Standardizing Variables in Multiplicative Choice Models

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To use multiplicative competitive interaction (MCI) models as part of a theory of the evaluative process in choice, we need a method to transform interval scale consumer judgments into positive, ratio scales. We develop a coefficient—zeta-squared—that possesses the needed scale requirements and other theoretically desirable properties, and report four research studies to demonstrate the diversity of applications of multiplicative choice models using zeta-squared. We also discuss the relations of MCI models to Luce choice models to illustrate the potential of zeta-squared for representing the effects of similarity on choice, and consider some of the benefits of standardizing variables in MCI models or multinomial logit models.

We are building a theory of the evaluative process in choice. Our goal is to predict marketplace activity (market shares) from the internal states (beliefs and values) of consumers. We believe this goal is achievable if we can represent the richness of systematic individual differences in the evaluation of market alternatives and show how psychological measures from individuals or homogeneous groups can be used in models of market share or choice probabilities. Of these two interrelated representational problems, we focus first on representing psychological measures in market share models.

In our theory, the core model for the compensatory process in the evaluation of alternatives is an attraction model we call a multiplicative competitive interaction (MCI) model (Nakanishi and Cooper 1974). Such a model requires special ratio scale properties that are not immediately available in the interval scale ratings typically collectible from consumers. Measures of travel time to retail centers and display space in those centers can easily be used in an MCI model to represent the attractiveness of the centers to consumers in different neighborhoods. But consumers' judgments of brand or store attributes have arbitrary origins and units of measures. Whether the rating scale goes from 1 to 9 or from -10 to 10 makes no difference. The properties of the scale dictate what operations can be performed on the numbers without distorting whatever meaning they originally possessed. For interval scale ratings, we can assume that the differences in scale units are constant throughout the range and thus that the ratios of differences are meaningful. However, a multiplicative model requires values which themselves form meaningful ratios. The meaning of the original measures will be distorted unless a legitimate transformation is employed to imbue interval scale ratings with ratio scale properties.

While the heart of the problem is epistemological (how can one maintain the meaning inherent in the original measures?), the necessity of solving the problem is easy to illustrate without dealing in abstractions. The estimation of parameters of MCI models and of many other multiplicative models requires taking logs, and one cannot take the log of a negative number. The meaning of interval scale judgments should be unchanged by a simple linear transformation of the original scale of measurement; but even if estimation procedures were available, each different linear transformation would have a different meaning in an MCI model.

This article compares the option of exponentially transforming the interval scale—which is equivalent to using a multinomial logit model—to other options. The primary objective is to develop the zeta-squared transformation as an alternative; several other alternatives result as well.

We first introduce MCI models as representations of the compensatory process in the evaluation of alternatives. Next we develop the components of the zeta-squared transformation from their foundation in physics and show how

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they represent the comparative information in a set of measures. Through an empirical comparison with Murdock's (1960) index of distinctiveness, we then demonstrate that the information an observer perceives in an object is preserved by the components of the transformation, and develop the final transformation, showing its applicability to interval scale ratings as well as to binary measures. We apply zeta-squared to the study of choice among political candidates, the relation of fashion perception to fashion choice, the study of retail patronage behavior, and as part of the cognitive algebra used to forecast test-market performance from consumers' reactions to new product offerings. We show how representing comparative contextual information makes MCI models depart from Luce (1959) choice models, and present some potential uses of zeta-squared for representing the effects on choice of changes in the composition of competitive choice sets. Finally, we discuss the benefits of using standardized variables in multiplicative choice models for the simultaneous analysis of several choice situations.

MULTIPLICATIVE COMPETITIVE INTERACTION MODELS

MCI models have certain advantages over general linear models when it comes to predicting market shares, or choice probabilities. MCI models are logically consistent, that is, the estimates of choice probabilities which they produce are always between zero and one and always sum to one over all choice alternatives. General linear models can produce estimates of probabilities which are less than zero or greater than one, and which do not sum to one over all alternatives. The logical consistency of MCI models flows from their basic structure rather than from a constrained optimization of some sort. MCI models can be transformed so that they are just about as easy to estimate as are general linear models (Nakanishi and Cooper 1982). But transformations to simplify estimation do not eliminate the need to transform interval scale judgments into ratio scales.

Our desire to use MCI models in a consumer context comes from six basic assertions we make about the compensatory evaluative process in a Competitive Interaction Theory of Choice:

1. Choice alternatives are evaluated in competitive sets (relevant sets or evoked sets).
2. Choice alternatives can be represented as bundles of attributes (manifest or latent attributes; psychological, sociological, physical or economic attributes).
3. Individuals evaluate choice alternatives in terms of the comparative possession of attributes in the competitive set.
4. Attributes interact to form the attractiveness of each alternative to each individual.
5. Individuals are inclined to choose alternatives according to their shares of the total attractiveness of the competitive set.
6. Individuals differ in largely systematic and analyzable ways.

A mathematical representation of these principles is:

$$\pi_{ij} = \frac{\prod_{h=1}^{H} f_i(X_{hij})^{\beta_h}\delta_{ij}}{\sum_{j=1}^{m_i}\left(\prod_{h=1}^{H} f_i(X_{hij})^{\beta_h}\delta_{ij}\right)} \tag{1}$$

The subscript i refers to choice situations, e.g., time periods, geographic areas, individuals, or homogeneous groups: it allows one to model the systematic structure of individual differences mentioned in point 6. Subscript j refers to choice alternatives. There are m_i choice alternatives in the ith choice situation: this allows the competitive sets mentioned in point 1 to vary over choice situations. X_{hij} is the original measure of explanatory variable h, that is, a particular attribute in the bundle mentioned in point 2 as it relates to choice alternative j in situation i. And $f_i(X_{hij})$ is a positive, ratio scale function of the original measure. Equation 1 is a very basic model encompassing both MCI and multinomial logit (MNL) models, depending on the choice of representation for $f_i(X_{hij})$. For this function to satisfy point 3, it must reflect the comparative information inherent in the set of original measures. This will make the MCI model a non-Lucian model (to be discussed later). Murdock's (1960) experiments will be considered in a later section, which demonstrates that the proposed transformation conveys the comparative information in the original measures. β_h is a parameter to reflect the sensitivity of choice probability π_{ij} to variable h; δ_{ij} is a specification error term which reflects the possibility that H attributes may not include all the explanatory influences. The product of the H terms $f_i(X_{hij})^{\beta_h}$ represents the concept that the attributes interact to form the attractiveness of each alternative in the competitive set, as in point 4. This product divided by the sum of products for all alternatives in a competitive set reflects the "share of the pie" notion in point 5.

