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EVIDENCE ON THE "GROWTH-OPTIMUM" MODEL

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I. INTRODUCTION

A PORTFOLIO OWNER may hope to maximize the long run growth rate of his real wealth. In 1959, Latané suggested maximum growth as an operational criterion for portfolio selection, contending that its (possible) suboptimality on theoretical grounds was practically unimportant and emphasizing Roy's [1952] warning that "A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility."

In the past ten years, however, little work on growth-maximization of portfolio value has appeared in the academic literature of finance or economics. The neglect was due to competing norms for asset selection, particularly to norms based on two-parameter, two-period portfolio models deriving from the work of Markowitz [1959], Tobin [1958], Sharpe [1964], and Lintner [1965]. These were developed into full theories of capital market equilibrium and the empirical evidence collected in their support seemed at least sufficient to justify continued research along the lines of relaxing assumptions and performing more tests on observed portfolio behavior.

Recently, Hakansson [1971] and Hakansson and Liu [1970] again brought the growth maximization criterion to our attention. Hakansson presented a persuasive theoretical argument that ". . . the mean-variance model [a special case of the aforementioned two-parameter model, was] severely compromised by the capital growth model in several significant respects."[1]

Most readers will find the following a significant respect: Given temporally independent returns, a number of mean-variance efficient portfolios can be shown to bring complete ruin after an infinite sequence of re-investments. It is true, of course, that such sequences may indeed be optimal from an expected

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1. Hakansson [1971, p. 517].

utility viewpoint, despite ultimate ruin. No mathematician can prove the contrary, especially when it is recognized that a typical investor's horizon will probably fall short of infinity and that he will likely consume a fraction of his assets each period. However, one of our basic concerns should not be with formal proof, but with practical and intuitively credible models of investor behavior. In this regard, "growth-optimum" portfolios possess appealing features even during a *finite* time span. For example, such portfolios maximize the probability of exceeding a given level of wealth within a fixed time.[2]

Hakansson also pointed out other features of the growth-optimum model that may be more or less appealing. It implies a logarithmic (in wealth) utility function, which displays decreasing absolute risk aversion, and it implies optimal decision rules that are myopic. As a further embellishment, Hakansson and Liu derived a "separation theorem" that holds the optimal sequence of investments to be independent of the sequence of wealth levels even when returns are stochastically *dependent* across time. (Myopia also holds with temporal dependence).

Although these are strong challenges to the practical superiority of two-parameter portfolio models, we should not abandon the latter too hurriedly. They have been successfully used in many empirical contexts and their competitors should be required to weather empirical examination; so the purpose of this paper is to report on some empirical tests of growth-optimum theory using common stock returns.

Briefly, the growth-optimum model receives mixed support. In some tests it performs extremely well while the results of other tests are puzzling. In comparison to the mean-variance model it also performs well but the test results are clouded by the close operational similarity of the two models.

II. A Test Statistic for the Growth-Optimum Model

The quantitative derivation of the growth-optimum rule will employ the following convenient

NOTATION:

n_j—number of shares purchased of security j initially

$p_{j,t}$—price per share of security j in period t.

$V_t \equiv \sum_{j=1}^{N} n_j p_{j,t}$—Value of a portfolio of N distinct securities in period t.

$X_{j,t} \equiv n_j p_{j,t} / \sum_j n_j p_{j,t}$—fraction of resources invested in security j in t.

$R_{j,t} \equiv \{[(p_{j,t} + D_{j,t})/p_{j,t-1}] - 1\}$—rate of return to security j from $t-1$ to t.

$D_{j,t}$—per share dividend or coupon paid to security j between $t-1$ and t.

E—Mathematical expectation.

2. Breiman [1961, section 5]. Hakansson seems to have erred slightly when he states that "Breiman has shown that if the objective is to achieve a certain level of capital as soon as possible, then the optimal-growth portfolio . . . minimizes the expected time to reach the given level," Hakansson [1971, p. 540]. Breiman conjectured that this was true but was unable to state a proof for a fixed level of wealth. As wealth grows indefinitely large, however, the limited expected minimum time is in fact achieved by the "growth-optimum" portfolio.

Since the rule for maximizing the expected growth of portfolio value is myopic, one only needs to optimize between successive periods. Thus, the growth-optimum rule is

$$\underset{X_{t-1}}{\text{maximize}} \quad E[\log_e(\tilde{V}_t/V_{t-1})] = E\,\{\log_e[\Sigma_i X_{i,t-1}\,(1 + \tilde{R}_{i,t})]\} \tag{1}$$

subject to $\Sigma_i X_{i,t-1} = 1$. Negative values of X represent short sales which, by assumption, can be accomplished without penalty. Transaction costs are neglected.

Although the problem is easily solved without further qualifications, for historical comparison purposes it is worth assuming that the N^{th} asset, denoted F, is risk-free and that it receives the residual portion from the amounts invested in all risky assets; i.e.,

$$V_t/V_{t-1} = \sum_{j=1}^{N-1} X_{j,t-1}\,(1 + R_{j,t}) + (1 + R_{F,t})\left(1 - \sum_{j=1}^{N-1} X_{j,t-1}\right).$$

First-order conditions from problem (1) show that the investor's growth-optimum portfolio will be determined by proportions X^* such that

$$E\left[\frac{\tilde{R}_{j,t} - R_{F,t}}{1 + R_{F,t} + \sum_{i=1}^{N-1} X^*_{i,t-1}\,(\tilde{R}_{i,t} - R_{F,t})}\right] = 0;\ j = 1, \ldots, N-1. \tag{2}$$

As illustrated by Hakansson [1971] and Breiman [1960], these optimal investment fractions generally will imply a diversified portfolio but their exact values cannot be determined without specifying the joint probability distribution of the R_i's.

