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Assets, Money, and Commodity Price Inflation under Uncertainty

Demand Theory

I. INTRODUCTION

Financial assets produce no consumption goods. They cannot be used to satisfy such human desires as food or shelter. Instead, their value stems from claims on future consumption. As these claims fluctuate in expected magnitude, in risk of being satisfied, and in the time until fulfillment, asset prices also vary. That much economists and men of affairs take for granted. But the precise relations between asset prices and individual expectations, risk preferences, and time preferences are not well understood.

Economic tradition has assigned to financial assets the quality of "value storage." Money has received additional credit as a "medium of exchange," that is, a completely "liquid" asset; but both characteristics can be subsumed under the phrase, "claim on consumption." For example, the "liquidity" value of a currency arises from its immediate recognition.[1] One can easily verify this by noting the differences in waiter response when paying for lunch with dongs in New York and Hanoi; or,

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[1]Cf. Hicks: "the imperfect 'moneyness' of those bills which are not money is due to their lack of general acceptability; it is this lack of general acceptability which causes the trouble of investing in them, and that causes them to stand at a discount" [6, p. 166].

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by attempting to acquire immediate possession of a new automobile with a bundle of real estate titles, or corporate bonds, or "negotiable" warehouse receipts. The car will remain in the showroom for the present, but give the salesman an hour for some certificates and several weeks for others and he will be glad to complete the transaction. Thus, "medium of exchange" connotes "a claim on instantaneous consumption," which is another way of saying that currency is particularly useful when one intends to store value for a short time.

Since nearly all financial assets are quoted and traded in domestic currency units, changes in the goods value of currency affect the real return obtained from holding assets. This seems trivially obvious, yet inflation of commodity prices has been almost completely neglected in the literature of asset pricing. The next section develops a capital-asset pricing model that includes the risk of currency inflation (or deflation) explicitly. The model is simple and it seeks a modest objective: to specify a little further how assets and commodities acquire equilibrium prices in competitive markets.

II. A THEORY OF ASSETS AND INFLATION

Specification of the Consumption-Investment Environment

To illustrate the basic mechanism of asset price equilibrium, we outline the most elementary circumstance that still retains the essential features of uncertainty and inflation: a two-period world devoid of taxes, where no individual can alter the currently announced price of anything. Everyone maximizes expected utility of consumption and assesses all available information about the future while doing his calculation (which is assumed to be costless).

In the first period, N distinct commodities are available for consumption. For simplicity, we assume that each person is endowed with an initial financial resource, W (for wealth),[2] that may be divided into consumption and savings. The latter may in turn be used to purchase a portfolio from among the $\eta + 1$ available real or financial assets, including money, or it may be used to purchase inventories of storable commodities. In period two, the individual consumption program is limited by two sets of prevailing prices: one for commodities to be sold from inventory or purchased and consumed in period two, and another for assets held from period one.[3] All labor income is neglected.[4]

The Formal Symbolic Problem. The individual operates under the following fi-

[2] Initial financial wealth would, of course, be an endogenous variable in a system completed by the inclusion of a production function. Here, however, the production decision is assumed to have been made in advance.

[3] For symbolic simplicity, the set of available commodities will be assumed to contain the same elements in both periods. However, the analysis is readily (but messily) extendable to accommodate the appearance of new commodities and the disappearance of old ones.

[4] Labor income could be designated as a return to a "human capital" asset if legal restrictions against such assets were disregarded. Alternatively, it could be included directly in the constraints with no difficulty other than the tedious algebra of additional terms. An explicit analysis of human capital and other nonmarketable assets in a portfolio context has been provided by Mayers [10].

nancial constraints:

