SOME PROPERTIES OF SYMMETRIC STABLE DISTRIBUTIONS

EUGENE F. FAMA AND RICHARD ROLL*
University of Chicago and Carnegie-Mellon University

This paper takes a few steps toward alleviating problems of data analysis that arise from the fact that elementary expressions for density and cumulative distribution functions (c.d.f.'s) for most stable distributions are unknown. In section 2 results of Bergstrom [3] are used to develop numerical approximations for the c.d.f.'s and the inverse functions of the c.d.f.'s of symmetric stable distributions. Tables of the c.d.f.'s and their inverse functions are presented for twelve values of the characteristic exponent. In section 3 the usefulness of the numerical c.d.f.'s and their inverse functions in estimating the parameters of stable distributions and testing linear models involving stable variables is discussed. Finally, section 4 presents a Monte Carlo study of truncated means as estimates of location. In every case but the Gaussian, some truncated mean is shown to have smaller sampling dispersion than the full mean.

1. STABLE DISTRIBUTIONS[1]

THE class of symmetric stable distributions may be defined by the log characteristic function,

$$\log_e \phi_x(t) = \log_e\left[\int_{-\infty}^{\infty} e^{itx} dF(x)\right] = i\delta t - \gamma \mid t \mid^\alpha = i\delta t - \mid ct \mid^\alpha, \tag{1}$$

where t is a real number, and F is the c.d.f.

A symmetric stable distribution has three parameters,[2] α, δ ,and $\gamma = c^\alpha$. The parameter $\alpha (0 < \alpha \leq 2)$ is called the characteristic exponent. The normal distribution is stable with $\alpha = 2$ and is the only stable distribution for which second and higher absolute moments exist. When $\alpha < 2$, absolute moments of order less than α exist, while those of order equal to and greater than α do not.

The parameter δ (δ = any finite real number) is the location parameter of a stable distribution, while c, ($\gamma = c^\alpha$), is the scale parameter. When $\alpha = 2$ (the normal distribution), γ is one-half the variance. For the Cauchy distribution ($\alpha = 1$), $\gamma = c$ is the semi-interquartile range.

Stable distributions have two important properties:

1) By definition, stable distributions are invariant under addition. That is, a sum of independent stable variables with characteristic exponents α, will be stable with the same exponent.

2) A distribution possesses a domain of attraction if and only if it is stable (cf. [7], or [9]). Thus the stable non-Gaussian distributions generalize the

---

* We have benefited from the comments of the editor, an associate editor, and several referees of this *Journal* as well as from the comments of our colleagues, F. J. Gould, R. Graves, S. J. Press, and especially H. V. Roberts. The research was supported in part by a grant from the Ford Foundation to the Graduate School of Business, University of Chicago and in part by a grant from the National Science Foundation.

1 The definition and original treatment of the class of stable distributions is due to Lévy [9]. References [7], [8], and [12] provide compact treatments of the available statistical theory, including derivations of the properties of stable distributions summarized here.

2 The stable class also includes asymmetric distributions. A fourth parameter (for skewness) is zero in the symmetric case.

Central Limit Theorem to the case where the second moments of the underlying variables do not exist.

This Generalized Central Limit Theorem accounts for the growing interest in stable distributions as data models, especially in economics. It says that if a sum of independent, identically distributed random variables has a limiting distribution, the limiting distribution will be a member of the stable class [7, p. 168, pp. 302–306]. Under these conditions, empirical variables which are sums of random variables may be expected to conform to stable laws. Some examples of such variables are common stock price changes, changes in other speculative prices, and interest rate changes. These have already been shown to have properties that conform closely to those of stable distributions, and in most cases to stable non-Gaussian distributions (cf. [6], [12], and [13]).

2. TABLES OF SYMMETRIC STABLE DISTRIBUTIONS

Consider the standardized variable[3]

$$u = \frac{x - \delta}{c}. \tag{2}$$

If x is symmetric stable with parameters α, δ, and $\gamma = c^{\alpha}$, it follows from (1) by application of standard properties of characteristic functions that u is stable with parameters α (unaffected by the transformation), $\delta = 0$, and $\gamma = c = 1$. The log characteristic function for this "standardized" symmetric stable variable is

$$\log_e \phi_u(t) = - |t|^{\alpha}. \tag{3}$$

Elementary expressions for the density and cumulative distribution functions of symmetric stable variables are unknown, except for the normal ($\alpha = 2$) and Cauchy ($\alpha = 1$) cases, but Bergstrom [3] presents series expansions which can be used to approximate densities corresponding to (3). When $\alpha > 1$, his results yield the convergent series,[4]

$$f_\alpha(u) = \frac{1}{\pi\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2k+1}{\alpha}\right)}{(2k)!} u^{2k}. \tag{4}$$

For $\alpha > 1$ Bergstrom also provides a finite series, which for $u > 0$ is

$$f_\alpha(u) = -\frac{1}{\pi} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha k + 1)}{u^{\alpha k + 1}} \sin\left(\frac{k\pi\alpha}{2}\right) + R(u), \tag{5}$$

where the remainder

$$R(u) = 0[u^{-\alpha(n+1)-1}]. \tag{6}$$

That is, for some positive constant M,

[3] In the remainder of this paper, "standardized" will always refer to the transformation of (2). In the normal case when γ is *one half* the variance, the standardization (2) does not correspond to the usual normal standardization of unit variance. Instead, (2) results in the "standardized" normal variable having $\sigma^2 = 2$.

[4] It is important to note that Bergstrom's definition of a standardized stable variable is only equivalent to (2) in the symmetric case. For a discussion see Lukacs [10], pp. 102–105.

$$|R(u)| < Mu^{-\alpha(n+1)-1}. \tag{7}$$

For large u equation (5) is asymptotic in the sense that for every n the error of approximation, $R(u)$, has smaller order of magnitude than the last term in the partial sum [3, p. 376].

Term by term integration of (4) yields a convergent series for the c.d.f. of a standardized, symmetric stable variable with $\alpha > 1$.

$$\begin{aligned} F_\alpha(u) &= \frac{1}{2} + \int_0^u f_\alpha(z)dz \\ &= \frac{1}{2} + \frac{1}{\pi\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma\left(\frac{2k-1}{\alpha}\right)}{(2k-1)!}u^{2k-1}. \end{aligned} \tag{8}$$

Similarly, integration of (5) yields, for large u, the asymptotic series

$$\begin{aligned} F_\alpha(u) &= 1 - \int_u^\infty f_\alpha(z)dz \\ &= 1 + \frac{1}{\pi}\sum_{k=1}^{n}(-1)^k\frac{\Gamma(\alpha k)}{k!u^{\alpha k}}\sin\left(\frac{k\pi\alpha}{2}\right) - \int_u^\infty R(u)du, \end{aligned} \tag{9}$$

where the remainder,

$$\int_u^\infty R(u)du = 0\left(\frac{u^{-\alpha(n+1)}}{\alpha(n+1)}\right). \tag{10}$$

Table 1 presents the c.d.f.'s of "standardized," symmetric stable distributions for twelve different values of α in the interval $1 \leq \alpha \leq 2$. The numbers are accurate to three decimal places and accurate except for rounding error in the fourth. Problems that arise in computing (8) or (9) and deciding which is better for given values of u and α are discussed briefly in the Appendix.