To obtain a testable proposition from (2), however, it will not be necessary to specify that distribution. We can rely instead on the fact that in a given period, the value of

$$\frac{1 + \tilde{R}_{j,t}}{1 + R_{F,t} + \sum_i [X_{i,t-1}\,(\tilde{R}_{i,t} - R_{F,t})]},$$

expected by each individual is equal for every risky security. Advancing to an aggregate level will require either of two traditional assumptions: (a) that all investors hold identical probability beliefs or (b) that a "representative" investor holds the expectations of equation (2) with the invested proportions (X^*'s) being equal to relative values of existing asset supplies. In either case, the denominator of (2) is equal to $1 + R_{m,t}$, a "market return" defined as a value-weighted average of all individual asset returns. The approximately observable[3] variable

$$Z_{j,t} \equiv \frac{1 + R_{j,t}}{1 + R_{m,t}} \tag{3}$$

3. It is only approximately observable because no comprehensive value-weighted asset indexes exist.

will have the same mean for all securities. It will also be true, of course, that some unexplained variation may occur about these expectations so that the quantities $Z_{j,t}$ will not be exactly equal in every period. However, an appropriate test of the identity of expectations only requires that corresponding sample means be statistically equal; that is, the temporally averaged means

$$\bar{Z}_j = \sum_{t=1}^{T} \frac{1 + R_{j,t}}{1 + R_{m,t}} \Big/ T \tag{4}$$

must be insignificantly different across securities. Testing the equality of N sample means is an analysis of variance problem. Its application to New York and American Stock Exchange listed securities will be reported in the next section.

III. A Simple Test of the Growth-Optimum Model's Basic Validity

Data used in this section are rates of return obtained from the Wells Fargo rate-of-return tape prepared by M. Scholes. The tape contains daily price changes for all common stocks listed on the New York and American exchanges from June 2, 1962 through July 11, 1969. It is a condensation and thus a tractable version of the ISL Quarterly Historical Stock Price Tapes. The Standard & Poor's Composite Price Index (The "500") is used to obtain the "market return."

In the following test, returns were taken over weekly intervals. Thus, the statistic

$$Z_{j,t} = (1 + R_{j,t})/(1 + R_{m,t})$$

was calculated for stock j at the end of week t; where $1 + R_{j,t} = (p_{j,t} + D_{j,t})/p_{j,t-1}$ and $1 + R_{m,t}$ was similarly calculated using the S&P Composite Index. The null hypothesis requires the expected return ratios to be equal, $E(Z_{j,t}) = E(Z_{i,t})$, for all i, j, and t. Usually, this would be tested by one-way analysis of variance on the temporally-averaged means, $\bar{Z}_j \equiv \frac{1}{T} \Sigma_t Z_{j,t}$ but simple analysis of variance procedures requires that all the $Z_{j,t}$'s be uncorrelated cross-sectionally and have equal variances under the null hypothesis. These assumptions are obviously too strong and are not required by the (null) growth-optimum hypothesis anyway. Furthermore, we have the evidence of many previous studies to confirm the existence of positive covariation between stock returns and returns on a market index. Some researchers (notably King, [1966]) have calculated directly a substantial co-movement among stock returns. Although the covariation between two individual stocks returns, say R_j and R_k, may be reduced in the return ratios (Z's) as the result of division by $1 + R_m$, it would be too audacious to assert a complete elimination. Furthermore, there is no *a priori* reason to suppose that $\text{Var}(\tilde{Z}_j) = \text{Var}(\tilde{Z}_k)$, as is required by a simple one-way of variance.

Fortunately, the Hotelling T^2 statistic[4] is available for precisely those cases

4. See Morrison [1967, pp. 117-124] or Graybill, [1961, pp. 205-206].

where independence among observations and equal variances do not hold. At the expense of considerable computer time, which is patiently borne by Carnegie-Mellon's undergraduates and the Nation's taxpayers, and is thus free to its current beneficiaries, this statistic was calculated for groups of 31 stocks selected alphabetically according to the procedure described below.

Hotelling's T^2 is computed as a quadratic form in the sample vector of differences between adjacent return ratios and the sample covariance matrix of those differences.[5] It is distributed as an F distribution with $k - 1$ and $N - k + 1$ degrees of freedom where N is the sample size (weeks) and k is the number of stocks in a group.[6] Since stocks are not always traded over coincidental calendar periods, and since coincidental observations are required in order to calculate sample covariances, the sample size N of each group was reduced to the number of weeks when all 31 stocks had been traded and recorded. At most, this was the number of weeks for the stock that had the minimum in its group and it was generally somewhat less. In fact, realizing in advance that some groups might be reduced to a very low number of coincidental observations, I decided to discard a stock if keeping it in the group meant reducing the second number of degrees of freedom, $N - k + 1$, below 30. When a stock was discarded, the next alphabetical one on the tape was added to the group. Of course this meant that a stock was then missing from the subsequent group and another had to be taken from the group following that and similarly to the end of the tape. Finally, 68 groups of 31 stocks remained for analysis and this number comprises all the stocks on the tape with sufficient coincidental observations for the test procedure. These 68 F statistics are depicted in Figure 1 and tabulated in Table 1.

