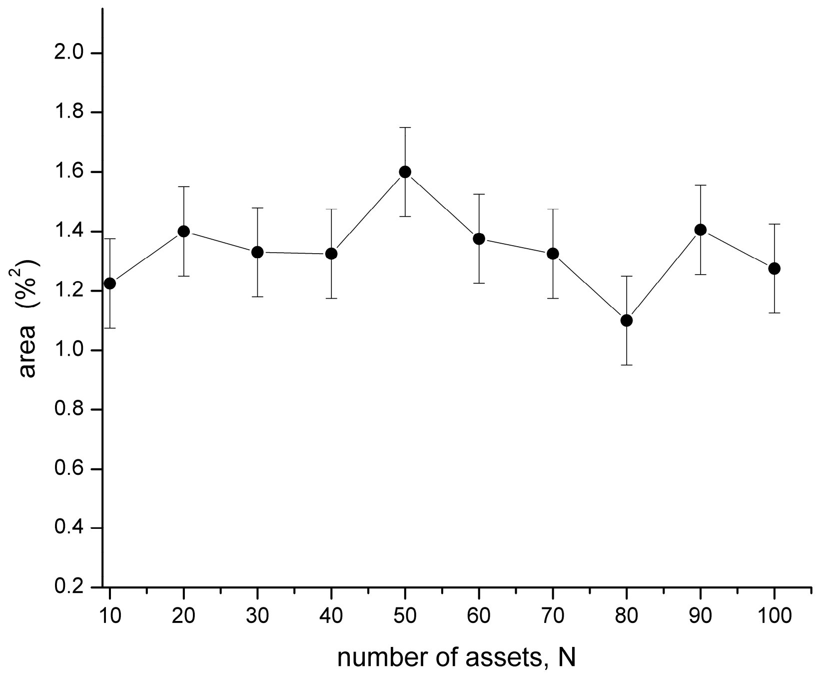

Figure 5: The Area of Admissible Proxy Portfolio Parameters as a Function of the Number of Assets.

For each value of N, starting with the largest ten stocks, the area of a consistency polygon is computed analogous to the one shown in Figure 4. This area measures the range of proxy portfolio return parameters consistent with the CAPM and the given proxy portfolio. This is an approximation of the precise area, because it depends on a finite set of parameter points in the MV plane. The error bars reflect this possible estimation error. The figure shows that the area of admissible parameters does not change systematically with the number or the identity of the stocks included in the market index proxy.