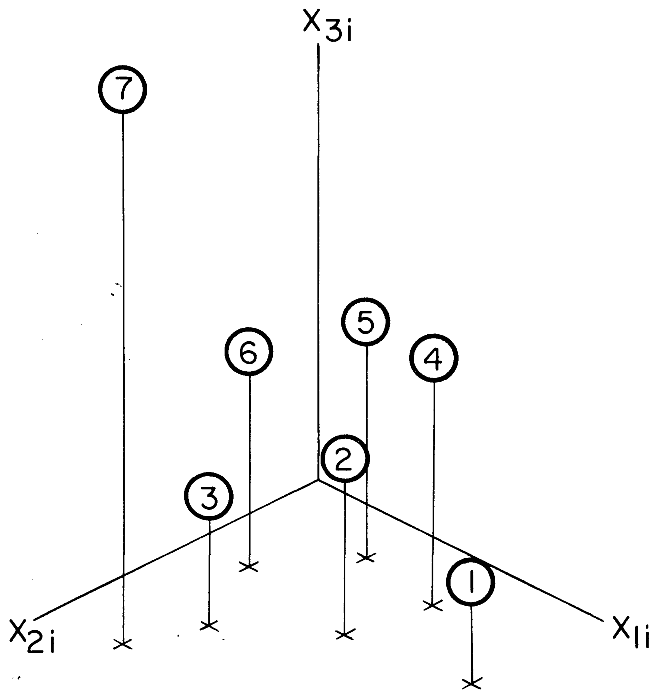
Thus, if we can construct a legitimate transformation which retains the comparative information in the original measures, we have captured in Equation 1 these six basic aspects of the compensatory evaluation in our theory. This problem forms the focus of the next section.

REPRESENTING COMPARATIVE INFORMATION

The H explanatory variables in Equation 1 could be thought of as the axes of a space. Each choice alternative would be a point in this space, as depicted in Figure A. We

FIGURE A

SEVEN BRANDS IN A THREE-DIMENSIONAL SPACE



could imagine that each alternative is a ring held in place not by the line tying it to the plane at the bottom of the figure, but by invisible rods connecting the rings to each other. Say each ring weighs eight ounces and the rods are weightless. Imagine sticking your finger through a ring and trying to twirl all the rings around your finger. Feel how much easier it is to twirl all the rings from rings in the middle (5 or 6) than from rings at the extreme (1 or 7). Such feelings equate to the physical concepts of moments of inertia or radii of gyration. A ring at the center of gravity in this figure would be the easiest of all to twirl. One can construct a two-dimensional example of this by cutting finger holes at different places in a rectangular piece of cardboard, or a one-dimensional example by drilling holes at various places along the length of a wooden dowel.

These physical concepts have statistical counterparts, as pointed out by Hays (1963, p. 181):

> The reader acquainted with elementary physics may recognize not only that the mean is the center of gravity of a physical distribution of objects, but also that the variance is the *moment of inertia* of a distribution of mass. Furthermore, the standard deviation corresponds to the radius of gyration of mass distribution; this is the real basis for regarding the standard deviation as analogous to a resultant force away from the mean. These physical conceptions and their associated mathematical formulations have influenced the course of theoretical statistics very strongly, and have helped to shape the form of statistical inference as we will encounter it.

Let the moment of inertia about alternative j^* on measure h be denoted I_{hij^*}:

$$I_{hij^*} = \frac{1}{m_i} \sum_{j=1}^{m_i} (X_{hij^*} - X_{hij})^2 \qquad (2)$$

where the other symbols have the same meaning as in Equation 1. As indicated by Hays, the central moment of inertia, denoted $I_{hi\mathrm{O}}$, is the variance of the distribution:

$$I_{hi\mathrm{O}} = \frac{1}{m_i} \sum_{j=1}^{m_i} (X_{hij} - \overline{X}_{hi.})^2 \qquad (3)$$

The ratio of the noncentral moment of inertia to the central moment of inertia is a comparative index with solid foundation in physics and statistics. Note that the origin of the space is immaterial to this index. An overall rescaling will affect the numerator and denominator in the same way, leading to no change in the ratio. In sum, then, a general linear transformation of the measure does not affect this ratio, making it an appropriate transformation of interval scale ratings. It has a minimum value of one for choice alternatives at the center, and increases as a particular alternative gets farther away from the center. Just as the standard deviation is in the scale units of the original measures, the square root of this ratio, which we denote:

$$D_{hij} = \frac{I_{hij}^{1/2}}{I_{hi\mathrm{O}}^{1/2}} \qquad (4)$$

relates to the original unit of measures rather than to squared units. It is this ratio which we will compare to Murdock's (1960) index of distinctiveness.

An Empirical Comparison to Murdock's Index of Distinctiveness

Murdock desired "a quantitative measure of distinctiveness that could be derived without recourse to experimentation so that the general effects of the variable could be determined" (1960, p. 16). He saw a role for such a measure in understanding very diverse areas of psychology. His research is of interest here because he validated his measure "by determining the extent to which it can predict accuracy of identification in the method of absolute judgments" (1960, p. 18). Absolute judgment tasks are tasks in which an observer "of a particular stimulus identifies that stimulus with a single name or number. He does not make a judgment of whether a stimulus is greater than, less than, or the same as another" (Garner and Hake 1951, p. 446). Murdock reasoned that performance in such tasks would primarily be a function of stimulus distinctiveness: the more distinctive stimuli would be correctly identified relatively often, while the less distinctive stimuli would be correctly identified less often.

Three things need to be noted here. First, Murdock implies that the information perceived by an observer is comparative information even in an absolute judgment task, that is, that judgments are dependent on the comparative context. But early research indicated that "information conveyed by stimuli varying on a single dimension is likely to fall between 2 and 3 bits" (Attneave 1959, p. 72), corresponding to perfect perceptual identification of five to nine stimuli. Increasing the number of stimuli beyond this leads to little or no improvement in the amount of information perceived. Second, if Murdock's measure of distinctiveness accurately reflects the percent correct identification in absolute judgments, then it accurately reflects the information perceived by an observer in the stimulus objects. Third, if our index D_{hij} accurately reflects the percent correct identification in absolute judgments, it too will reflect the information perceived. As Attneave (1959) pointed out, this paradigm and some of the data Murdock reanalyzed form part of the foundation of information theory in psychology.