The distribution of Figure 1 is stochastically below the expected null distribution. For example, using a Chi-square goodness-of-fit test with 17 classes to compare the F sampling distribution with the null distribution, the test statistic is about 70, which is far above the .005 level of significant difference between the two distributions. It should be emphasized, however, that only ***high*** F values reject the null growth-optimum hypothesis. Thus, growth-optimum theory is strongly, even too strongly, supported by this test of its basic validity.[7]

5. As further explanation, recall that $Z_{j,t} \equiv (1 + R_{j,t})/(1 + R_{m,t})$ is the ratio of return on stock j to the market return. For a given group of 31 stocks, the differences in return ratios are calculated as $y_{j,t} \equiv Z_{j,t} - Z_{j+1,t}$ for $j = 1, \ldots, 30$. The means of $y_{j,t}$ and covariances of $y_{j,t}$ and $y_{k,t}$ are then calculated over time. Hotelling's statistic is based on the sample quadratic form $\bar{y}'S^{-1}\bar{y}$ where $\bar{y}$ is the vector of sample mean differences and S is the sample covariance matrix of differences. Differences were calculated between adjacent alphabetic pairs but any other random scheme for selecting pairs would have been equally acceptable. Cf. Morrison [1967, pp. 135-138].

6. This is the rationale for using a group size of 31 stocks: since the number of degrees of freedom for the test is $k-1$, where k is the number of stocks, a group size of 31 is both large and makes tabular comparison easy. If the group size had been chosen larger, the second degrees of freedom parameter, $N-k+1$, (where N is the number of available time points), would be reduced to a low number. Thus, I thought $k=31$ would balance the two d.f. parameters and still leave them quite large.

7. Because the Hotelling test supports the null hypothesis *too* strongly, I decided to check several potential causes. One obvious possibility is the thick-tailed distributions of stock returns that have been pointed out by many researchers, (Cf. Fama [1965], Blume [1970]). The appropriate way to check this problem is to use a non-parametric analysis of variance, Friedman's multi-sample test, (Bradley, [1968, p. 127]). This was done for exactly the same sample of stocks grouped in

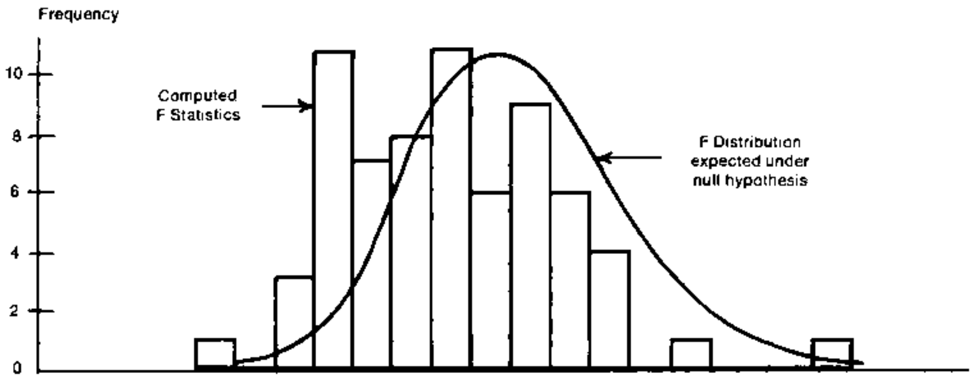

FIGURE 1
Distribution of F Statistics from Hotelling's T^2 Test of Equality Among Mean Return Ratios

IV. RELATIONS BETWEEN THE GROWTH-OPTIMUM AND THE SHARPE-LINTNER MODELS

A. *Theory*

When one compares the first order conditions for the growth-optimum model

$$E\frac{1+\tilde{R}_j}{1+\tilde{R}_m} = (1+R_f).\,E\frac{1}{1+\tilde{R}_m}; \quad j=1,\ldots N-1 \tag{5}$$

with those for the Sharpe [1964]-Lintner [1965] model

$$\frac{E(1+\tilde{R}_j)}{E(1+\tilde{R}_m)} = \beta_j + (1-\beta_j)\frac{(1+R_f)}{E(1+\tilde{R}_m)}; \quad j=1,\ldots,N-1 \tag{6}$$

(where $\beta_j = \text{Cov}(\tilde{R}_j, \tilde{R}_m)/\text{Var}(\tilde{R}_m))$, some correspondence appears but it is rather difficult to evaluate fully just by inspection. For example, when the Sharpe-Lintner risk coefficient, β_j, is equal to zero, equation (6) becomes

$$\frac{E(1+\tilde{R}_j)}{E(1+\tilde{R}_m)} = \frac{1+R_f}{E(1+\tilde{R}_m)}.$$

But when $\beta_j = 0$, $\text{Cov}(\tilde{R}_j, \tilde{R}_m) = 0$, and the growth optimum condition expressed in equation (5) becomes

the same way. The results were identical to those obtained by using Hotelling's T^2, the growth-optimum model was too strongly supported.

A second possible misspecification is a deficiency in the market price index. The Standard & Poor's Composite Price Index, used in the preceding tests, is heavily-weighted in favor of a few stocks. Also, since it is essentially a "buy-and-hold" portfolio, weights of individual stocks change over time as relative prices change. Evans [1968] and Cheng and Deets [1971] have provided empirical evidence that such an index performs quite differently from a "fixed-investment proportion" or "rebalanced" index. Therefore, a rebalanced index composed of stocks on the tape was constructed and used in reporting the tests already done. Again, no difference was detected.

Thirdly, the possibility that an unrepresentative episode biased the entire sample period of seven years of weekly observations was checked by repeating the tests for annual sub-periods. There was no perceptible difference among the sub-periods or between the overall period and any sub-period. In every case the growth-optimum model was strongly supported.

Further details of all the tests in this footnote are available in an earlier working paper. (It was edited to conserve *Journal* space.)