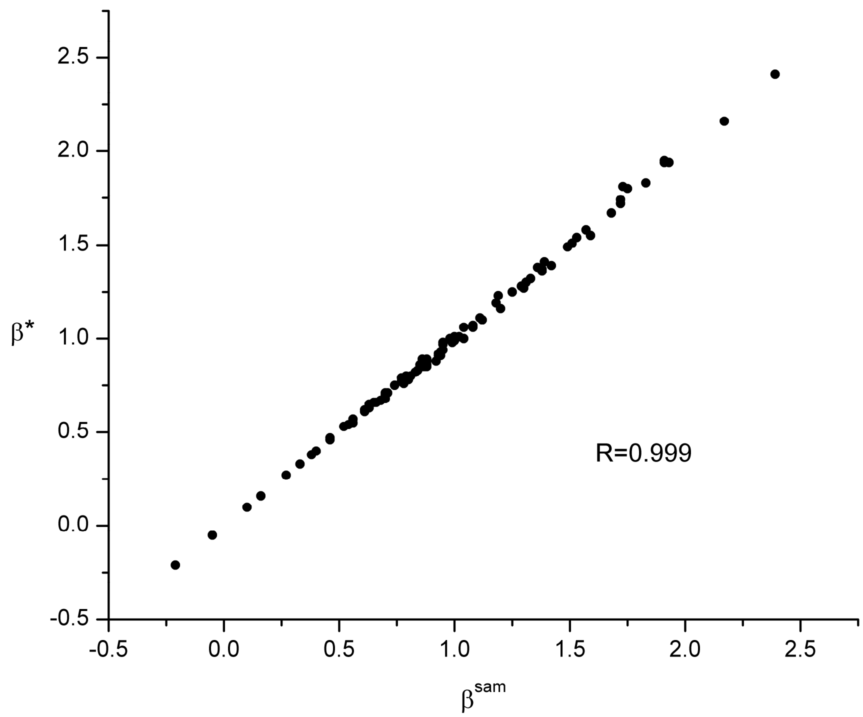

Figure 6: The Relation Between Sample Betas and the "True" Betas.
The "true" parameters are those that solve Optimization Problem 1 and satisfy the CAPM: $\left(\mu^*,\sigma^*\right)$. The sample parameters are $\left(\mu^{sam},\sigma^{sam}\right)$. The true and sample betas are given by eq.(3). The figure shows that the sample betas are very close to the true betas, and thus yield excellent estimates of the expected returns.

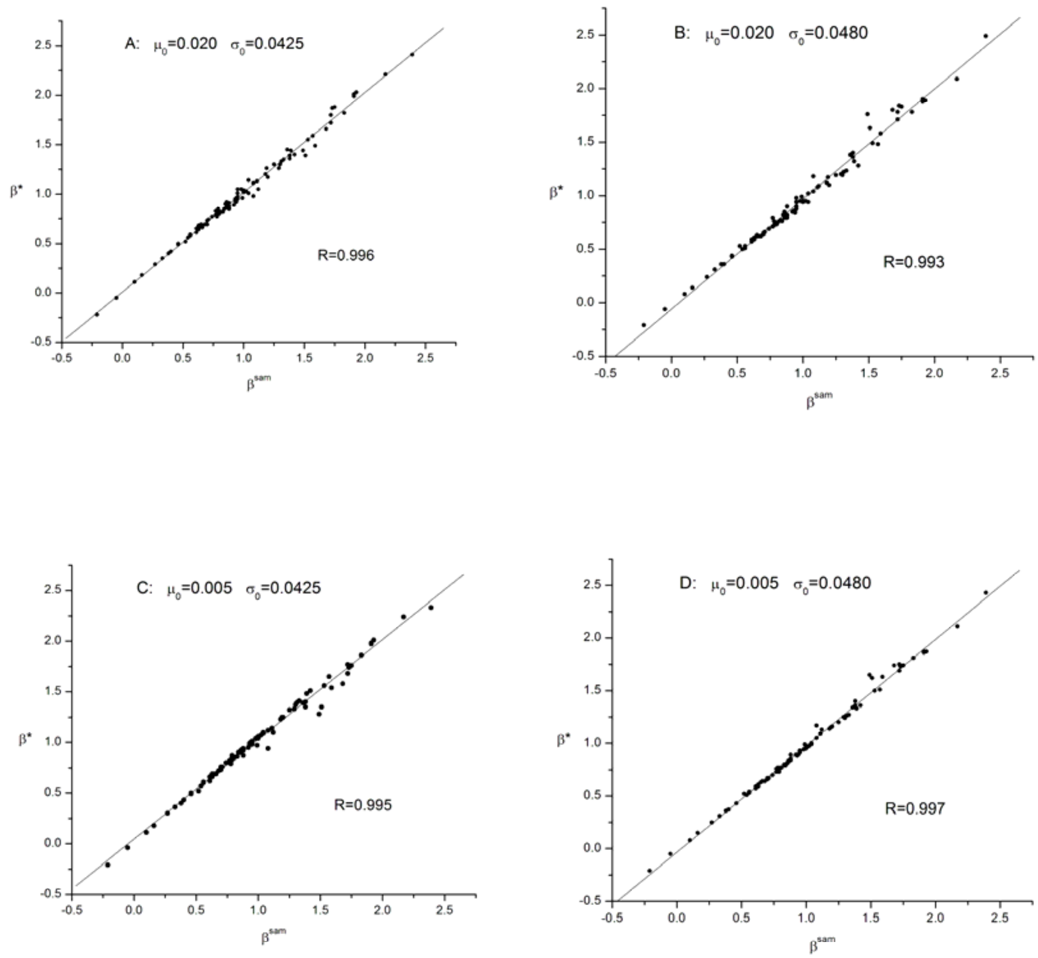

Figure 7: The Relation Between Sample Betas and "True" Betas.