Table 1 provides a direct comparison of the shapes of symmetric stable distributions standardized according to (2). The distributions for $\alpha < 2$ depart from the corresponding normal distribution in the following ways. (a) For some range of u close to δ, a stable distribution with $\alpha < 2$ is more peaked (has higher densities) than the normal. (b) For $|u|$ "large," the stable distributions for $\alpha < 2$ have higher densities than the normal distribution, i.e., for $|u_0|$ large, $\Pr(|u| > |u_0|)$ is larger the smaller the value of α. (c) For some intermediate range of $|u|$, stable distribution with $\alpha < 2$ have lower densities than the normal distribution. These properties are illustrated in Figure 1 which presents the density functions of the normal and Cauchy distributions, standardized according to (2).

The differences between symmetric stable distributions with different values of α are further illustrated in Table 2 and Figure 2. Table 2 shows, for many levels of cumulative probability F, the fractiles $u(\alpha, F)$. For example, for the

TABLE 1. CUMULATIVE DISTRIBUTION FUNCTIONS $F_{\alpha}(U)$

U \ α	1.0	1.1	1.2	1.3	1.4	1.5
.05	.5159	.5153	.5150	.5147	.5145	.5144
.10	.5317	.5306	.5299	.5294	.5290	.5287
.15	.5474	.5458	.5447	.5439	.5434	.5430
.20	.5628	.5608	.5594	.5584	.5577	.5572
.25	.5780	.5756	.5740	.5728	.5719	.5713
.30	.5928	.5902	.5883	.5869	.5860	.5853
.35	.6072	.6044	.6024	.6009	.5998	.5991
.40	.6211	.6183	.6162	.6146	.6135	.6127
.45	.6346	.6318	.6297	.6281	.6270	.6262
.50	.6476	.6449	.6428	.6413	.6402	.6394
.55	.6601	.6576	.6557	.6542	.6532	.6524
.60	.6720	.6698	.6681	.6668	.6658	.6651
.65	.6835	.6817	.6802	.6790	.6782	.6776
.70	.6944	.6930	.6919	.6909	.6902	.6898
.75	.7048	.7039	.7031	.7025	.7020	.7017
.80	.7148	.7144	.7140	.7136	.7134	.7133
.85	.7242	.7244	.7244	.7244	.7244	.7245
.90	.7333	.7340	.7345	.7348	.7351	.7355
.95	.7418	.7432	.7441	.7449	.7455	.7461
1.00	.7500	.7519	.7534	.7545	.7555	.7563
1.10	.7651	.7682	.7707	.7727	.7744	.7759
1.20	.7789	.7831	.7865	.7894	.7919	.7940
1.30	.7913	.7965	.8010	.8048	.8080	.8108
1.40	.8026	.8088	.8142	.8188	.8228	.8263
1.50	.8128	.8194	.8261	.8316	.8364	.8406
1.60	.8222	.8300	.8370	.8433	.8487	.8536
1.70	.8307	.8393	.8470	.8539	.8600	.8655
1.80	.8386	.8477	.8560	.8635	.8702	.8763
1.90	.8458	.8554	.8643	.8723	.8795	.8861
2.00	.8524	.8625	.8719	.8802	.8879	.8950
2.20	.8642	.8750	.8850	.8941	.9025	.9103
2.40	.8743	.8856	.8961	.9057	.9146	.9228
2.60	.8831	.8948	.9055	.9155	.9246	.9331
2.80	.8908	.9027	.9136	.9236	.9329	.9415
3.00	.8976	.9096	.9205	.9306	.9399	.9484
3.20	.9036	.9156	.9265	.9365	.9457	.9542
3.40	.9089	.9209	.9318	.9417	.9507	.9590
3.60	.9138	.9257	.9365	.9462	.9550	.9631
3.80	.9181	.9299	.9406	.9501	.9587	.9665
4.00	.9220	.9338	.9442	.9536	.9619	.9694
4.40	.9289	.9403	.9504	.9593	.9672	.9742
4.80	.9346	.9458	.9555	.9640	.9714	.9778
5.20	.9395	.9504	.9597	.9678	.9747	.9807
5.60	.9438	.9543	.9633	.9709	.9774	.9830
6.00	.9474	.9576	.9663	.9736	.9797	.9848
7.00	.9548	.9643	.9721	.9786	.9839	.9882
8.00	.9604	.9692	.9764	.9821	.9868	.9905
10.00	.9683	.9760	.9820	.9868	.9905	.9934
15.00	.9788	.9847	.9891	.9923	.9947	.9965
20.00	.9841	.9888	.9923	.9947	.9965	.9977

FOR STANDARDIZED SYMMETRIC STABLE DISTRIBUTIONS

1.6	1.7	1.8	1.9	1.95	2.0
.5143	.5142	.5142	.5141	.5141	.5141
.5285	.5284	.5283	.5282	.5282	.5282
.5427	.5425	.5424	.5423	.5423	.5422
.5568	.5566	.5564	.5563	.5563	.5562
.5709	.5706	.5704	.5702	.5702	.5702
.5848	.5844	.5842	.5841	.5840	.5840
.5985	.5982	.5979	.5978	.5978	.5977
.6122	.6118	.6115	.6114	.6114	.6114
.6256	.6252	.6250	.6249	.6248	.6248
.6389	.6385	.6383	.6382	.6382	.6382
.6519	.6516	.6514	.6513	.6513	.6513
.6647	.6644	.6643	.6643	.6643	.6643
.6772	.6770	.6770	.6770	.6770	.6771
.6895	.6894	.6894	.6895	.6896	.6897
.7015	.7015	.7016	.7018	.7019	.7021
.7133	.7134	.7136	.7139	.7140	.7142
.7247	.7250	.7253	.7257	.7259	.7261
.7358	.7363	.7367	.7372	.7375	.7377
.7467	.7472	.7479	.7485	.7488	.7491
.7572	.7579	.7587	.7595	.7599	.7602
.7772	.7784	.7795	.7806	.7811	.7817
.7959	.7976	.7991	.8006	.8013	.8019
.8133	.8155	.8175	.8193	.8202	.8210
.8294	.8322	.8346	.8369	.8379	.8389
.8443	.8475	.8505	.8531	.8544	.8556
.8579	.8617	.8651	.8682	.8697	.8711
.8703	.8747	.8786	.8821	.8838	.8853
.8817	.8865	.8909	.8949	.8967	.8985
.8920	.8973	.9021	.9065	.9085	.9104
.9013	.9071	.9123	.9170	.9192	.9214
.9174	.9238	.9298	.9352	.9377	.9401
.9304	.9374	.9438	.9497	.9525	.9552
.9409	.9482	.9550	.9612	.9642	.9670
.9495	.9569	.9638	.9702	.9732	.9761
.9564	.9638	.9707	.9771	.9801	.9831
.9620	.9692	.9760	.9823	.9853	.9882
.9666	.9736	.9802	.9862	.9891	.9919
.9704	.9771	.9834	.9892	.9919	.9945
.9736	.9800	.9859	.9914	.9939	.9964
.9762	.9823	.9879	.9930	.9954	.9977
.9804	.9859	.9908	.9951	.9972	.9991
.9834	.9883	.9927	.9964	.9981	.9997
.9858	.9902	.9939	.9972	.9986	.9999
.9876	.9916	.9949	.9977	.9989	1.0000
.9891	.9927	.9956	.9980	.9991	1.0000
.9918	.9946	.9969	.9986	.9994	1.0000
.9935	.9958	.9976	.9990	.9995	1.0000
.9956	.9972	.9985	.9994	.9997	1.0000
.9977	.9986	.9993	.9997	.9999	1.0000
.9986	.9992	.9996	.9998	.9999	1.0000