Murdock provided results for six experiments. The first three involved absolute judgments of the loudness of 1,000 cycle tones that varied over a 40-decibel range. Respondents used a 1 to 9 scale to label the loudness of the tones. There were 60 presentations of a two-second tone followed by five seconds in which to write the label for the tone. These three loudness experiments are referred to as LI, LII, and LIII. The fourth experiment involved judgments concerning weights. There were eight weights in identical containers, ranging from one to 16 pounds. The respondents had to learn the correct color name for each weight in a paired associates learning paradigm. They lifted each weight for five seconds, had 10 seconds to associate the color with the weight, and then had a two-minute rest interval. The criterion was one perfect trial through a randomly ordered eight-weight sequence. Murdock also reanalyzed data from two prior studies. Eriksen and Hake (1957) collected absolute judgments of the area of 20 squares that ranged from two millimeters on a side to 40 mm. in two millimeter steps. Bugelski (1950) had subjects study lists of eight words, and recorded the percent correct recall for the first word in each list, the second word in each list, and so on up to the last word in each list (the serial position effect, or SPE).

To understand Murdock's index, one must realize that he was dealing with pounds and decibels—physical measures he called "energy values." He cited the Weber-Fechner Law as well as Helson's (1947) adaptation-level theory as justification for working with the logs of the energy values. His index is the sum of the absolute value of the differences in the logs of the energy values from one stimulus to all others. This index is expressed as a percentage of the total sum of log differences:

$$\%D_{j*} = \frac{\sum_{j=1}^{m} |\log X_{j*} - \log X_j|}{\left(\sum_{j*=1}^{m} \sum_{j=1}^{m} |\log X_{j*} - \log X_j|\right)} \qquad (5)$$

where m is the number of stimuli, X is the single measure of "energy value," and $j*$ is a particular stimulus from the set. This index cannot be used with interval scale measures because of the problem mentioned earlier in taking the logs of negative numbers. It also would pose difficulties in interpretation if the measures fell between zero and one. Large negative values for the logs could distort the index. But in the cases Murdock considered, these difficulties did not arise.

The predictions from Murdock's index (Equation 5), and our index (Equation 4) are based entirely on information available before the human judgments in these psychophysical experiments were collected. "Energy values" measure loudness in decibels for the loudness experiments LI, LII, and LIII, pounds for the weight experiment, the length of a side in millimeters for the squares experiment, and the log of the serial position for the SPE experiment. Since the values from Equation 5 are ratio numbers, fit was assessed by a congruence coefficient which allows for rescaling alone (i.e., no additive constant; Tucker 1951):

$$\Phi_{op} = \frac{\sum_{j=1}^{m} o_j p_j}{\left(\sum_{j=1}^{m} o_j^2 \sum_{j=1}^{m} p_j^2\right)^{1/2}} \qquad (6)$$

where o_j is an observed score and p_j is a predicted score for stimulus j. The results are presented in Table 1.

TABLE 1

CONGRUENCE COEFFICIENTS BASED ON PREDICTION OF PERCENT CORRECT IDENTIFICATION FROM DISTINCTIVENESS: MURDOCK'S INDEX AND D_{hij}

	LI	LII	LIII	Weights	SPE	Squares
m	9	9	9	8	8	20
Murdock	.994	.986	.991	.988	.987	.983
D_{hij}	.994	.989	.987	.996	.969	.945

All of these congruence coefficients appear significant if the simulation distributions developed by Korth and Tucker (1975) are used as a guide. These distributions are for matching two vector spaces, not merely two vectors. We are safe, however, in assuming that values greater than 0.9 are significantly greater than zero. In Murdock's original experiments, his index of distinctiveness fits slightly better than D_{hij} one out of four times, D_{hij} fits slightly better two out of four times, and the indices are tied once for all practical purposes. These results provide evidence that D_{hij} preserves the information perceived by observers in a set of objects.

The significant result for the SPE is interesting in that this is one of the few well-known effects that cannot be predicted by Landauer's (1975) computer simulation model for memory without organization. Using a naïve memory model with random storage and undirected retrieval, Landauer was still able to reproduce well-known learning and forgetting curves, the effects of massed versus distributed practice on learning, and other results for which complex memory models have been proposed. But his naïve model cannot account for memorability due to the comparative position of words in serial lists. This lends further support to the notion that our index represents something basic about human judgments that is not easily explained by other means.

COEFFICIENT ZETA-SQUARED

One desirable property is missing from the index D_{hij}. Values that are a given number of units above the mean of the original interval scale measure X_{hij} are mapped into the same D_{hij} value as are values that are a given number of units below the mean. We can make the mapping unique, instill a sense of direction to a coefficient, and still maintain the ratio properties that an MCI model requires by inverting the index D_{hij} at the mean of the original measure. We define:

$$\zeta^2_{hij} = \begin{cases} D^2_{hij} \text{ if } X_{hij} \geq \bar{X}_{hi.} \\ 1/D^2_{hij} \text{ if } X_{hij} \leq \bar{X}_{hi.} \end{cases} \quad (7)$$

It might seem unusual to compress everything below the mean of the original measure into the range from just above zero to one, and to allow everything above the mean of the original measure to range from one to just below infinity. Yet this is a proper representation in multiplicative models because every value in a multiplicative model is balanced by its inverse value. To understand this and other relations between multiplicative models and additive models, it is useful to express Equation 7 in terms of standard scores. Since the standard deviation is $I^{1/2}_{hiO}$, the standard score, z_{hij}, can be expressed as:

$$z_{hij} = \frac{(X_{hij} - \bar{X}_{hi.})}{I^{1/2}_{hiO}} \quad (8)$$

Note that:

$$\begin{aligned} I_{hij*} &= \frac{1}{m_i}\sum_{j=1}^{m_i}(X_{hij*} - \bar{X}_{hi.} - X_{hij} + \bar{X}_{hi.})^2 \\ &= (X_{hij*} - \bar{X}_{hi.})^2 + \frac{1}{m_i}\sum_{j=1}^{m_i}(X_{hij} - \bar{X}_{hi.})^2 \quad (9) \\ &= (X_{hij*} - \bar{X}_{hi.})^2 + I_{hiO} \end{aligned}$$

so that:

$$\zeta^2_{hij} = \begin{cases} \dfrac{(X_{hij} - \bar{X}_{hi.})^2 + I_{hiO}}{I_{hiO}} \text{ if } X_{hij} \geq \bar{X}_{hi.} \\ \dfrac{I_{hiO}}{(X_{hij} - \bar{X}_{hi.})^2 + I_{hiO}} \text{ if } X_{hij} \leq \bar{X}_{hi.} \end{cases} \quad (10)$$

In terms of z-scores we have:

$$\zeta^2_{hij} = \begin{cases} (1 + z^2_{hij}) \text{ if } z_{hij} \geq O \\ \dfrac{1}{(1 + z^2_{hij})} \text{ if } z_{hij} \leq O \end{cases} \quad (11)$$

The "origin" of a z-score is zero. In linear, additive models, the differences in scale units below the origin must be the same as the differences in scale units above the origin. In multiplicative models, the "origin" is one. The ratio of scale values below the origin of a multiplicative coefficient must be the same as ratios of the scale values above the origin. For example, the meaning of the interval from 0.125 to 0.250 must be the same in a multiplicative model, as the interval from 8 to 4.