The "true" parameters are those that satisfy the CAPM and solve Optimization Problem 2. Each panel corresponds to a different combination of values of the pre-specified expected return and standard deviation of the proxy portfolio, μ_0 and σ_0. (The points corresponding to these four panels are indicated by A,B,C, and D, respectively, in Figure 3). The true and sample betas are given by eq.(3). The figure shows that the sample betas are very close to the true betas, and thus yield excellent estimates of the true expected returns, even when μ_0 and σ_0 are not close to the values obtained with the sample parameters.

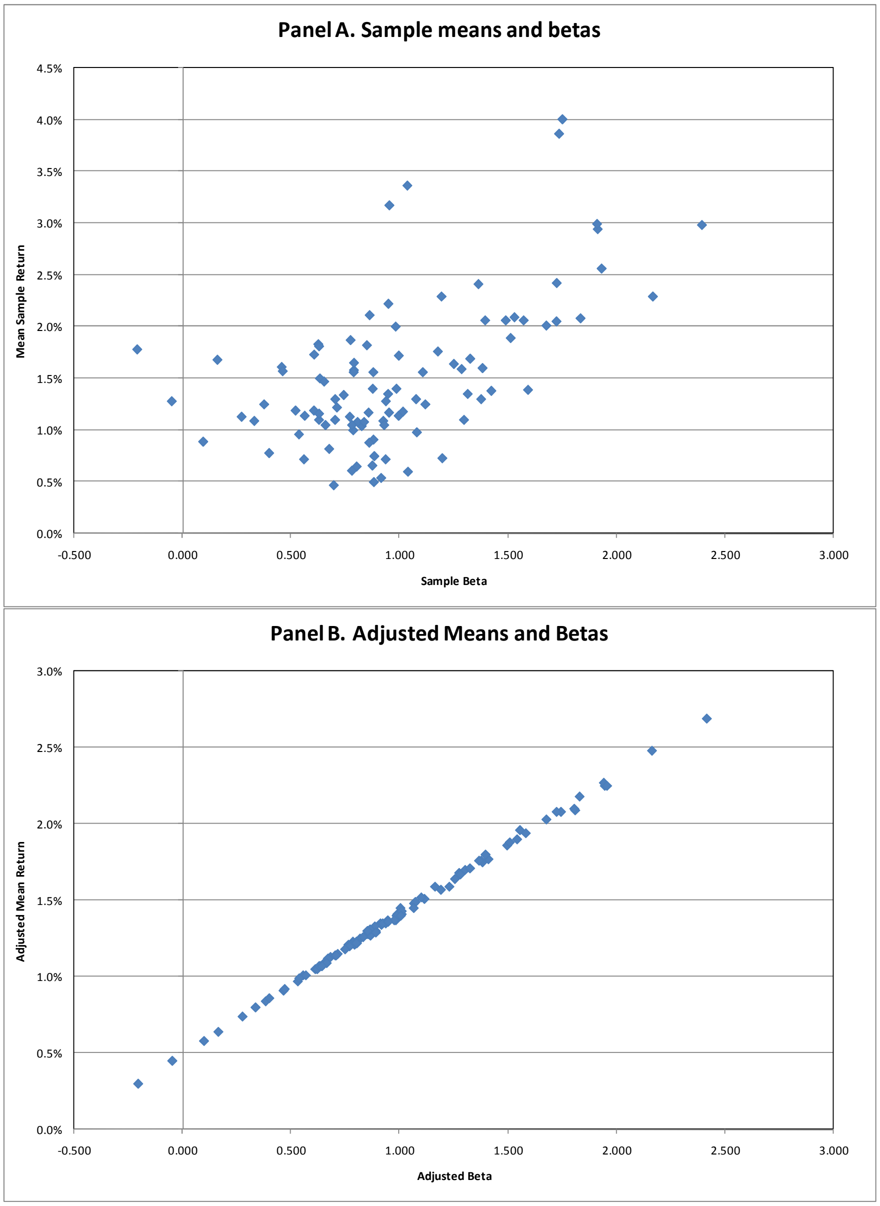

Figure 8: The Securities Market Line Scatter for Sample vs. Adjusted Means and Betas

Sample estimates of means and betas for our 100 stocks are plotted against each other in Panel A. Panel B plots the corresponding adjusted means and betas that are obtained from optimization problem 1.