TABLE 2. FRACTILES OF STANDARDIZED SYMMETRIC STABLE DISTRIBUTIONS

F \ α	1.0	1.1	1.2	1.3	1.4	1.5
.5200	.063	.065	.067	.068	.069	.070
.5400	.126	.131	.134	.136	.138	.139
.5600	.191	.197	.202	.205	.208	.210
.5800	.257	.265	.271	.275	.279	.281
.6000	.325	.334	.341	.347	.350	.353
.6200	.396	.406	.414	.420	.424	.427
.6400	.471	.481	.489	.495	.499	.502
.6600	.550	.560	.567	.573	.577	.580
.6800	.635	.643	.649	.654	.658	.660
.7000	.727	.732	.736	.739	.742	.743
.7200	.827	.828	.829	.830	.830	.830
.7400	.939	.932	.928	.926	.924	.921
.7600	1.065	1.048	1.037	1.030	1.024	1.018
.7800	1.209	1.179	1.158	1.143	1.131	1.122
.8000	1.376	1.327	1.293	1.268	1.249	1.235
.8200	1.576	1.505	1.447	1.409	1.380	1.358
.8400	1.819	1.709	1.628	1.571	1.528	1.496
.8600	2.125	1.964	1.847	1.762	1.700	1.653
.8800	2.526	2.290	2.122	1.996	1.905	1.837
.9000	3.078	2.729	2.480	2.297	2.161	2.061
.9200	3.695	3.366	2.964	2.708	2.503	2.351
.9400	5.242	4.379	3.774	3.331	3.002	2.763
.9500	6.314	5.165	4.370	3.798	3.369	3.053
.9600	7.916	6.319	5.230	4.453	3.882	3.448
.9700	10.579	8.189	6.596	5.476	4.659	4.049
.9750	12.706	9.651	7.645	6.251	5.240	4.485
.9800	15.895	11.802	9.164	7.359	6.063	5.099
.9850	21.205	15.300	11.589	9.100	7.341	6.043
.9900	31.820	22.071	16.160	12.313	9.659	7.737
.9950	63.657	41.348	28.630	20.775	15.595	11.983
.9995	636.609	334.595	193.989	120.952	79.556	54.337

.99 fractile of the standardized normal distribution,[5] $u(2, .99)=3.29$ whereas for the standardized Cauchy, $u(1, .99)=31.82$. Figure 2 presents the c.d.f.'s as normal probability plots.

3. THREE APPLICATIONS OF THE NUMERICAL C.D.F.'S AND THEIR INVERSE FUNCTIONS

The numerical approximations of the c.d.f. and of the inverse function of the c.d.f. have many potential uses in data analysis. Here we discuss three such uses.

a. *Estimating the Scale Parameter, c*

In Table 1 for $u=1$ we have $F_\alpha(\delta+c)$. Over the range $1\leq\alpha\leq 2$, $F_\alpha(\delta+c)$ increases monotonically, but only from .75 for $\alpha=1$ to .76 for $\alpha=2$. Since the distributions are symmetric, $F_\alpha(\delta-c)$ will decrease monotonically from .25 for $\alpha=1$ to .24 for $\alpha=2$. Thus the interval $\delta\pm c$ of a symmetric stable distribution is approximately the interquartile range, so that the standardization of

[5] Note again that the .99 fractile of the normal variable standardized according to (2) is 3.29 rather than the 2.326 it would be with the ordinary unit variance standardization.

TABLE 2. (*Continued*)

1.6	1.7	1.8	1.9	1.95	2.0
.070	.070	.071	.071	.071	.071
.140	.141	.141	.142	.142	.142
.211	.212	.213	.213	.214	.214
.283	.284	.285	.286	.286	.286
.355	.357	.357	.358	.358	.358
.429	.430	.432	.432	.432	.432
.504	.506	.506	.507	.507	.507
.581	.583	.583	.583	.583	.583
.661	.662	.662	.662	.661	.661
.744	.744	.743	.743	.742	.742
.830	.829	.828	.826	.825	.824
.919	.917	.915	.912	.911	.910
1.014	1.010	1.006	1.003	1.001	.999
1.115	1.108	1.102	1.097	1.095	1.092
1.223	1.213	1.204	1.197	1.194	1.190
1.341	1.326	1.314	1.304	1.299	1.295
1.471	1.450	1.433	1.419	1.413	1.407
1.616	1.587	1.564	1.544	1.536	1.528
1.785	1.744	1.711	1.684	1.672	1.662
1.985	1.927	1.880	1.843	1.827	1.813
2.237	2.150	2.084	2.030	2.007	1.988
2.581	2.444	2.341	2.261	2.228	2.199
2.816	2.638	2.505	2.404	2.363	2.327
3.127	2.687	2.708	2.576	2.522	2.477
3.577	3.234	2.980	2.795	2.722	2.661
3.901	3.478	3.160	2.933	2.846	2.772
4.357	3.799	3.394	3.104	2.996	2.905
5.056	4.283	3.728	3.330	3.191	3.070
6.285	5.166	4.291	3.670	3.461	3.290
9.332	7.290	5.633	4.375	3.947	3.643
37.967	26.666	18.290	11.333	7.790	4.653

(2) and (3) is equivalent to making the interquartile ranges approximately equal for stable distributions with different α's.

Though the semi-interquartile range provides a rough-and-ready measure of c, in large samples other interfractile ranges are less biased. When $\alpha > 1.8$, Table 3 shows that

$$u(\alpha, .75) < .96c.$$

Thus the asymptotic bias of the semi-interquartile range as an estimate of c can be more than 4 per cent of the true value. Now consider the estimator

$$\hat{c} = \frac{1}{.827(2)} \left[\hat{u}(\alpha, .72) - \hat{u}(\alpha, .28)\right], \tag{11}$$

where $\hat{u}$ is the sample fractile. In Table 3 the fractiles $u(\alpha, .72)$ all fall within the interval $.824 \leq u \leq .830$, so that for $1 \leq \alpha \leq 2$ the asymptotic bias in estimating c by means of the .72 and .28 sample fractiles is less than .4 per cent of the true value.[6]

[6] From Table 3 the minimum value of $[u(\alpha, .72) - u(\alpha, .28)]/2$ is .824, obtained when $\alpha = 2.0$. Since sample fractiles are asymptotically unbiased and consistent estimates of population fractiles, for $\alpha = 2.0$ the "large sample" value of $\hat{c}$, computed according to (11), is thus $.824/.827 = .9963$. Since $c = 1$, the asymptotic bias of $\hat{c}$ in this case is less than .4 per cent. Similarly, the maximum asymptotic value $\hat{c} = .830/.827 = 1.0036$ occurs when $1.3 \leq \alpha \leq 1.6$, and again the asymptotic bias is less than .4 per cent.

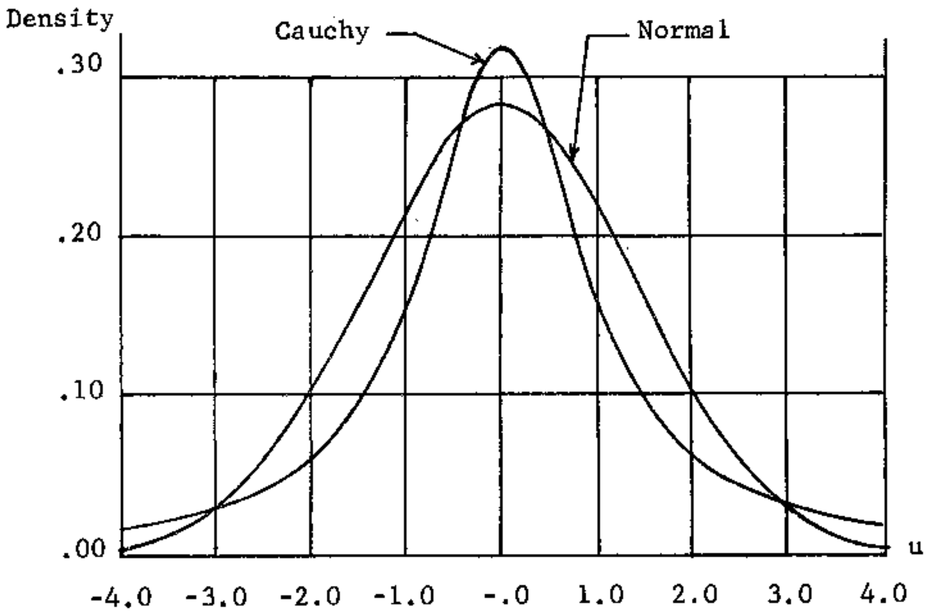

FIG. 1. Cauchy and Gaussian Density functions, standardized according to (2) and (3).