TABLE 1
CALCULATED VALUES OF HOTELLING'S T^2 STATISTIC FROM COMMON STOCK MEAN RETURN RATIOS, 1962-1969

Group	Sample size (weeks)	T^2	Group	Sample size (weeks)	T^2
1	113	36.0	36	98	30.8
2	81	20.2	37	76	35.0
3	61	53.3	38	101	29.2
4	61	45.4	39	79	54.1
5	143	28.4	40	69	56.3
6	76	35.3	41	107	32.3
7	132	55.5	42	61	68.9
8	61	14.4	43	61	72.0
9	148	29.7	44	78	32.0
10	131	24.1	45	64	35.9
11	90	22.6	46	97	46.3
12	68	28.7	47	66	56.9
13	108	41.3	48	62	57.1
14	76	27.5	49	91	24.9
15	109	41.5	50	60	50.1
16	273	29.4	51	61	37.6
17	111	33.5	52	64	40.3
18	110	35.0	53	63	38.3
19	175	36.5	54	65	25.6
20	75	42.0	55	60	35.1
21	95	27.9	56	67	63.8
22	76	51.3	57	80	48.2
23	112	22.4	58	60	107.0
24	74	64.5	59	96	24.9
25	112	23.4	60	79	39.0
26	127	31.8	61	73	57.8
27	158	38.6	62	82	56.2
28	78	23.0	63	72	28.5
29	123	39.6	64	66	65.7
30	112	40.2	65	61	47.3
31	104	49.7	66	65	53.5
32	61	32.3	67	75	42.3
33	115	35.2	68	63	52.9
34	62	65.6			
35	70	30.6			

Note: The F statistic is calculated as $\frac{N-30}{(N-1)\,30}T^2$.

$$E(1+R_j)\,E\left(\frac{1}{1+\tilde{R}_m}\right) = (1+R_f)\,E\left(\frac{1}{1+\tilde{R}_m}\right).$$

Since the terms containing market returns cancel in both of the displayed equations just above, the growth optimum model provides a market diversification result which is well-known from Sharpe-Lintner theory; namely, a security whose portfolio risk is zero will sell at an expected return equal to the riskless rate.

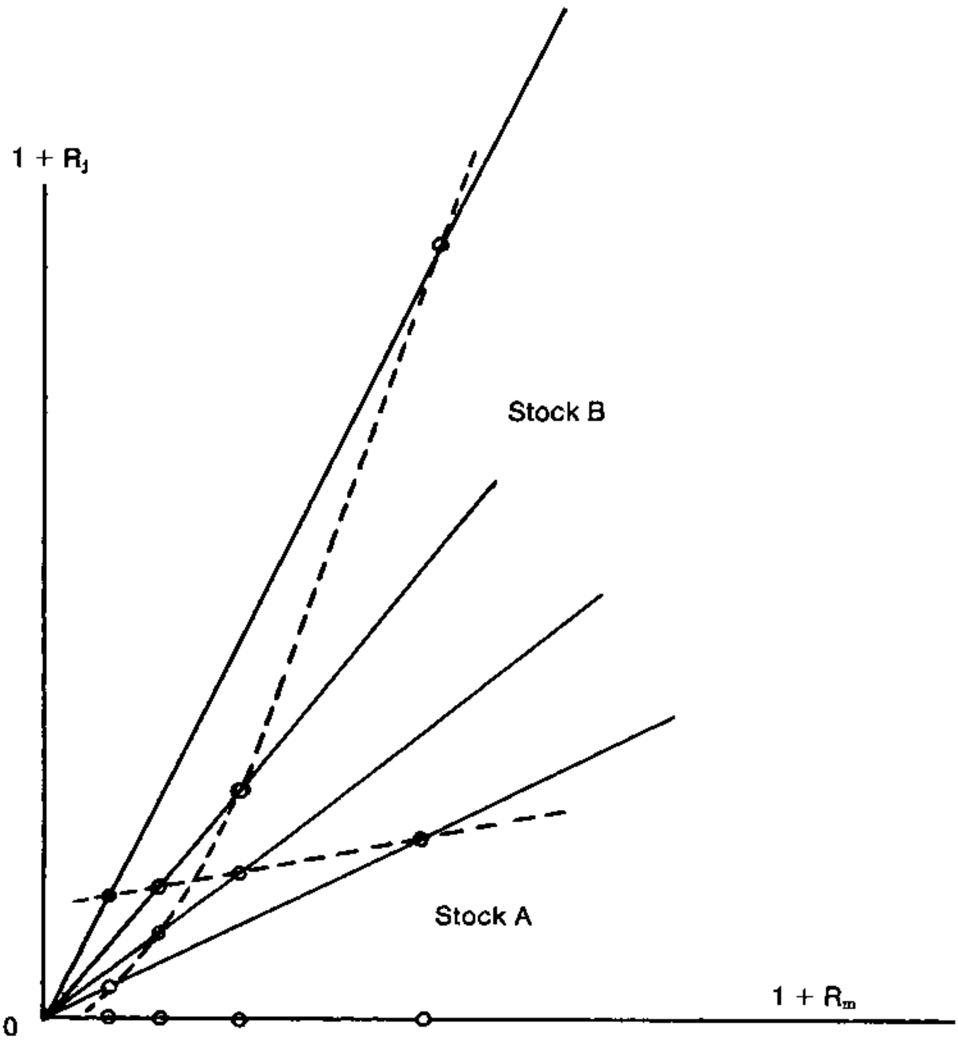

FIGURE 2
Stocks Satisfying the Growth-Optimum Model with Different Degrees of Market Response

It is *not* true that the growth-optimum model implies a constant β coefficient although it may seem to after a first glance at equation (6).[8] A simple example is sufficient to show the contrary. Figure 2 illustrates the *discrete* probability

8. Because (6) is

$$\frac{E(1+\tilde{R}_j)}{E(1+\tilde{R}_m)} = \frac{1+R_F}{E(1+\tilde{R}_m)} + \beta_j\left[1 - \frac{1+R_F}{E(1+\tilde{R}_m)}\right]$$

and (5) is

$$E\left(\frac{1+\tilde{R}_j}{1+\tilde{R}_m}\right) = E\left(\frac{1+\tilde{R}_F}{1+\tilde{R}_m}\right),$$

a reader not cautious about quotients and reciprocals of random variables might think that the latter model implies $\beta_j =$ zero, independent of j.