Appendix- Table AI

The Sample Parameters and Closest Parameters Ensuring that the Market Proxy is Mean/variance Efficient

This is the complete version of Table I given in the text, where here the data is provided for all 100 stocks. The sample parameters are given in the second and fourth columns. The expected returns and standard deviations which are closest to these parameters and ensure that the market proxy is efficient (i.e. the parameters that solve Optimization Problem 1) are given in columns (3) and (5). The t-values for the expected returns are given in column (6), which shows that none of these values are significant at the 95% level. Column (7) reports the ratio between the variances $(\sigma^*)^2$ and the sample variances. The 95% confidence interval for this ratio is [0.790-1.319] (see footnote 5). All of the ratios in the table fall well within this interval.

(1) Stock # (i)	(2) μ_i^{sam}	(3) μ_i^*	(4) σ_i^{sam}	(5) σ_i^*	(6) t-value for μ_i^*	(7) $(\sigma_i^*)^2 / (\sigma_i^{sam})^2$ (the 95% confidence interval for this value is [0.790-1.319])
1	0.024	0.018	0.165	0.167	-0.423	1.019
2	0.021	0.019	0.115	0.115	-0.170	1.003
3	0.011	0.017	0.106	0.104	0.588	0.963
4	0.029	0.023	0.158	0.160	-0.444	1.028
5	0.039	0.022	0.150	0.156	-1.228	1.077
6	0.005	0.011	0.075	0.073	0.952	0.953
7	0.007	0.013	0.072	0.070	0.938	0.942
8	0.012	0.010	0.051	0.052	-0.433	1.028
9	0.013	0.015	0.070	0.069	0.286	0.978
10	0.016	0.018	0.099	0.098	0.185	0.986
11	0.010	0.013	0.067	0.066	0.344	0.977
12	0.016	0.009	0.092	0.093	-0.819	1.025
13	0.015	0.011	0.071	0.072	-0.627	1.035
14	0.019	0.012	0.100	0.102	-0.702	1.034
15	0.011	0.011	0.061	0.061	-0.029	1.006
16	0.032	0.014	0.159	0.162	-1.215	1.044
17	0.023	0.025	0.158	0.157	0.145	0.990
18	0.024	0.021	0.146	0.147	-0.232	1.016
19	0.011	0.012	0.086	0.085	0.199	0.988
20	0.007	0.010	0.067	0.066	0.477	0.979
21	0.011	0.011	0.065	0.065	0.082	0.996
22	0.018	0.016	0.080	0.081	-0.225	1.018
23	0.012	0.008	0.067	0.068	-0.652	1.023
24	0.013	0.004	0.059	0.059	-1.533	0.995
25	0.017	0.014	0.088	0.088	-0.361	1.021
26	0.014	0.013	0.081	0.082	-0.128	1.007
27	0.006	0.012	0.077	0.075	0.810	0.955
28	0.018	0.011	0.077	0.078	-1.058	1.044
29	0.010	0.012	0.087	0.086	0.276	0.989
30	0.010	0.010	0.065	0.064	0.055	0.999
31	0.012	0.013	0.086	0.085	0.147	0.991
32	0.009	0.006	0.082	0.082	-0.406	1.004
33	0.016	0.009	0.082	0.083	-0.862	1.026
34	0.017	0.006	0.077	0.078	-1.461	1.018