This discussion illustrates the usefulness of the numerical c.d.f. and the inverse of the c.d.f. Until now, measures of c were unavailable except for the normal and Cauchy cases. Of course, (11) may not ultimately be judged a "good" or "best" estimator. But later studies that will decide this interesting question are made possible by the availability of the numerical c.d.f. and its inverse function.

b. *Linear Functions of Stable Variables*

Table 1 is also useful for making probability statements about linear functions of symmetric stable variables. Let x_j, $j=1, 2, \cdots, N$, be independent, identically distributed with log characteristic function given by (1). Then

$$b = \sum_{j=1}^{N} a_j x_j \tag{12}$$

is symmetric stable with log characteristic function

$$\log_e \phi_b(t) = i\left(\delta \sum_{j=1}^{N} a_j\right) t - \gamma \sum_{j=1}^{N} | a_j |^\alpha | t |^\alpha. \tag{13}$$

For example, the sample mean $\bar{x}$ is a random function in the form of (12) with $a_j = 1/N$ for all j. Thus its distribution is symmetric stable with the same α, with $\delta(\bar{x}) = \delta$, and with $\gamma = \gamma/N^{\alpha-1}$.

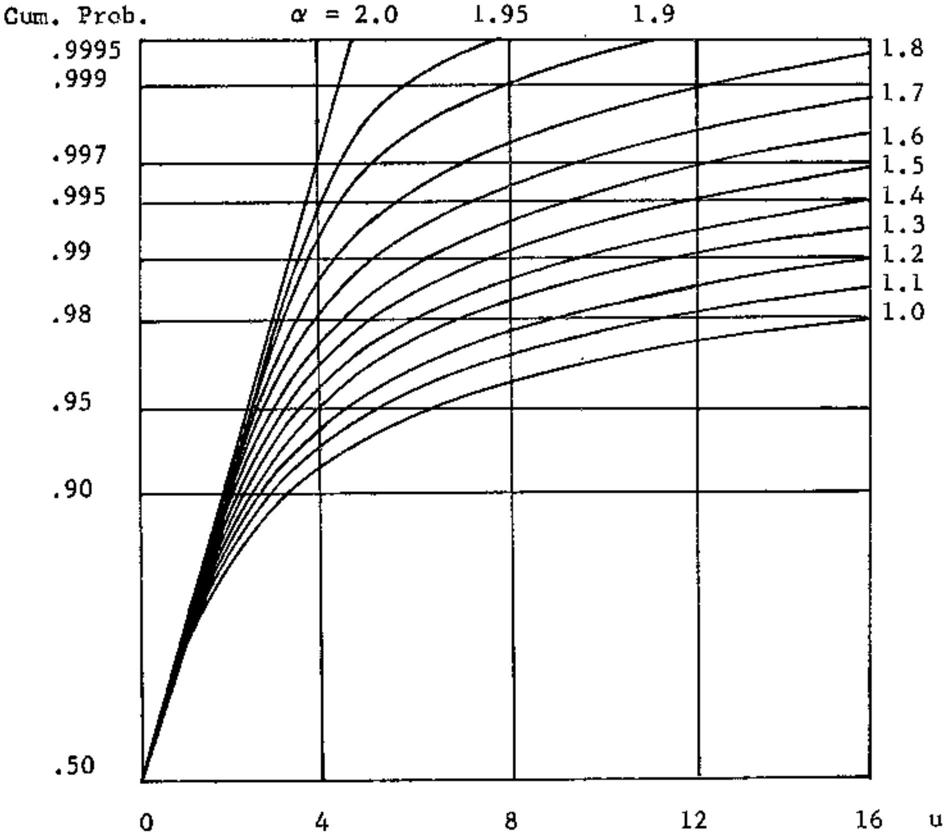

FIG. 2. Normal probability plots of standardized symmetric stable distributions.

TABLE 3. FRACTILES OF STANDARDIZED SYMMETRIC STABLE VARIABLE FOR VALUES OF THE C.D.F. IN THE INTERVAL $.70 \le F \le .75$.

$u(\alpha, F)$

F \ α	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
.70	.727	.732	.736	.739	.742	.743	.744	.744	.743	.743	.742
.71	.776	.779	.782	.784	.785	.786	.786	.786	.785	.784	.783
.72	.827	.828	.829	.830	.830	.830	.830	.829	.828	.826	.824
.73	.882	.879	.878	.877	.876	.875	.874	.872	.871	.869	.867
.74	.939	.932	.928	.926	.924	.921	.919	.917	.915	.912	.910
.75	1.000	.989	.982	.977	.973	.969	.966	.963	.960	.957	.954

Wise [15] considers the class of estimators of the form (12) for the coefficients b_k in

$$x_j = \sum_{k=1}^{K} b_k y_{jk} + \epsilon_j \qquad j = 1, 2, \cdots, N \tag{14}$$

where the y_{jk} are non-stochastic and the ϵ_j are independent, identically distributed with characteristic function given by (1), but with $\delta(\epsilon)=0$. All such estimators (including least squares as a special case) have symmetric stable distributions and their scale parameters are simple functions of $\gamma(\epsilon)$ and the weights a_j. Thus when $\gamma(\epsilon)$ is known, Wise's results, along with Table 1, can be used in standard ways to make inferences about the coefficients b_k in (14). When $\gamma(\epsilon)$ is unknown, of course, such probability statements will only be approximate, with the error of approximation smaller the larger N.

c. *Rough Measures of Skewness and of α*

Finally, in empirical work one would usually want to check (at least in a rough way) the assumption that the data are generated by a symmetric stable distribution. The following procedure suggests itself. First, the data points are standardized according to (2), using the sample mean, or, better, one of the other measures to be discussed in the next section to estimate δ and (11) to estimate c. Both tails of the standardized data distribution can then be plotted on Figure 2. The coincidence of the data plots with the theoretical curves of Figure 2 will provide a check on symmetry as well as a rough measure of α (and thus of the degree of departure from normality in the data distribution).

4. SAMPLING PROPERTIES OF TRUNCATED MEANS AS ESTIMATES OF δ

The stability or invariance under addition of stable variables makes it easy to derive the distributions of random functions of the form (12). But there are many interesting functions that do not have this form and the absence of elementary expressions for the density and c.d.f.'s of stable variables makes it difficult to derive analytically the properties of such functions. Given the numerical c.d.f. and its inverse function, however, it is possible to study these functions with Monte Carlo techniques.[7] Here we shall use the Monte Carlo approach to examine the sampling properties of truncated means[8] as estimators of the location parameter δ.

a. *The Data*

The data for the Monte Carlo study were generated as follows. First, 30,401 "cumulative probabilities" (i.e., random numbers from the uniform distribution $U(0, 1)$) were generated. For each randomly chosen cumulative probability U, the numerical inverse function $u(\alpha, U)=F_\alpha^{-1}(U)$ was used to obtain the seven standardized, symmetric stable random deviates corresponding to $\alpha=1.0$, 1.1, 1.3, 1.5, 1.7, 1.9, 2.0. The result is seven samples of 30,401

[7] A limitation of the Monte Carlo approach is that it will usually not *suggest* good methods of estimation. Rather it is limited to establishing sampling properties of estimates chosen by the experimenter.