distributions of stocks A and B and of the market. In this example, only four equally likely returns are possible for the market and each stock can return corresponding amounts to those four. The growth-optimum model requires that

$$E\left(\frac{1+\tilde{R}_A}{1+\tilde{R}_m}\right) = E\left(\frac{1+\tilde{R}_B}{1+\tilde{R}_m}\right)$$

which in geometric terms implies that the expected *slopes* of rays from the origin through each possible point in the $1 + R_j$, $1 + R_m$ plane is equal for both stocks. This is clearly satisfied by the radially symmetric solid lines which pass through the points of possible occurrence in Figure 2. Nevertheless, the slopes of regression lines of R_j on R_m, indicated by dashes, are quite different. Stock A has low portfolio risk because its β is low, while stock B has much greater response to the market and thus higher risk. Both securities' returns are perfectly functionally related to the market return but it is already apparent that correlation per se will have the same relatively unimportant role in a growth-optimum as it has in the Sharpe-Lintner framework (i.e., the expected slopes of rays can be equal no matter what correlation occurs between a stock's return and the market's).

A close correspondence between the two models can be made more apparent by rearranging a few terms. The result (derived in the footnote[9]), is

$$E(\tilde{R}_j - R_F) = \left[\operatorname{Cov}\left(\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right) \Big/ \operatorname{Cov}\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)\right] E(\tilde{R}_m - R_F); \tag{7}$$

which is very similar indeed to the Sharpe-Lintner equilibrium equation. In fact, since $\operatorname{Cov}\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)$ is negative, one is tempted to suggest that a second implication of Sharpe-Lintner theory is also satisfied by the growth-optimum model: namely, that security j's expected return will exceed the riskless return if and only if $\operatorname{Cov}(\tilde{R}_j, \tilde{R}_m)$ is positive. One would need to prove,

9. To obtain (7), note that equation (5) is equivalent to

$$\operatorname{Cov}[\tilde{R}_j, 1/(1+\tilde{R}_m)] = (1+R_F)K - E(1+\tilde{R}_j)K \tag{5a}$$

where $K \equiv E[1/(1+\tilde{R}_m)]$.

Multiplying both sides of (5a) by X_j, the proportion of wealth invested in security j, (or the fraction of aggregate economic wealth represented by security j),

$$\operatorname{Cov}[\sum_j X_j\tilde{R}_j, 1/(1+\tilde{R}_m)] = -K\sum_j [X_j E(\tilde{R}_j - R_F)]$$

and substituting for the definition of R_m, i.e., for $R_m \equiv R_F + \Sigma X_j(R_j - R_F)$,

$$\operatorname{Cov}[\tilde{R}_m - R_F(1 - \sum_j X_j), 1/(1+\tilde{R}_m)] = -K\,E(\tilde{R}_m - R_F).$$

Since $R_F(1-\Sigma x_j)$ is a constant,

$$K = -\operatorname{Cov}[\tilde{R}_m, 1/(1+\tilde{R}_m)]/E(\tilde{R}_m - R_F).$$

Substituting this for K in (5a) provides equation (7).

however, that Cov $(R_j, R_m) > 0$ implies Cov $\left(R_j, \frac{1}{1+\tilde{R}_m}\right) < 0$ and this is definitely not true in general. Counterexamples can be displayed for highly-skewed distributions.[10]

B. *Testing Sharpe-Lintner vs. Growth-Optimum*

The two models will provide approximately equivalent empirical implications if certain restrictions are placed on the ranges of individual rates of return. To verify this, one only needs to note the following facts: (a) quadratic utility functions and homogenous anticipations will lead to the Sharpe-Lintner equilibrium result of equation (6) and (b) the logarithmic utility function implied by portfolio growth maximization can be approximated by a quadratic as

$$\log(1 + R_T) \approx R_T - 1/2\, R_T^2$$

provided that the portfolio's total return is restricted to less than 100 per cent per period.[11] It is indeed trivial to show that the two models are identical when this approximation to the logarithm is made. Thus, given the truth of one theory, we should not be surprised to find that an empirical test of the other supports it very well, especially when the observed rates of return used in testing fall predominantly near zero. Of course, this leads us to ask whether the strong empirical support for the growth-optimum model reported in the preceding section is really damnation for Sharpe-Lintner or just an accidental stroke of choosing a short time interval (one week) which guaranteed that returns were never observed far from zero.

An obvious way to test this is suggested by the logarithmic approximation. If growth-optimum theory appears to satisfy the data only because the logarithm approximates a quadratic when returns are near zero, one should choose a longer time interval for empirical testing so that many more large and small returns are observed.[12] This was done for both four week and twenty-six week periods with the same common stock data as used previously and the conclusions were identical to those for weekly periods already reported.

A more refined and direct test to discriminate between the two models can be based on equilibrium conditions of the two competing theories,

$$E(\tilde{R}_j - R_F) = \left\{ \mathrm{Cov}\left[\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right] \Big/ \mathrm{Cov}\left[\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right] \right\} E(\tilde{R}_m - R_f) \equiv \gamma_j \quad (8)$$

10. However, for at least one special asymmetric case, (lognormal distributions) the growth-optimum model does agree completely with the Sharpe-Lintner result that a security's expected return will be a linear function of systematic risk; i.e., of β_j. I am indebted to Robert Litzenberger for demonstrating this point.

11. And more than minus 100 per cent per period. Without short sales, this last restriction is presumably satisfied for common stocks by the existence of limited liability. Samuelson [1970] has derived a broader "fundamental approximation theorem" which shows the close match of mean-variance to *any* correct portfolio theory when the joint distribution of returns has a small dispersion.