35	0.011	0.012	0.072	0.072	0.243	0.984
36	0.009	0.013	0.064	0.062	0.658	0.954
37	0.012	0.011	0.064	0.064	-0.228	1.012
38	0.026	0.023	0.203	0.204	-0.142	1.006
39	0.011	0.010	0.065	0.065	-0.195	1.009
40	0.006	0.012	0.087	0.085	0.749	0.960
41	0.010	0.015	0.115	0.114	0.480	0.978
42	0.016	0.017	0.119	0.119	0.011	1.001
43	0.018	0.003	0.100	0.099	-1.615	0.986
44	0.013	0.017	0.105	0.104	0.364	0.976
45	0.009	0.013	0.088	0.087	0.499	0.974
46	0.006	0.014	0.085	0.082	1.067	0.932
47	0.013	0.018	0.124	0.122	0.409	0.978
48	0.011	0.011	0.084	0.083	0.057	0.997
49	0.008	0.009	0.077	0.077	0.112	0.998
50	0.017	0.011	0.082	0.084	-0.884	1.036
51	0.012	0.014	0.081	0.081	0.265	0.984
52	0.021	0.018	0.105	0.106	-0.277	1.019
53	0.016	0.012	0.072	0.073	-0.517	1.030
54	0.011	0.014	0.106	0.105	0.281	0.984
55	0.011	0.012	0.074	0.074	0.118	0.993
56	0.007	0.013	0.076	0.074	0.889	0.945
57	0.011	0.013	0.072	0.071	0.379	0.975
58	0.014	0.019	0.102	0.100	0.581	0.952
59	0.023	0.016	0.089	0.091	-0.807	1.056
60	0.014	0.018	0.090	0.088	0.489	0.961
61	0.012	0.012	0.070	0.070	-0.095	1.005
62	0.012	0.011	0.093	0.093	-0.095	1.000
63	0.008	0.011	0.075	0.074	0.436	0.979
64	0.021	0.019	0.106	0.107	-0.172	1.012
65	0.016	0.013	0.077	0.078	-0.336	1.018
66	0.013	0.014	0.074	0.074	0.110	0.993
67	0.016	0.017	0.076	0.075	0.130	0.988
68	0.011	0.008	0.052	0.052	-0.610	1.020
69	0.020	0.020	0.134	0.133	0.029	0.994
70	0.014	0.014	0.076	0.076	0.009	0.997
71	0.010	0.013	0.094	0.094	0.346	0.983
72	0.015	0.011	0.070	0.071	-0.560	1.028
73	0.018	0.013	0.088	0.089	-0.658	1.033
74	0.022	0.014	0.096	0.098	-0.934	1.049
75	0.011	0.007	0.059	0.059	-0.705	1.018
76	0.005	0.013	0.083	0.081	1.013	0.937
77	0.007	0.013	0.083	0.081	0.718	0.957
78	0.005	0.013	0.083	0.081	1.032	0.938
79	0.013	0.014	0.086	0.086	0.028	0.997
80	0.016	0.015	0.090	0.090	-0.046	1.006
81	0.012	0.015	0.074	0.072	0.392	0.964
82	0.011	0.013	0.070	0.069	0.290	0.983
83	0.021	0.022	0.117	0.116	0.099	0.992
84	0.019	0.019	0.089	0.088	-0.004	0.993
85	0.018	0.011	0.098	0.100	-0.800	1.029
86	0.013	0.012	0.073	0.073	-0.228	1.012
87	0.021	0.021	0.130	0.130	0.031	0.996
88	0.007	0.016	0.095	0.092	0.968	0.939
89	0.021	0.020	0.100	0.100	-0.109	1.009
90	0.040	0.022	0.193	0.199	-1.035	1.052
91	0.034	0.015	0.161	0.164	-1.274	1.046
92	0.030	0.027	0.170	0.171	-0.163	1.014

93	0.012	0.014	0.086	0.086	0.310	0.982
94	0.013	0.011	0.080	0.080	-0.204	1.009
95	0.030	0.023	0.130	0.133	-0.579	1.045
96	0.016	0.012	0.147	0.147	-0.245	1.009
97	0.017	0.012	0.087	0.088	-0.523	1.024
98	0.017	0.017	0.102	0.102	0.035	0.997
99	0.020	0.014	0.089	0.090	-0.704	1.041
100	0.021	0.013	0.087	0.089	-0.997	1.057