[8] The g truncated sample mean is the average of the middle $100g$ per cent of the ordered observations in the sample. That is, in computing the mean, $100(1-g)/2$ per cent of the observations in each tail of the data distribution are discarded.

observations, but the samples are not independent since the i'th observation in each corresponds to the same cumulative probability. This is actually an advantage since it facilitates comparisons of the sampling properties of a given estimator for different values of α.[9]

To study the sampling properties of truncated means, the seven samples were subdivided, first into samples of size 15 and then into samples of sizes 21, 51, and 101. For *each* value of α there are 2026 subsamples of 15, 1447 subsamples of 21, 596 subsamples of 51, and 301 subsamples of 101 observations each. Next, the sample mean, median and three truncated means, corresponding approximately to the middle 25, 50, and 75 per cent of the ordered sample observations, were calculated for each of the subsamples. For each combination of α and sample size n the frequency distribution of each estimator was tabulated and various summary statistics for the distributions were computed.

The results are presented in Tables 4–7. Each line of Table 4 summarizes the Monte Carlo distribution obtained when samples of size $n=15$ from a stable distribution with characteristic exponent α were used to get $N=2026$ estimates of δ by a particular estimation procedure E. Columns (2) and (3) show two measures of location: the mean ($\bar{E}$) and the median ($\hat{M}_E$) of the Monte Carlo distribution of E. Columns (4)–(8) present five measures of dispersion: the standard deviation $s(E)$, and the four interfractile ranges, $R.5$, $R.8$, $R.9$ and $R.98$, which correspond respectively to one-half of the range of the estimates covering the middle 50, 80, 90 and 98 per cent of the Monte Carlo distribution of E. Thus, $R.5$ is one-half the difference between the estimated values of E at the .75 and .25 fractiles of its empirical distribution. In column (1) of Table 4 we see that for each of the seven values of α, five estimators (E's) are examined—the median, truncated means corresponding to the averages of the middle 3, 7, and 11 observations in each sample of $n=15$, and the sample mean. Tables 5–7 present the same analysis of Monte Carlo sampling distributions as Table 4, but for samples of size $n=21$ (Table 5), $n=51$ (Table 6), and $n=101$ (Table 7). The numbers of observations used in computing truncated means also change with the sample size, but for each n the numbers chosen correspond approximately to the .25, .50, and .75 truncated means.

b. *Analysis of Results*

Except for the mean in the Cauchy case ($\alpha=1$) all of the estimators in Tables 4–7 are at least asymptotically unbiased. Thus it is not surprising that

[9] The Monte Carlo results will depend on whether the 30,401 cumulative probabilities (a) were independently generated and (b) conform well to $U(0, 1)$. Both conditions seem to be met by the data. The first order serial correlation coefficient of the cumulative probabilities is $-.009$. The population mean, variance, second moment and third moment for $U(0, 1)$ are, respectively, .5, .0833, .3333 and .25; the smple values are .5013, .0830, .3343, and .2506. Finally, the interval (0, 1) has been divided into twenty equal subintervals of length .05. If n_i, $i=1, 2, \cdots, 20$, is the actual number of observations in each interval, and if the data are in fact a random sample from $U(0, 1)$, then

$$X^2 = \frac{\sum_{i=1}^{20} (n_i - 1520.05)^2}{1520.05}$$

has a chi-square distribution with 19 degrees of freedom. The observed value is 19.457—just slightly above the median (18.3) of the chi-square distribution.

TABLE 4. SUMMARY STATISTICS FOR DISTRIBUTIONS OF ESTIMATORS OF δ: $n=15$, $N=2026$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
E	Sampling Dist of E	$\bar{E}$	$\hat{M}_E$	s(E)	R.5	R.8	R.9	R.98
$\alpha = 1.0$	Median	.0098	.0164	.449	.276	.535	.730	1.166
	3	.0096	.0079	.434*	.264*	.519*	.713*	1.116*
	7	.0084	.0007	.469	.270	.550	.749	1.239
	11	.0108	.0038	.645	.345	.728	.999	1.797
	Mean	-.1399	-.0225	21.810	1.014	3.103	6.205	32.728
$\alpha = 1.1$	Median	.0099	.0170	.449	.285	.545	.735	1.141
	3	.0097	.0083	.432*	.270	.525*	.713*	1.079*
	7	.0083	.0020	.448	.264*	.538	.726	1.153
	11	.0100	.0024	.562	.318	.654	.888	1.523
	Mean	-.3242	-.0193	12.588	.787	2.147	4.009	18.101
$\alpha = 1.3$	Median	.0100	.0178	.451	.296	.559	.743	1.110
	3	.0099	.0087	.429	.277	.534	.714	1.043*
	7	.0082	.0018	.425*	.265*	.527*	.698*	1.054
	11	.0089	.0022	.468	.288	.571	.752	1.196
	Mean	-.1890	-.0070	5.001	.519	1.201	2.010	6.813
$\alpha = 1.5$	Median	.0102	.0183	.452	.302	.566	.746	1.092
	3	.0099	.0092	.428	.281	.538	.714	1.024
	7	.0081	.0014	.413*	.262*	.516*	.672*	1.000*
	11	.0084	-.0007	.422	.269	.529	.688	1.028
	Mean	-.0625	.0023	1.886	.390	.815	1.214	3.309
$\alpha = 1.7$	Median	.0102	.0185	.451	.305	.569	.747	1.080
	3	.0100	.0093	.427	.284	.540	.710	1.015
	7	.0080	.0029	.405	.259	.512	.664	.976
	11	.0080	.0012	.397*	.254*	.505*	.654*	.953*
	Mean	-.0058	-.0032	.740	.313	.607	.861	1.692
$\alpha = 1.9$	Median	.0102	.0185	.450	.306	.569	.746	1.070
	3	.0099	.0093	.425	.284	.541	.706	1.005
	7	.0080	.0028	.400	.258	.507	.657	.957
	11	.0078	.0005	.382*	.248*	.487*	.633*	.917*
	Mean	.0047	-.0056	.424	.263	.497	.653	1.023
$\alpha = 2.0$	Median	.0102	.0185	.449	.306	.569	.744	1.065
	3	.0099	.0093	.424	.285	.540	.703	1.000
	7	.0080	.0028	.398	.257	.505	.654	.951
	11	.0077	.0008	.377	.245	.483	.622	.905
	Mean	.0048	-.0026	.364*	.242*	.461*	.586*	.886*