The presentation of zeta-squared in Equation 11 is useful for several other purposes. First, it emphasizes that only interval scale properties are required of the original measures. Since z_{hij} is invariant over linear transformations of the original scale of measurement, zeta-squared must also be invariant. Second, it shows that zeta-squared is easily computed from standard scores. Third, as noted by Miettinen (1970), if the original measures are normally distributed, one can use the underlying normal distribution and/or the underlying chi-square distribution to test hypotheses of interest. Thus the compression of half the scale information into the range from 1.0 to just above zero does not inhibit one's ability to make statistical comparisons. Fourth, it shows that zeta-squared is a kind of standardization of variables for multiplicative models.

Zeta-squared is also a very justifiable coefficient for use with nominal scale variables. To see this, we let p_{hi} be the proportion of choice alternatives which possess attribute h in choice situation i, then:

$$\begin{aligned} p_{hi} &= \bar{X}_{hi.} \\ p_{hi}(1 - p_{hi}) &= I_{hiO} \end{aligned} \quad (12)$$

and:

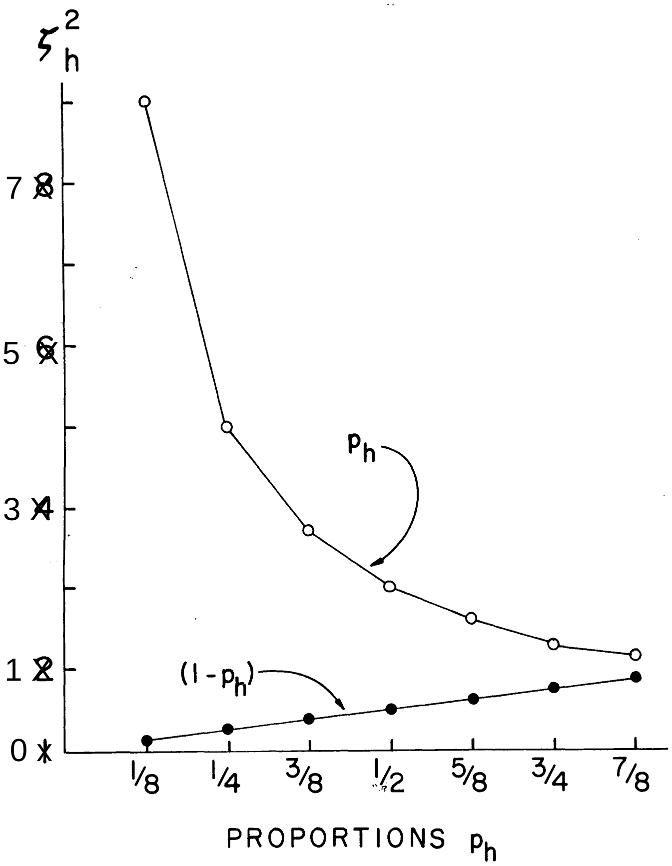
$$\zeta^2_{hij} = \begin{cases} \left(1 + \dfrac{(1 - p_{hi})^2}{p_{hi}(1 - p_{hi})}\right) = \dfrac{1}{p_{hi}} & \text{if } j \text{ possesses the characteristic} \\ \left(1 + \dfrac{(-p_{hi})^2}{p_{hi}(1 - p_{hi})}\right)^{-1} = 1 - p_{hi} & \text{if } j \text{ does not possess the characteristic} \end{cases} \quad (13)$$

For binary variables, consider that there are m_i alternatives in choice situation i, and that c_h of those alternatives possess the attribute. Those alternatives which possess the attribute receive a value of m_i/c_h, which is equal to $1/p_{hi}$, and those who do not possess the attribute receive a value of $(m_i - c_h)/m_i$, which is equal to $(1 - p_{hi})$.

Three things should be noted. First, in the binary case the value of zeta-squared comes from a legitimate operation on a binary variable: counting the frequency of occurrence. Second, the binary form highlights the symmetry expected from a coefficient in a multiplicative model. Figure B plots

FIGURE B

VALUES OF ζ_h^2 CORRESPONDING TO PROPORTIONS POSSESSING (p_h) OR NOT POSSESSING ($1 - p_h$) AN ATTRIBUTE



the value of zeta-squared for an eight-object example. If only one of the eight objects possesses the attribute, that object receives a value of 8. If only one object does not possess the attribute, that object gets a value of 1/8. If three objects possess the attribute, each receives a value of 8/3, and if only three objects do not possess the attribute, they receive a value of 3/8. If seven objects possess the attribute, they each receive a value of 8/7; if seven objects do not possess the attribute, they receive a value of 7/8. Third, this reconnects zeta-squared with the original definition of the NCK distinctiveness index (Nakanishi, Cooper, and Kassarjian 1974). For binary data, zeta-squared is the NCK distinctiveness index. For interval scale measures, zeta-squared is the generalization of that index, which we have sought.[1]

Some interesting relations exist between zeta-squared and $\exp(z_{hij})$. First note that if, in Equation 1, $f_i(X_{hij}) = \exp(X_{hij})$, the well-known multinomial logit model is specified. If $f_i\,(X_{hij}) = \exp(z_{hij})$, a different multinomial logit model is specified. Since MCI models using zeta-squared and $\exp(z)$ are both multiple regression models when properly log transformed (cf. Nakanishi and Cooper 1982), comparison can be done between z and the log of zeta-squared. Figure C plots this relation between ± 4 standard squared deviations. In this range, ζ^2 increased more slowly than z for values near the mean. Then there is an inflection point. So within the ± 3 standard deviation range the relation is almost linear, with a slope of 0.77. Outside this range, large increases in extreme values of z do not result in proportional increases in the log of ζ^2.

These results indicate that there will be some similarity between an MCI model with zeta-squared and an MCI model with $\exp(z)$—i.e., a multinomial logit model using z—but that values very close to the mean and values beyond 3 standard deviations will have less impact using zeta-squared than using $\exp(z)$.

FOUR APPLICATIONS

This section briefly summarizes four studies using zeta-squared. Its purpose is to reflect some of the diversity of applications of multiplicative models using zeta-squared.