12. This completely ignores the crucial question of investor horizon period that may have a significant effect on the *form* of the Sharpe-Lintner market model. Cf. Jensen [1969, pp. 186-191].

which is the growth-optimum condition (7), and

$$E(\tilde{R}_j - R_F) = \{Cov[\tilde{R}_j, \tilde{R}_m]/Var(\tilde{R}_m)\} E(\tilde{R}_m - R_F) \equiv \delta_j \tag{9}$$

which is the familiar Sharpe-Lintner condition. A suggested test procedure obtains the best estimates of all components of (8) and (9) from time series and then performs cross-sectional regression with those estimates.

A series of comparative tests of these two equations was conducted with the common stock returns mentioned previously. In each case, time series were used to calculate $\bar{R}_j$, the mean return, and $\hat{\gamma}_j$ or $\hat{\delta}_j$, the estimated risk measures implied by the growth-optimum or the Sharpe-Lintner model, respectively.[13] Then, cross-sectional computations were performed for the regression models

$$\bar{R}_j = \hat{a}_0 + \hat{a}_1\hat{\delta}_j \tag{10}$$

and

$$\bar{R}_j = \hat{b}_0 + \hat{b}_1\hat{\delta}_j. \tag{11}$$

Estimated coefficients from (10) and (11) should be compared to their theoretical counterparts: depending on which theory is correct, $\hat{a}_0$ or $\hat{b}_0$ should equal R_F, the average risk-free interest rate, and $\hat{a}_1$ or $\hat{b}_1$ should equal unity.

In the first test, all calculations were carried out with *individual* security returns during the same time period.[14] For example, 1192 separate values of $\bar{R}$, $\hat{\gamma}$, and $\hat{\delta}$ were obtained from weekly data covering the annual sub-period July, 1962 through June, 1963. Cross-sectional regressions using these 1192 estimates gave $\hat{a}_0 = 20.1$ and $\hat{b}_0 = 20.3$ per cent. The value of $\bar{R}_F$, as measured by the weekly average interest rate on short-term government debt obligations during the year, was only 3.27 per cent; so neither model satisfied its theoretical prediction very well in this particular annual sub-period.[15] Over all the seven years of available data, the estimated values of $\hat{a}_1$ and $\hat{b}_1$ were very significantly positive and they were scattered around unity in nice accord with their expected level. $\hat{a}_0$ and $\hat{b}_0$ were also reasonably close to $\bar{R}_F$, at least on average. As a distinguishing test of the two competing theories, however, these results failed miserably; for the estimates were very highly correlated between the models. The two competitive intercepts were practically identical in every

13. To be precise,

$$\hat{\gamma}_j \equiv \left[\hat{C}ov\left(R_j, \frac{1}{1+R_m} \right) \Big/ \hat{C}ov\left(R_m, \frac{1}{1+R_m} \right) \right] (\bar{R}_m - \bar{R}_F)$$

and

$$\hat{\delta}_j \equiv [\hat{C}ov(R_j, R_m)/\hat{V}ar(R_m)](\bar{R}_m - \bar{R}_F)$$

where ˆ indicates the sample analog of a population parameter, calculated from weekly observations over a specified period, and — indicates sample mean from the same period.

14. To save computation expense, only New York Exchange listed stocks with at least 30 weekly quotations during a year were included in the sample. The risk-free rate was measured by a weekly "average of short-term government debt obligations" taken from *Standard & Poor's Trade Statistics*. The market indexes used were: The S&P Composite Index and a rebalanced index constructed by weighting all NYSE Stock Returns equally each week. The results were very similar but only the results for the rebalanced index are quoted in the text.

15. To save space, only one annual sub-period is reported here but *all* the results are available from the author upon request.

period as were the two competitive slope coefficients. They deviated much more from theoretical predictions than from each other.

In order to sharpen the discriminatory resolution of the data, a different procedure[16] was necessary. The first problem to be alleviated was an extremely low explanatory power which was indicated by low R^2's (on the order of .05), for the cross-sectional models with individual stock returns. A technique to remedy this is to form portfolios of stocks and conduct cross-sectional tests on portfolios rather than on individual securities. A difficulty arises, however, because randomly-selected portfolios would have very similar values of the risk measures $\hat{\gamma}$ and $\hat{\delta}$. To create a cross-sectional spread in risk measures, portfolios must be selected on the basis of risk by grouping stocks with the lowest $\hat{\gamma}$'s or $\hat{\delta}$'s in one portfolio, stocks with higher $\hat{\gamma}$'s or $\hat{\delta}$'s in the next portfolio, and so on.

This procedure for forming portfolios makes another econometric problem obvious: In the cross-sectional regression, the explanatory variables contain errors which will tend to bias slope coefficient estimates toward zero.[17] To alleviate this problem somewhat, stocks were placed in portfolios based on their risk measures calculated from weekly data of a given year and then the mean portfolio return and risk measures for the portfolio as a whole were calculated from weekly data in the subsequent year. Cross-sectional models (10) and (11) were then estimated using these latter estimates and the results are given in Table 2 and Figure 3. To recapitulate, the procedure which resulted in the output of Table 2 and Figure 3 was as follows:

1. For each stock (j) in year t, the total risk premia $\hat{\gamma}_j$ and $\hat{\delta}_j$ were calculated from the time series of year t using the rebalanced market index (see note 7).
2. Stocks were ranked from smallest to largest $\hat{\gamma}_j$ and from smallest to largest $\hat{\delta}_j$.
3. Twenty portfolios were selected by assigning the lowest five per cent of ranked stocks to one portfolio, the next lowest five per cent to a second portfolio, etc. Thus, two sets of 20 portfolios each were formed; one set based on γ rankings and one set on δ rankings.
4. Each portfolio's mean return, $\bar{R}_p$, was calculated from time series in year t + 1. For each portfolio that had been formed on the basis of γ rankings, the growth-optimum risk premium $\hat{\gamma}_p$ was calculated from year t + 1 data. Similarly, the Sharpe-Lintner premium $\hat{\delta}_p$ was calculated from year t + 1 data for each portfolio that had been formed by δ rankings.
5. For each of six years (1963-64, 1964-65, . . . 1968-69), these calculated portfolio mean returns and risk measures are plotted in Figure 3 and regressions across portfolios are reported in Table 2.