TABLE 5. SUMMARY STATISTICS FOR DISTRIBUTIONS OF ESTIMATORS OF δ: $n=21$, $N=1447$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
E	Sampling Dist of E	$\bar{E}$	$\hat{M}_E$	s(E)	R.5	R.8	R.9	R.98
$\alpha = 1.0$	Median	.0080	.0102	.364	.235	.443	.582*	.905
	5	.0071	-.0022	.357*	.231*	.435*	.584	.895*
	11	.0101	.0015	.391	.240	.485	.632	1.006
	17	.0131	-.0074	.607	.325	.677	.968	1.597
	Mean	-.1400	.0065	18.287	1.051	2.976	6.433	31.522
$\alpha = 1.1$	Median	.0083	.0105	.368	.242	.454	.592	.901
	5	.0074	-.0021	.357*	.235*	.442*	.586*	.884*
	11	.0097	.0012	.375	.237	.469	.605	.953
	17	.0113	-.0058	.516	.298	.599	.832	1.313
	Mean	-.3243	.0014	10.582	.780	2.061	3.971	17.397
$\alpha = 1.3$	Median	.0086	.0110	.374	.252	.468	.604	.897
	5	.0077	-.0022	.359	.240	.450*	.587	.857*
	11	.0091	.0022	.355*	.232*	.453	.584*	.867
	17	.0092	-.0046	.413	.260	.507	.670	1.014
	Mean	-.1891	-.0027	4.210	.501	1.123	1.878	6.916
$\alpha = 1.5$	Median	.0087	.0113	.377	.257	.475	.611	.894
	5	.0079	-.0024	.359	.244	.454	.588	.845
	11	.0088	.0019	.345*	.230*	.441*	.570*	.824*
	17	.0081	.0010	.363	.237	.454	.599	.861
	Mean	-.0625	-.0053	1.584	.359	.765	1.099	3.087
$\alpha = 1.7$	Median	.0088	.0114	.378	.260	.478	.613	.891
	5	.0080	-.0025	.359	.245	.457	.588	.839
	11	.0087	.0016	.338	.229	.436	.560	.798
	17	.0074	-.0005	.335*	.224*	.428*	.553*	.774*
	Mean	-.0058	-.0051	.618	.278	.544	.752	1.420
$\alpha = 1.9$	Median	.0089	.0115	.378	.261	.479	.614	.887
	5	.0081	-.0026	.358	.245	.457	.587	.833
	11	.0085	.0018	.334	.228	.432	.553	.780
	17	.0071	-.0025	.320*	.216*	.410*	.523*	.735*
	Mean	.0046	-.0094	.356	.223	.431	.571	.835
$\alpha = 2.0$	Median	.0089	.0015	.378	.262	.480	.614	.885
	5	.0081	-.0026	.358	.245	.457	.585	.830
	11	.0085	.0021	.332	.227	.430	.551	.775
	17	.0070	-.0011	.314	.214	.404	.515	.720
	Mean	.0048	-.0042	.307*	.207*	.391*	.503*	.708*

TABLE 6. SUMMARY STATISTICS FOR DISTRIBUTIONS OF ESTIMATORS OF δ: $n=51$, $N=596$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
E	Sampling Dist of E	$\bar{E}$	$\hat{M}_E$	s(E)	R.5	R.8	R.9	R.98
α = 1.0	Median	.0111	.0159	.223	.141	.279	.380	.539*
	13	.0110	.0119	.218*	.137*	.266*	.346*	.570
	25	.0109	.0138	.232	.144	.274	.367	.601
	37	.0103	.0147	.289	.186	.344	.454	.721
	Mean	-.1398	-.0822	11.838	1.017	3.291	6.645	38.563
α = 1.1	Median	.0114	.0164	.230	.146	.287	.391	.549*
	13	.0112	.0122	.222*	.140*	.270*	.352*	.571
	25	.0106	.0138	.228	.143	.270*	.359	.590
	37	.0098	.0084	.267	.176	.323	.420	.665
	Mean	-.3241	-.0467	6.843	.705	2.056	3.768	24.766
α = 1.3	Median	.0117	.0172	.237	.153	.298	.404	.562*
	13	.0114	.0132	.226	.145	.277	.359	.573
	25	.0103	.0118	.223*	.141*	.267*	.356*	.577
	37	.0089	.0119	.238	.160	.293	.384	.587
	Mean	-.1892	.0105	2.721	.403	.976	1.617	10.190
α = 1.5	Median	.0120	.0177	.241	.156	.304	.411	.569
	13	.0115	.0135	.228	.148	.280	.361	.574
	25	.0101	.0116	.221*	.141*	.264*	.350*	.568
	37	.0085	.0145	.223	.151	.274	.363	.557*
	Mean	-.0625	.0104	1.030	.256	.585	.818	3.590
α = 1.7	Median	.0120	.0178	.243	.158	.307	.414	.572
	13	.0115	.0134	.229	.149	.282	.362	.574
	25	.0100	.0110	.219	.141*	.262*	.349*	.563
	37	.0083	.0155	.214*	.144	.267	.349*	.529*
	Mean	-.0057	.0136	.406	.180	.383	.526	1.128
α = 1.9	Median	.0121	.0179	.244	.159	.309	.416	.573
	13	.0115	.0136	.229	.149	.282	.362	.573
	25	.0099	.0114	.217	.141*	.260	.348	.559
	37	.0081	.0153	.209*	.141*	.259*	.337*	.511*
	Mean	.0048	.0114	.233	.141*	.281	.375	.577
α = 2.0	Median	.0121	.0179	.244	.159	.309	.416	.573
	13	.0115	.0137	.229	.149	.282	.362	.573
	25	.0099	.0115	.216	.140	.259	.347	.557
	37	.0080	.0150	.207	.139	.258	.332	.504
	Mean	.0049	.0087	.200*	.130*	.253*	.325*	.494*

TABLE 7. SUMMARY STATISTICS FOR DISTRIBUTIONS OF ESTIMATORS OF δ: $n=101$, $N=301$

	Sampling Dist of E \ E	(2) $\bar{E}$	(3) $\hat{M}_E$	(4) $s(E)$	(5) R.5	(6) R.8	(7) R.9	(8) R.98
$\alpha = 1.0$	Median	.0085	.0108	.149	.098	.199	.250	.330
	25	.0079	.0138	.141*	.092*	.177*	.238*	.325*
	51	.0100	.0039	.148	.103	.179	.245	.343
	75	.0096	.0106	.184	.116	.240	.315	.416
	Mean	-.1397	-.0020	8.359	.874	3.029	5.777	25.721
$\alpha = 1.1$	Median	.0088	.0111	.155	.102	.206	.258	.340
	25	.0081	.0138	.146*	.094*	.181	.244	.331*
	51	.0099	.0052	.147	.103	.177*	.243*	.335
	75	.0092	.0084	.171	.109	.222	.288	.383
	Mean	-.3243	.0103	4.813	.574	1.707	3.221	15.748
$\alpha = 1.3$	Median	.0092	.0117	.162	.106	.214	.268	.352
	25	.0084	.0132	.150	.096*	.187	.251	.338
	51	.0096	.0069	.145*	.102	.176*	.241*	.321*
	75	.0085	.0088	.154	.103	.199	.258	.342
	Mean	-.1897	-.0009	1.902	.305	.766	1.330	6.338
$\alpha = 1.5$	Median	.0094	.0120	.165	.109	.219	.274	.359
	25	.0086	.0129	.152	.097*	.190	.253	.343
	51	.0095	.0078	.144*	.101	.176*	.236*	.317*
	75	.0081	.0075	.145	.099	.185	.243	.321
	Mean	-.0628	-.0061	.713	.185	.441	.656	2.256
$\alpha = 1.7$	Median	.0096	.0121	.167	.110	.221	.277	.362
	25	.0087	.0126	.153	.097	.191	.255	.344
	51	.0094	.0077	.143	.100	.175*	.234	.315
	75	.0079	.0049	.140*	.094*	.178	.232*	.308*
	Mean	-.0058	.0160	.275	.133	.271	.375	.765
$\alpha = 1.9$	Median	.0096	.0121	.167	.110	.223	.278	.364
	25	.0087	.0125	.153	.097	.192	.255	.345
	51	.0093	.0076	.142	.100	.174	.231	.314
	75	.0077	.0059	.137*	.091*	.173*	.225*	.300*
	Mean	.0048	.0112	.154	.094	.185	.242	.361
$\alpha = 2.0$	Median	.0096	.0121	.168	.111	.223	.278	.364
	25	.0087	.0125	.153	.097	.192	.255	.345
	51	.0093	.0075	.142	.099	.174	.230	.313
	75	.0077	.0046	.135	.090	.171	.222	.296*
	Mean	.0050	.0086	.131*	.082*	.167*	.211*	.297

the location parameters $\bar{E}$ and $\hat{M}_E$ of the Monte Carlo distributions of the estimators examined are trivially different from the true value $\delta=0$.[10]

We turn now to the question: Among the estimators of the location parameter δ examined in Tables 4–7, which is "best" for given values of α and n? By "best," we mean that the Monte Carlo distribution of the estimator has *minimum dispersion as measured by the four interfractile range,* $R.5$, $R.8$, $R.9$, and $R.98$, reported in columns (5)–(8) of Tables 4–7. To facilitate identification of the "best" estimators in the tables, for each α and n the lowest value of each interfractile range is starred.