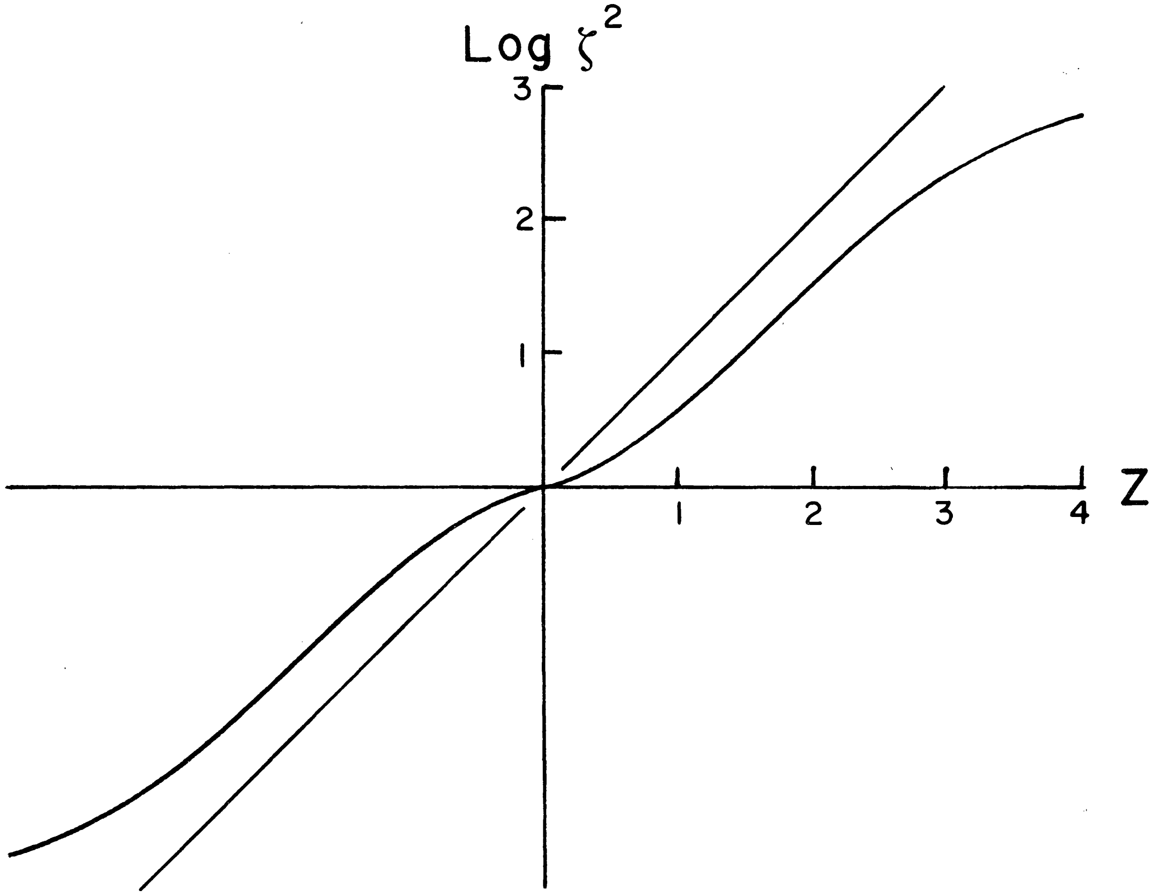
Study 1: Choice Among Political Candidates

The initial use of zeta-squared involved modeling how voters' choices were related to binary characteristics of the candidates in an election with many candidates and little information (Nakanishi et al. 1974). Five seats on the community college board of trustees were being sought by 14, 24, 7, 12, and 7 candidates, respectively. The choice alternatives in each of the five choice situations were different, but the analysis assumed that characteristics of the candidates had the same influence in each of the situations. Eighteen binary variables reflecting occupation, sex, religion, ballot position, campaign effort, and a variety of endorsements were used. An MCI model was compared to a multiple regression model analogous to the one proposed by Mueller (1970) for such elections. The maximum likelihood solution was used to estimate parameters of the MCI model. In calibration, both models fit extremely well, but on cross-validation, major differences appeared. The pa-

[1] It is very unusual to have a single transformation that creates a meaningful positive ratio scale out of either binary or interval scale measures. The only other obvious candidate is $\exp(z_{hij})$, which is discussed below. In Cooper and Nakanishi (1977), we proposed a generalization of the NCK distinctiveness index to the case of multistep discrete scales. That generalization possesses neither the foundation in physics nor the statistics that zeta-squared displays. The discrimination ratio approach of Mahajan and Jain (1977) and Mahajan, Jain, and Ratchford (1978) sets the value of possession of an attribute and the value of nonpossession at values other than one and zero, but does not generalize from the binary to the interval scale case. What Jain and Mahajan (1979, p. 224) referred back to as a "Likert Scale . . . transformation" was originally referred to as a "Semantic Differential Scale Transformation" (Mahajan and Jain 1977, p. 322). This transformation, however, only rescales a binary variable so that it takes on two values in the range of an interval scale variable. No generalization was suggested.

FIGURE C

THE RELATION BETWEEN Z AND LOG ζ^2



rameters of these models were developed on three of the offices and used to predict the results in the other two offices. Averaged over the 10 different ways this cross-validation can be performed, the MCI model had a squared cross validity correlation of 0.87, compared to 0.53 for the multiple regression model. Linear additive models have the ability to fit well in calibration, but without the logical consistency of the MCI and without the comparative, office-by-office standardization explicit in zeta-squared, linear models do not seem to hold up as well in actual prediction. What zeta-squared reflects in this multiple office voting example is that if several candidates possess an attribute, they share whatever value possession conveys. Zeta-squared gives a set of rules for dividing up the shares. To the extent that these rules are accurate, this frees parameters to reflect the fundamental sensitivity of choice probabilities to possession of the attribute. Otherwise, the parameters reflect an effect which confounds the pattern of possession and nonpossession of the attribute in a particular research context with the fundamental importance of the attribute.

Study 2: Choice Among Retail Centers

In a reanalysis of data from Huff (1962) concerning how residents from three different neighborhoods were attracted to retail shopping centers, Nakanishi and Cooper (1982) used only (1) distance from each neighborhood to each center, and (2) a measure of the total space for display in each center to forecast choice probabilities. The variables were originally measured on ratio scales. The zeta-squared transformation of these variables reduces the information in them to just the comparative base. But further analysis shows that when the parameters of the MCI model are developed on two neighborhoods and used to forecast results for the third, the average square of the cross validity correlation is 0.85 for the MCI model with explanatory variables transformed by zeta-squared, compared to 0.83 for the MCI model on the original ratio scale measures. Zeta-squared imposes a scale of measurement that is relative to each choice situation. In this context, and perhaps in many others, it seems that the comparative information conveyed by zeta-squared relates at least as closely to the information used in the human judgments as the absolute information in the original measures does.