In Figure 3, the scatters of mean portfolio returns versus the two competing risk measures are displayed for six different years. The solid lines mark

16. Blume [1970], Miller and Scholes [1972], Black, Jensen, and Scholes [1972], and Fama and MacBeth [1972], have originated and developed the procedures used here in their work with the empirical validity of the two-parameter portfolio model.

17. This is true, of course, for regressions using individual stocks as well as for those using portfolios.

TABLE 2

RELATIONS BETWEEN AVERAGE RETURNS AND RISK COEFFICIENTS OF 20 PORTFOLIOS SELECTED ON THE BASIS OF INDIVIDUAL RISK COEFFICIENTS CALCULATED ONE PERIOD EARLIER

(Rebalanced Index)

			Growth Optimum Model			Sharpe-Lintner Model			
Period (July through June)[a]	$\bar{R}_F$[e] % per annum	$\bar{R}_m$	$\hat{a}_0$ % per annum	$\hat{a}_1$[d]	R^2	$\hat{b}_0$ % per annum	$\hat{b}_1$[d]	R^2	Size of Portfolio[c] (No. of Stocks)
1963-1964	3.71	14.4	15.8 (6.29)	−.0817 (−.374)	.0077	15.8 (6.05)	−.0781 (.343)	.0065	59
1964-1965	3.81	12.6	12.4 (6.53)	.0195 (.0918)	.0005	12.5 (6.90)	.00604 (.0297)	.0001	58
1965-1966	4.34	21.0	−9.91 (−5.14)	1.85 (16.8)	.940	−9.68 (−5.62)	1.83 (18.6)	.951	62
1966-1967	4.61	29.4	−2.72 (−1.04)	1.27 (12.5)	.896	−3.02 (−1.14)	1.28 (12.4)	.895	63
1967-1968	5.15	29.2	21.7 (8.20)	.261 (2.43)	.247	22.2 (7.23)	.241 (1.93)	.172	62
1968-1969	5.36	−2.45	13.3 (3.55)	1.97 (4.24)	.500	13.2 (3.84)	1.96 (4.57)	.537	59

[a] The last date in 1969 was July 11. In other years the first date was the first Thursday in July, 196**X**, and the last date was the last Thursday in June, 196(X + 1).

[b] t-ratios are in parentheses.

[c] This is equal to the total number of stocks that have at least 30 available prices in periods **t − 1** and t, N, divided by 20. The remainder from N/20, J = MOD(N,20), was distributed such that one extra stock was included in each of the first **J** portfolios.

[d] Means of sample values $\hat{a}_1 = .881$; $\hat{b}_1 = .873$; $\bar{R}_F = 4.50$; $\hat{a}_0 = 8.43$; $\hat{b}_0 = 8.50$.

[e] Mean of weekly observations of short-term government debt obligations, *Standard and Poor's Trade Statistics*.