For the Cauchy distribution ($\alpha=1$), the "best" estimator of those considered is the .25 truncated mean.[11] For $n=15$ and $n=101$, the .25 truncated mean is uniformly less disperse (i.e., has smaller $R.5$, $R.8$, $R.9$, and $R.98$) than any of the other estimators. For $n=21$ and $n=51$, three of the four interfractile ranges in the Monte Carlo distributions of the .25 truncated mean are less than the corresponding ranges for any of the other estimators. For higher values of the characteristic exponent α, however, it is better to average an increasingly larger proportion of the central observations in estimating δ. When $\alpha=1.1$, the .25 truncated means are still dominant for all n. For $\alpha=1.3$ and $\alpha=1.5$ the .50 truncated means are generally better than the other estimators, but when $\alpha=1.7$ the .75 truncated means are generally best, and when $\alpha=1.9$ the distributions of the .75 truncated means are uniformly less disperse than those of the other estimators. Finally, when the generating process is Gaussian ($\alpha=2$), the mean is the "best" estimator. Of course, it is also minimum-variance, unbiased in this case.

But in applications, α is likely to be unknown. Fortunately, the .5 truncated mean performs very well over the entire range $1\le\alpha\le 2$. If it were used exclusively to estimate δ, the increases in sampling dispersion relative to a "best" truncated mean would occur at the extremes, i.e., when the true value of α is either 1.0 or 2.0. Even for these values of α, the interfractile ranges of the distributions of the .50 truncated mean are on average less than 10 per cent larger than the corresponding ranges of the "best" estimators (the mean when $\alpha=2$ and the .25 truncated mean when $\alpha=1$). For intermediate values of α, increases in sampling dispersion (if any) from using the .50 truncated mean are trivial, usually less than 2–3 per cent.

---

[10] Any observed consistency in the deviations of $\bar{E}$ and $\hat{M}_E$ from 0 can be traced to the underlying sample of cumulative probabilities which departs slightly from $U(0, 1)$ in ways that any random sample (even one of 30,401) can be expected to depart from the population distribution from which it is drawn. For example, except for the mean, $\bar{E}$ and $\hat{M}_E$ are generally positive for all the estimators. In the sample of 30,401 from $U(0, 1)$, 15,344 observations are greater than .5, while 15,057 are less. Moreover, the intervals in the range .5–.65 account for much of the "excess" of observations to the right of .5, and these intervals have heavy representation in computing medians and truncated means.

On the other hand, there are 1539 cumulative probabilities in the interval 0–.05, an excess of 18.95 over the expected number 1520.05, while the interval .95–1 contains 1525 observations, just 4.95 more than expected. For low values of α this very slight asymmetry in the extreme tails of the random sample from $U(0, 1)$ is sufficient to swing the location parameters of the distributions of the mean to the left of 0. That this phenomenon is due to a very few observations in the extreme left tail of the sample from $U(0, 1)$ is also indicated by the fact that for low values of α the medians ($\hat{M}_E$) of the Monte Carlo distributions of the mean are much closer to 0 than the means ($\bar{E}$.)

[11] This agrees with the analytic results of Rothenberg, *et al.* [14] for the asymptotic case. The results in [14] for truncated means in Cauchy samples have recently been extended in [2] and [4], which consider more general estimators based on the order statistics. Since our purpose is simply to illustrate the usefulness of the Monte Carlo approach in studying sampling properties of estimators involving stable variables, we feel justified in limiting attention to the class of truncated means.

Finally, of the estimators considered in Tables 4–7, the sampling distribution of the mean is most sensitive to the value of α. Also, when $\alpha \leq 1.7$ the sample mean is dominated in rather striking fashion by each of the other estimators considered. This is not the case, however, when α is close to two.

c. *The "Asymptotic" Normality of the Estimators*

Finally, medians and truncated means of samples from symmetric stable distributions are *asymptotically* normal. But, it is not known how fast normality is approached nor how large a sample size is necessary for the limiting property to hold, at least as an approximation. This section presents some answers to these problems.[12]

For each combination of α and n the Monte Carlo frequency distributions of the median and each of the truncated means have been tabulated. The Monte Carlo distributions were subdivided into 22 intervals. The middle 20 intervals are each of length .25 $s(E)$ and cover the range $\overline{E} \pm 2.5\ s(E)$, where $\overline{E}$ and $s(E)$ are the sample mean and standard deviation of the estimator for the particular combination of α and n being considered. The two remaining intervals cover, of course, the ranges $E < \overline{E} - 2.5\ s(E)$ and $E > \overline{E} + 2.5\ s(E)$. For each frequency distribution the chi-square statistic

TABLE 8. CUMULATIVE CHI-SQUARE DISTRIBUTION FOR 19 DEGREES OF FREEDOM

$F(\chi^2)$	.250	.500	.750	.900	.950	.975	.990	.995
χ^2	14.6	18.3	22.7	22.7	30.1	32.9	36.2	38.6

$$X^2 = \sum_{i=1}^{22} \frac{(A_i - \epsilon_i)^2}{\epsilon_i}$$

was computed. A_i is the actual number of values of the estimator in interval i of its frequency distribution and ϵ_i is the expected number if the distribution were Gaussian. Since two parameters $[\overline{E}$ and $s(E)]$ estimated from the data were used in computing ϵ_i, there are $22-2-1=19$ degrees of freedom. The cumulative chi-square distribution for 19 degrees of freedom is shown in Table 8. The values of the chi-square statistic for the Monte Carlo distributions of estimators of δ are shown in Table 9.

The results in Table 9 indicate that distributions of medians and truncated means are close to normal even in small samples. The .95 fractile of the chi-square distribution with 19 degrees of freedom is 30.1. For $n > 15$ the sample chi-square statistics for the medians and .25 truncated means are all less than 30.1, and even for $n = 15$ high chi-square values occur only for the lowest values of α. In general, for $n > 15$ the remaining sample values of the chi-square statistic for the medians and .25 truncated means are close probabilistically to 18.3, the median of the chi-square distribution with 19 degrees of freedom. Similar

[12] These experiments can tell us nothing new about the sample mean. Its distribution is known exactly. See section 3-b.