Study 3: Choice Among Women's Fashions

Cooper and Midgley (1982) report an analysis in which zeta-squared is used to link the results of an individual- and group-differences model for multidimensional scaling of perceptual dissimilarities to purchases of women's fashions. From a large sample of Australian women studied over the course of a fashion season, a group of 57 women were chosen. They responded on three measurement occasions (early, middle, and late) to a set of paired comparisons among seven examples of women's fashions. The interval scale dissimilarity judgments were multidimensionally scaled using RASCAL (Cooper 1981). Three perceptual dimensions were used to represent the comparative judgments in each of four groups discovered in the analysis. A logit model for external analysis of preferences (Cooper and Nakanishi 1983) was used to imbed ideal points into the perceptual space for each group. Then, using the groups to define four choice situtations, dimension-by-dimension distances of each fashion from the ideal points were transformed by zeta-squared and used as attributes in an MCI model. Ratings of "friends' likely reaction" to each fashion were also transformed by zeta-squared and included as another attribute. Each fashion's share of wardrobe purchases was the dependent measure. The logit ideal point model fit extremely well. Adjusted R^2 ranged from 0.87 to 0.91, and all were statistically significant beyond the 0.001 level. The final MCI model gave a statistically significant account of fashion purchases ($R^2 = 0.61$, $F = 2.5$, $p < 0.05$).

Study 4: The Cognitive Algebra of MCI Models

The final analysis is based on the use of Equation 1 not as a model with parameters—β_h—to be estimated, but as a set of guidelines for the integration of attribute ratings and importance ratings. Cooper and Finkbeiner (1982) report a reanalysis of aggregate responses from 130 people involved in the testing of a new consumer product. In essence, the test group rated six existing brands and the test brand on around two dozen product attributes. They also rated the subjective importance of each attribute. Factor analysis was used to select five common and two specific factors among the attributes. Variables loading highly on each of the five common factors were averaged to form five scales. Including the two specific attributes, there were a total of seven variables used to represent the aggregate test group's reactions to the seven brands. Each of the seven variables was transformed to zeta-squared values. Use of Equation 1 would be ineffectual because there are more parameters to be estimated than the available degrees of freedom. But the weights—β_h—are attribute importances in a sense, so that one could use some function of the interval scale importance ratings in the place of each β_h. In the primary model, the standardized importance ratings (i.e., z-scores to reflect relative importance of each attribute) were used to form a composite with the zeta-squared transformed attribute rating. The algebra of Equation 1 dictated how the transformed attributes and the standardized importances were combined. The probabilities used to calibrate the model came from a rescaling of measures of overall brand acceptability. The composite MCI was compared to several other composite models, including the traditional linear additive composite that would be formed in this situation. The composite MCI model provided the best fit in calibration, with a root mean squared error (RMSE) of 0.050 between calibration probabilities and the estimated probabilities. The traditional linear additive model had an RMSE of 0.070. The calibrated models were used—along with estimates provided by the sponsor of what portion of the test market would be made familiar with the test brand (i.e., a market penetration estimate)—to forecast the shares of these brands during test market. The composite MCI model had the smallest RMSE in actual prediction (RMSE = 0.042), compared to a RMSE of 0.050 for the linear additive model. The composite MCI model also came closest in forecasting the test market share for the new brand, being off by 1.2 share points compared to 4 share points with the linear additive model. While this is an aggregate analysis, it indicates that at least some of the disparity between self-report importance measures and analytically estimated importance weights has been due to the form of the model used in the comparison, rather than to any inherent problem in the meaning of importance measures.

A side benefit of this analysis was the demonstration that zeta-scores (the square root of zeta-squared) gave a better representation of the cognitive algebra of the consumers. In standard applications of MCI models, the use of zeta-scores or zeta-squared will give parallel results. There will simply be an overall rescaling of the parameter values and the standard errors. But in forming composites we see, as expected, that zeta-scores relate more closely to the scale units consumers use. This is not surprising because zeta-scores are in standard deviation units while zeta-squared is in variance units.

These examples show the usefulness of zeta-squared and zeta-scores in multiplicative models for relating explanatory variables to choice probabilities at the aggregate level and at the level of homogeneous subgroups. We now discuss the class of probabilistic choice models formed by integrating zeta-scores with MCI models.

USING ZETA-SCORES TO REPRESENT THE EFFECTS OF CHANGES IN CHOICE SETS

We have seen that zeta-scores are built from comparative contextual information and that they can be thought of as standardizations of variables for multiplicative models or as measures of distinctiveness. Comparison, standardization, and distinctiveness are three constructs with overlapping domains. They all deal with relating individual stimuli, objects, or choice alternatives to the context in which judgments are made. Using zeta-scores amounts to a declaration that judgments are not context-free. This has implications for our theory and model of choice.

Consider an MCI model in which the original measures are on a ratio scale, and are used directly as the components of the model:

$$f_i(X_{hij}) = X_{hij} \tag{14}$$

Members of this class of MCI models are examples of Luce (1959) choice models. But models in which:

$$f_i(X_{hij}) = \zeta_{hij} \tag{15}$$

are not Luce choice models. The difference is that zeta-scores, by representing the contextual information, explicitly violate the constant utility assumption of Luce choice models.

As one would expect from the previous discussion of the relation of zeta-squared to exp(z), the relations in Equations 14 and 15 have other parallels. If:

$$f_i(X_{hij}) = \exp(X_{hij}) \tag{16}$$

then the MCI model is a Luce choice model. If:

$$f_i(X_{hij}) = \exp(z_{hij}) \tag{17}$$

then the MCI model is not a Luce choice model. Both of these specifications are logically consistent versions of the multinomial logit model. It is again the violation of the constant utility assumption that causes Equation 17 not to be a Luce choice model. So if one represents the contextual information in a choice set, one violates the constant utility assumption. This is basically another way to understand why the constant utility Luce choice models are criticized for doing a poor job at representing the effects of interobject similarity on choice probabilities. The independence from irrelevant alternatives (IIA) assumption (Arrow 1951; Luce 1959; Luce and Raiffa 1957) is at fault. According to Luce (1959, p. 9):

> The actual gist of the idea is that alternatives which should be irrelevant to the choice are in fact irrelevant, hence the present term. For example, the idea states that if one is comparing two alternatives according to some algebraic criterion, say preference, this comparison should be unaffected by the addition of new alternatives or the subtraction of old ones (different from the two under consideration). Exactly what should be taken to be the probabilistic analogue of this idea is not perfectly clear, but one reasonable possibility is the requirement that the ratio of the probability of choosing one alternative to the probability of choosing the other should not depend upon the total set of alternatives available.