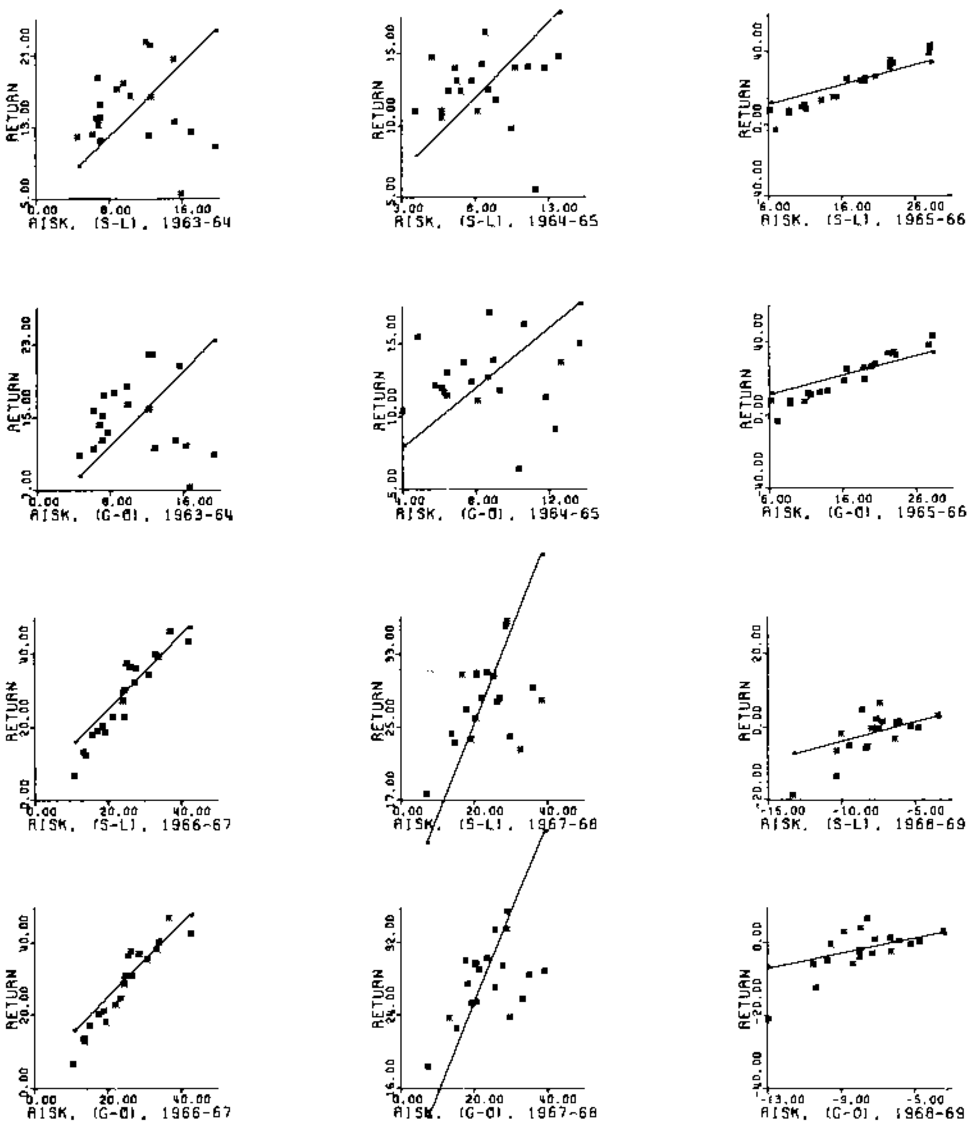

FIGURE 3
Mean Portfolio Returns and Estimated Portfolio Risk Premia, New York Exchange Stocks, 1962-1969, Rebalanced Index
* (S-L) Denotes Sharpe-Lintner Risk Premia and (G-O) denotes Growth-Optimum Premia.

the theoretical predictions, an intercept of $\bar{R}_F$ and a slope of unity.[18] Table 2 contains results from cross-sectional models (10) and (11) applied to these data.

On average across the six years, both the growth-optimum and the Sharpe-Lintner model seem to have excessive intercepts, $(\hat{a}_0, \hat{b}_0 > \bar{R}_F)$, and deficient slopes, $(\hat{a}_1, \hat{b}_1 < 1)$. The averages are given in footnote d of Table 2. These deviations from the anticipated can no doubt be attributed, at least in part,

18. Since the axes are scaled differently in each plot, the lines do not appear to have slopes of unity upon first examination. Note that the plotted lines are the theoretical predictions and are *not* the regression lines reported in Table 2.

to errors in the measurement of risk premia. In the first two years and in year 5 (1967-68) the wide scatter suggests that either the mean portfolio returns or the risk premia were measured inaccurately.

In the two years of best fit, 1965-66 and 1966-67, portfolios with low risk premia return less than anticipated while portfolios with high premia return more. This is unlikely to be just a sampling phenomena if the standard errors of $\hat{a}_1$ and $\hat{b}_1$ are credible. For example, the growth-optimum slope for 1965-66 is $\hat{a}_1 = 1.85$ which is nearly eight standard errors above unity.[19]

Perhaps the most striking characteristic of the two models is their very close relation over time. The slope coefficients and intercepts given in Table 2 vary widely across periods but are extremely close, between the two models, in each period. This is obvious from even a quick look at Figure 3.[20] Based on these results, one can only conclude that the two models are empirically identical. A qualification is in order, of course; assets whose returns are much more highly skewed (e.g., warrants), may permit a finer discriminatory test.

V. Summary and Conclusions

If investors wish to maximize the probability of achieving a given level of wealth within a fixed time, they should choose the "growth-optimum" portfolio; that is, the portfolio with highest expected rate of increase in value. This paper has examined the implications for observed common stock returns of all investors selecting such a portfolio.

Given some widely-used (and useful) aggregation assumptions, the growth-optimum model implies that the expected return ratio $E[(1 + \tilde{R}_j)/(1 + \tilde{R}_m)]$ is equal for all securities.[21] This implied equality of expected return ratios was utilized in analysis of variance tests with New York and American Exchange listed stocks from 1962-1969 in order to ascertain the basic validity of the growth-optimum model. The model was well-supported by the data.

The growth-optimum model was compared algebraically to Sharpe-Lintner theory, which is probably the most widely-used portfolio result in empirical work. A close correspondence was demonstrated between their qualitative implications. For example, both models imply that an asset's expected return will equal the risk-free interest rate if the covariance between the asset's return and the average return on all assets, $\mathrm{Cov}(\tilde{R}_j, \tilde{R}_m)$, is zero. For most cases, the growth-optimum model also shares the Sharpe-Lintner implication that an asset's expected return will exceed the risk-free rate if and only if $\mathrm{Cov}(R_j, R_m) > 0$. There are, however, some cases of highly-skewed probability distributions where this implication does not follow for the growth-optimum model.

A close empirical correspondence between the two models was demonstrated for common stock returns. The procedure (1) estimated returns and risk premia

19. For a more detailed discussion of this point, see Friend and Blume [1970].

20. The greatest difference between $\hat{a}_1$ and $\hat{b}_1$ is .02 which is only about 2.3 per cent of their average value. Between $\hat{a}_0$ $\hat{b}_0$, the greatest difference is about six per cent of their average value.

A close connection between the two models was previously implied by the work of Young and Trent [1969]. They showed that the geometric mean of portfolio returns was closely approximated by functions of the arithmetic mean and variance of returns. These functions were developed as approximations to the geometric mean. For accuracy, they require a minimal amount of skewness and are, therefore, analogous to the truncated (after two terms), Taylor series expansions of $\log_e(1+R)$.

21. R_j is the rate of return on security j and R_m is the rate of return on a portfolio of all assets.

implied by the two models from time series; (2) calculated cross-sectional relations between estimated returns and risks; and (3) compared the cross-sectional relations to the theoretical predictions of the two models. They could not be distinguished on an empirical basis. In every period, estimated corresponding coefficients of the two models were nearly equal; and indeed, they deviated much further from their theoretically anticipated levels than they deviated from each other.

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