TABLE 9. SAMPLE VALUES OF CHI-SQUARE STATISTICS TESTS OF NORMALITY OF TRUNCATED MEANS WHEN UNDERLYING VARIABLES ARE SYMMETRIC STABLE

Medians

n (N) \ α	1.0	1.1	1.3	1.5	1.7	1.9	2.0
15 (2026)	54.31	39.96	19.17	12.13	15.30	17.32	18.31
21 (1447)	24.88	23.15	17.45	16.69	16.18	19.16	17.42
51 (596)	22.19	21.53	24.78	24.59	24.30	25.94	25.94
101 (301	19.41	19.48	17.84	17.02	17.02	17.02	17.02

.25 Truncated Means

n (N) \ α	1.0	1.1	1.3	1.5	1.7	1.9	2.0
15 (2026)	63.65	47.23	31.60	22.78	21.53	20.63	19.06
21 (1447)	26.95	19.46	16.52	17.92	19.59	19.01	17.52
51 (596)	25.58	28.05	25.56	25.36	26.69	25.60	25.60
101 (301)	19.66	19.62	19.83	16.84	16.61	16.61	17.13

.50 Truncated Means

n (N) \ α	1.0	1.1	1.3	1.5	1.7	1.9	2.0
15 (2026)	87.64	72.03	42.78	28.11	21.72	23.33	21.90
21 (1447)	56.36	41.6	25.9	18.21	17.99	16.93	17.57
51 (596)	21.15	23.79	24.78	29.18	24.77	27.59	29.21
101 (301)	24.34	29.49	21.60	19.83	18.67	16.20	18.74

.75 Truncated Means

n (N) \ α	1.0	1.1	1.3	1.5	1.7	1.9	2.0
15 (2026)	160.54	117.34	48.90	23.28	21.01	19.75	20.50
21 (1447)	100.72	60.73	27.43	12.65	12.03	9.61	7.30
51 (596)	15.50	16.17	14.94	15.48	20.02	16.43	17.94
101 (301)	13.71	18.03	12.02	9.40	6.96	5.80	5.95

statements hold for the .50 and .75 truncated means, but here a combination of small α and n results in a greater departure from normality.[13]

APPENDIX

This appendix briefly discusses the numerical techniques used to approximate the c.d.f.'s and the inverse functions of the c.d.f.'s of symmetric stable distributions, standardized according to (2). The actual FORTRAN II computer subroutines are available on request.

[13] We thank the editor for pointing out that our calculation of the χ^2 statistic results in underestimating the probability of a Type I error. This is so because the mean and variance were estimated from the original sample rather than from grouped data. The size of this error is evidently very small with our sample sizes and numbers of classes [5]. Additionally, since the defect increases the probability that the hypothesis of normality will be rejected, it strengthens our conclusion that most of the sample medians and truncated means *are* normally distributed.

Bergstrom [3] indicates that the finite series (9) is appropriate for large absolute values of the standardized stable variate u, while the convergent formula (8) is appropriate for u close to zero. The only numerical analysis problems we had to consider were (a) increasing the accuracy of the formulas in the intermediate ranges of u, and (b) determining which formula to use for different combinations of α and u.

Problems in obtaining accurate approximations of the c.d.f.'s arose primarily because for large arguments the value of the gamma function $\Gamma(\cdot)$, which appears in both (8) and (9), exceeds the maximum allowed by the IBM 7094. We circumvented this difficulty in the following way. Note that the k'th term in the infinite sum of (8) can be written as

$$T_k = -T_{k-1}\left(\frac{u^2}{(2k-1)(2k-2)}\right)\frac{\Gamma\left(\frac{2k-1}{\alpha}\right)}{\Gamma\left(\frac{2k-3}{\alpha}\right)}, \qquad \left[T_1 = u\Gamma\left(\frac{1}{\alpha}\right)\right].$$

The ratio of gamma functions may be calculated from the asymptotic expansion[14]

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2z} + \frac{1}{12}\binom{a-b}{2}\left[3(a+b-1)^2 - (a+b-1)\right]\frac{1}{z^2} + \cdots.$$

This allowed us to use (8) to estimate the c.d.f.'s to a very high level of accuracy.

The problem of deciding exactly when to use (8), which is good when $|u|$ is "small," and (9), which is good when $|u|$ is "large," was solved by examining estimated cumulative probabilities produced by the two series for a wide range of values of α and u. Except for α close to 1, the estimates agreed to at least five decimal places over a wide range of values of u. For α close to 1, the agreement extended only over a narrow range of u, along the line $|u| = -4+5\alpha$. Since this was also within the range of overlap for higher values of α, the cutoff criterion chosen is to use (9) if $|u| > -4+5\alpha$ and (8) elsewhere.

Finally, the iterative procedure used to determine $u(\alpha, F)$ is as follows.

1. Make a first approximation Z to $u(\alpha, F)$ by taking a weighted average of the F fractiles of the Cauchy and Gaussian distributions.
2. If $|Z| > -4+5\alpha$ refine it by using the polynomial inverse of the first four terms of the finite series (9).
3. Iterate as follows:
   (a) Compute $F - F_\alpha(Z)$.
   (b) Change Z according to

$$\Delta Z = \frac{F - F_\alpha(Z)}{d}$$

[14] Formula 6.1.47 in Abramowitz and Stegun [1, p. 257]. The second set of parentheses inside the brackets are missing in the formula Abramowitz and Stegun report.

where d is a weighted average of the Cauchy and Gaussian densities evaluated at the point Z.

(c) Return to (a) and repeat the process until $F - F_\alpha(Z) < .0001$. The procedure rarely requires more than three iterations.

REFERENCES

[1] Abramowitz, Milton, and Stegun, Irene A., *Handbook of Mathematical Functions* (Washington: U. S. Government Printing Office, December, 1965).

[2] Barnett, V. D., "Order Statistics Estimators of the Location of the Cauchy Distribution," *Journal of the American Statistical Association*, LXI (1966), 1205–18.

[3] Bergstrom, Harald, "On Some Expansions of Stable Distributions," *Arkiv for Matematik*, II (1952), 375–78.

[4] Bloch, Daniel, "A Note on the Estimation of the Location Parameter of the Cauchy Distribution," *Journal of the American Statistical Association*, LXI (1966), 852–55.

[5] Chernoff, Herman and Lehmann, E. L., "The Use of Maximum Likelihood Estimates in χ^2 Tests for Goodness of Fit," *Annals of Mathematical Statistics*, XXV (1954), 579–586.

[6] Fama, Eugene F., "The Behavior of Stock-Market Prices," *Journal of Business*, XXXVIII (January, 1965), 34–105.

[7] Feller, William, *An Introduction to Probability Theory and Its Applications*, Vol. II (New York: John Wiley and Sons, 1966), chs. 6 and 17.

[8] Gnedenko, B. V. and Kolmogorov, A. N., *Limit Distributions for Sums of Independent Random Variables*, trans. by K. L. Chung (Reading, Mass.: Addison-Wesley, 1954), ch. 7.

[9] Lévy, Paul, *Calcul des Probabilities* (Pairs: Gauthier-Villars, 1925), Part II, ch. 6.

[10] Lukacs, Eugene, *Characteristic Functions* (London: Charles Griffin and Co., 1960), 97–107.

[11] Mandelbrot, Benoit, "The Stable Paretian Income Distribution When the Apparent Exponent Is Near Two," *International Economic Review*, IV (January, 1963), 111–14.

[12] Mandelbrot, Benoit, "The Variation of Certain Speculative Prices," *Journal of Business*, XXXVI (1963), 394–419.

[13] Roll, Richard. "The Efficient Market Model Applied to U. S. Treasury Bill Rates" (unpublished doctoral dissertation, Graduate School of Business, University of Chicago, 1968).

[14] Rothenberg, T. J., Fisher, F. M., and Tilanus, C. B., "A Note on Estimation from a Cauchy Sample," *Journal of the American Statistical Association*, LIX (June, 1964), 460–63.

[15] Wise, John, "Linear Estimators for Linear Regression Systems Having Infinite Residual Variances" (unpublished paper presented to the Berkeley-Stanford Mathematical Economics Seminar, October, 1963).