The evidence presented here concerning the comparative nature of judgments indicates that the composition of the competitive set does influence the ratios of choice probabilities for pairs of alternatives.

Luce himself supported this contention. In his 1977 Presidential Address to the Psychometric Society, he reviewed Thurstone's (1927a, b, and c) contributions after 50 years. It was the Thurstonian assumption most parallel to the IIA assumption that Luce criticized most directly: "Clearly, Thurstone's implicit assumption of a unique representation must go. In some sense the representation must vary with the experimental context" (Luce 1977a, p. 465). Luce also reviewed his own choice axiom after 20 years and concluded (1977b, p. 229; cf. Debreu 1960, Restle 1961):

> As a descriptive tool, it is surely imperfect; sometimes it works well, other times not very well. As Debreu and Restle made clear and as has been repeatedly demonstrated experimentally, it fails to describe choice behavior when the stimulus set is constructed in such a way that several alternatives are treated as substantially the same.

According to Luce, the choice axiom survives, despite its empirical failures (1977b, p. 229–230):

> as a canon of probabilistic rationality . . . a possible underpinning for rational, probabilistic theories of social behavior. Thus, in the development of economic theory based on the assumption of probabilistic individual choice behavior, it can play a role analogous to the algebraic rationality postulates of traditional theory.

But preserving convenient fictions counter to the weight of empirical evidence is not a proper role for theory.

To see how zeta-scores help in the cases where the choice axiom gives less than reasonable results, we have constructed a small example parallel to Debreu's (1960) classic counter-examples. We consider eight different competitive choice sets in which we portray choice as being influenced by a single binary attribute. We suppose, for illustration, that in a two-alternative choice set where one possesses and the other does not possess the desired attribute, possession of this attribute makes that alternative four times as likely to be chosen. This forms Case 1, the base case in Table 2. Cases 2 and 3 have three alternatives in the competitive choice set. In Case 2, the additional alternative does not possess the desired attribute, while in Case 3 the additional alternative does possess the desired attribute. Cases 4, 5, and 6 have four alternatives in the competitive choice set, and Cases 7 and 8 have five alternatives. The patterns of possession of the attribute are displayed as zeroes (non-possession) and ones (possession) in the left column of Table 2. The theoretical probabilities in the right column come from the notion that identical alternatives share equally the total probability of choice established in the base case. We compare the predictions of five models.

The first model is a basic MCI model using zeta-squared to standardize the binary variable separately for each of the competitive sets. The square of Equation 15 specifies this model, where the subscript i refers to the different competitive sets. The second model uses the exponential transformation of zeta-squared in a way that underscores the connection of these models to extreme value distributions (cf. Gumbel 1958; Johnson and Kotz 1970). The equation for this model is:

TABLE 2

COMPARISON OF PREDICTED PROBABILITIES

Model		1. MCI—ζ^2	2. MCI—exp(ζ^2)	3. MCI—exp(ζ)	4. MNL—X	5. MNL—Z	
Theoretical β		1.0	4/3 log 2 (.924)	$2\sqrt{2}$ log 2 (1.961)	log 4 (1.386)	log 2 (.693)	Theoretical probability
Empirical b:		.927	.817	1.869	1.386	.656	
	X_{hij}						
Case 1	0	.217	.227	.211	.2	.212	.2
	1	.783	.773	.789	.8	.788	.8
Case 2	0	.166	.115	.133	.167	.166	.1
	0	.166	.115	.133	.167	.166	.1
	1	.668	.771	.735	.666	.668	.8
Case 3	0	.110	.162	.130	.111	.111	.2
	1	.445	.419	.435	.444	.445	.4
	1	.445	.419	.435	.444	.445	.4
Case 4	0	.130	.058	.088	.143	.133	.067
	0	.130	.058	.088	.143	.133	.067
	0	.130	.058	.088	.143	.133	.067
	1	.611	.826	.735	.571	.602	.8
Case 5	0	.108	.114	.105	.1	.106	.1
	0	.108	.114	.105	.1	.106	.1
	1	.392	.386	.395	.4	.394	.4
	1	.392	.386	.395	.4	.394	.4
Case 6	0	.066	.121	.089	.077	.068	.2
	1	.311	.293	.304	.308	.311	.267
	1	.311	.293	.304	.308	.311	.267
	1	.311	.293	.304	.311	.267	.267
Case 7	0	.075	.096	.081	.071	.074	.1
	0	.075	.096	.081	.071	.074	.1
	1	.283	.269	.279	.286	.284	.267
	1	.283	.269	.279	.286	.284	.267
	1	.283	.269	.279	.286	.284	.267
Case 8	0	.095	.080	.083	.091	.094	.067
	0	.095	.080	.083	.091	.094	.067
	0	.095	.080	.083	.091	.094	.067
	1	.357	.379	.375	.364	.359	.4
	1	.357	.379	.375	.364	.359	.4
R^2		.849	.976	.938	.852	.850	
RMSE		.063	.023	.036	.067	.064	
F		14.7	105.9	39.4	15.1	14.9	
P		<.0001	<.0001	<.0001	<.0001	<.0001	

$$f_i(X_{hij}) = \exp(\zeta_{hij}^2) \tag{18}$$

The third model is a less extreme version using zeta-scores instead of zeta-squared:

$$f_i(X_{hij}) = \exp(\zeta_{hij}) \tag{19}$$

The final two models are the constant ratio, multinomial logit model in Equation 16 and the multinomial logit model using z-scores in Equation 17.

A single parameter is estimated for each model over the eight choice situations. This allows us to see how close the empirically estimated coefficient is to the theoretical parameter value (i.e., the parameter value that would calibrate each of these models to fit exactly the theoretical probabilities in the two-choice-alternative base case). It also allows us to assess overall fit of the models in terms of squared multiple correlation and F-statistics from the log forms in which all models are estimated. The root mean squared error (RMSE) is the more interesting measure because it directly matches the predicted probabilities and the theoretical probabilities. The resulting predictions and fits are reported in Table 2.

Note that the Luce choice model (model 4, the multinomial logit model using raw scores) has the largest RMSE.

As Luce indicated, constant ratio models do not do very well in this kind of example. Note that models 1 and 5—an MCI model using zeta-squared and an MNL model using z-scores—produce results very similar to each other. The relations plotted in Figure C between log(zeta-squared) and z-scores should make one guess that there would be similar results from models comparing zeta-squared to the exponential transformation of a z-score. There is a modest reduction in RMSE for both these models compared to the constant ratio model 4. Models 2 and 3 give far superior approximations to the theoretical probabilities: model 2 has a RMSE about one third the size of the constant ratio model 4.

There are many interesting and theoretically sound approaches to the problems created by the IIA assumption. Currim (1982) provides an excellent presentation of his own developments and the work of others on this multifaceted problem. An MCI model using zeta-scores simply avoids the IIA assumption by using a comparative representation of the judgment/choice context instead of estimating special parameters. We expect that MCI models and zeta-scores will be useful ingredients in future inquiries into the effects of similarity on choice.

SIMULTANEOUS ANALYSIS OF MULTIPLE CHOICE SITUATIONS

Heretofore, when using multiplicative choice models, the primary option has been to exponentially transform the interval scale consumer ratings. This was done either explicitly with an MCI model or inevitably with a multinomial logit (MNL) model. The virtue of the exponential transformation is that it makes numbers that add act like numbers that multiply. Its greatest liability is the failure to represent differences in comparative or contextual information across choice situations. If there were only one choice situation, the issue would not arise. A linear transformation to z-scores in a single choice situation will not affect the parameters of an MNL model. But the statistical power of MCI and MNL models comes mainly from their ability to analyze an increasing number of choice situations—the "i" in all the equations.

A multitude of choice situations can come from analysis of the systematic structure of individual differences in a domain of choice (cf. Tucker 1958; Tucker and Messick 1963): cross-sectional differences, over-time differences, or combinations of these. The leverage comes when individual differences models indicate that some parameters can be shared over choice situations. We believe that representing the comparative, contextual information in each choice situation can free parameters to reflect more of the fundamental importance of an attribute across choice situations. Simply standardizing the variables by forming z-scores in each choice situation should improve the representation of an MNL model. The research using zeta-squared and the relations between zeta-squared and z-scores in Figure C support this inference. The main differences should be observed at the extremes. In relation to z-scores, zeta-squared reflects a diminishing return to scale: becoming increasingly more extreme compared to the remainder of the objects has a constantly increasing impact with z-scores in an MNL model. The impact of the same increase for zeta-squared in an MCI model will taper off, as in Figure C.

From the perspective of psychometrics, we are seeking ways to understand the behavior of consumer markets from the beliefs, values, perceptions, and preferences of consumers. An econometrician's approach to consumer behavior places much more emphasis on the "objective market environment." McFadden proposed individual random utility models with double exponential error and showed that these models aggregate into Luce choice models (cf. McFadden 1974, 1980, 1981). His writing serves to underscore the differences in approaches. In a section entitled "An Econometrician's View of Marketing," McFadden states (1980, p. S14):

> The core of a model of market behavior will be an equation, consistent with the theory of the economic consumer, which specifies the probability of choices (e.g., brand, frequency, or volume of purchases) as a function of the objective market environment of the consumer.

Thinking in terms of the simultaneous analysis of multiple choice situations highlights some of the liabilities of McFadden's emphasis on the "objective market environment." The dependent measures, i.e., choice probabilities, are relative to each choice situation; that is, they sum to one over all the alternatives in each situation and they are range-constrained to be between zero and one. In general, unless the explanatory variables have scales of measurements that are also relative to each choice situation, we encounter the aggregation error of not representing "frog pond effects" (Davis 1966). Let us use price as an example and geographic regions as the multiple choice situations. The least expensive alternative in a particular geographic region might not be in the same relative position in the aggregate. If objective price, rather than comparative price, is used, the overall analysis could misconstrue the "true" influence of competitive pricing.

Focus on the "objective" measures might incline economists to think of the simultaneous analysis of several choice situations merely as a matter of aggregation, rather than in terms of the representation of systematic individual or group differences. Friedman framed the issue in theory (1957, p. 216):

> Suppose a regression were computed for a broad group of consumer units, say a sample of all units in the United States, and the corresponding elasticity estimate. Suppose this broad group were broken down into sub-groups, say by the communities in which they reside, and separate regressions computed for each community. An appropriately weighted average of the corresponding elasticities should then be smaller than the elasticity for the group as a whole . . . As the groups are more rigorously defined, the elasticity should approach zero.

Eisner (1958) provided empirical support for Friedman's argument. With this theory and evidence, we can understand why some researchers would be dissuaded from looking for more disaggregate representations. Yet we believe it would be improper to generalize methodological findings based on models of demand to models of choice probabilities or market shares.

As noted by Goldberger (1971), the aggregation issue Friedman addressed has its psychometric parallel in the effects of explicit and incidental selection on regression parameters (cf. Lord and Novick 1968, pp. 140–148). We, on the other hand, seek a disaggregation in order to represent more precisely the subjective context surrounding the compensatory process in the evaluation of alternatives. Different individuals and groups consider different competitive sets of alternatives. They systematically differ in what benefits they desire from choice alternatives, and even in what they perceive the attributes of the alternatives to be. Representing the richness of the structure of individual differences in values and beliefs can add power to the statistical model rather than reduce the strength of the overall relation.

Issues in aggregation (cf. Roberts and Burstein 1980) are important. Many mistakes can be made if the issues are not thought through. The most basic issue deals with the proper specification of the explanatory model at the most disaggregate level of analysis. We wish to model the individual choice process as close as is practical to the level at which the phenomena occur, and then to aggregate the results to the level of consumer markets. For models of choice probabilities or market shares, we believe that multiplicative competitive interaction models with standardized explanatory variables will be useful.

CONCLUSION

We have tried to address the interrelated problems of representing the systematic structure of individual differences, and representing how psychological measures can be used in ratio scale, market share models. While the requirements of MCI models have guided much of our development, we anticipate our results will have wider applicability. Zeta-scores can be used for incorporating nominal scale, interval scale, or ratio scale measures into any multiplicative model. The use of zeta-scores in multinomial logit models seems particularly promising. Zeta-scores seem promising for use in any model that seeks or emphasizes competitive explanation. Similarly, the representation of the systematic structure of individual differences can benefit many models other than MCI models. Models of the "average man" can be improved upon if that average masks systematic differences. The search for structure can reveal groups of individuals that are internally homogeneous with respect to the evaluative process. For such groups, averages of explanatory variables do not systematically distort the representation of the process. In relating those averages to choice probabilities, zeta-scores can play a very useful role.

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