In period one,

$$W = P_1'C_1 + M + p_1'(A_L - A_S) + P_1'(X_L - X_S); \tag{1}$$

and in period two,

$$\tilde{P}_2'\tilde{C}_2 = M + \tilde{p}_2'(\tilde{\lambda}_{AL}A_L - \tilde{\lambda}_{AS}A_S) + \tilde{P}_2'\tilde{d}[\tilde{\lambda}_{XL}(X_L - \tilde{X}_C) - \tilde{\lambda}_{XS}X_S] - \tilde{g}'X_L; \tag{2}$$

where

Symbol	*Denotes*	*Price per Unit*
W	Initial wealth endowment	
M	Nominal-money balances held	1
A_L	Assets purchased (Long)	p_1 and $\tilde{p}_2$
A_S	Assets sold short	p_1 and $\tilde{p}_2$
X_L	Commodity inventories purchased	P_1 and $\tilde{P}_2$
X_S	Commodities sold short	P_1 and $\tilde{P}_2$
$\tilde{X}_C$	Commodities consumed in period two from inventory held over from period one.	
C_t	Commodities purchased and consumed in period t, $(t = 1, 2)$	P_1 and $\tilde{P}_2$
$(1 - \tilde{d})$	Physical depreciation per unit of commodities held in inventory.	
$\tilde{g}$	Storage costs per unit for commodities held in inventory.	
$(1 - \tilde{\lambda})$	Transaction and liquidity charges per unit relative to "money." Subscripts refer to Assets: A, or commodities: X; Long positions: L; or Short positions: S.	

With N distinct consumption commodities in existence, $X_L, X_S, \tilde{X}_C, C_1, \tilde{C}_2, \tilde{g}, P_1$, and $\tilde{P}_2$ are column vectors with N elements each while $\tilde{\lambda}_{XS}$, $\tilde{\lambda}_{XL}$ and $\tilde{d}$ are $N \times N$ diagonal matrices with individual commodity transaction charges and physical depreciation factors along the diagonals. With η different assets available, $p_1, \tilde{p}_2$, and A are η-element column vectors while $\tilde{\lambda}_{AL}$ and $\tilde{\lambda}_{AS}$ are $\eta \times \eta$ diagonal matrices. A prime (′) denotes matrix transpose and a tilde (~) denotes random variable.

Uncertainty arises because asset and commodity prices for period two are random variables in the first period. Inventory storage and transaction-liquidity costs are also random because there is no good reason to presume otherwise. Since money acts as the unit of account, its nominal price is always one,[5] but this does not mean

[5] "Money" is implicitly defined as non-interest bearing currency and coin that will serve as unit of account in both periods one and two. Even demand deposits would have to be listed as a separate asset if they received interest, as they do in many countries [11].

that money is a riskless asset. Its relative command over period-two consumption is uncertain.

Non-negativity Restrictions. All the asset and commodity elements must be non-negative:

$$M, X_L, X_S, A_L, A_S \geqq 0 \tag{3}$$

This still permits a *net* short position in either non-money assets or in commodities because short sales could exceed long positions for any individual item or for all assets and commodities collectively. For the asset money, however, short-selling is not permissible because a short sale of money is nothing but personal borrowing and it is properly categorized as a separate short asset, one of the elements of A_S. As a "short-seller" of money, the individual would have to pay interest to the lender, and this also points to a distinction between long and short positions in money because it violates the requirement that money's price be unity in both periods.

Commodities also must be consumed in non-negative quantities, so the system requires the constraints

$$C_1, \tilde{C}_2 \geqslant 0. \tag{4}$$

Finally, commodities consumed in period two out of inventory held over from period one cannot exceed the actual amount held:[6]

$$\tilde{P}_2' \tilde{d} \tilde{\lambda}_{XL}(X_L - \tilde{X}_C) \geqq 0. \tag{5}$$

Explanation of Commodity Storage Depreciation. There is little logical distinction between an asset and a commodity held in inventory. The commodity is simply a store of value which can be consumed without being exchanged. But since commodities are bulkier than assets, they are more costly to store and are more subject to deterioration and obsolescence. The diagonal element d_i is supposed to measure all physical depreciation for commodity i. In addition, there are pecuniary storage fees such as those charged for warehousing which must be paid whether the commodity is sold or consumed. The element g_i contains these costs for commodity i.[7] For any good that is not directly storable at all, (e.g., a service), physical deterioration is complete and thus d_i is equal to zero. In this degenerate case, the good may be thought of as being held in inventory but period-two wealth will not be increased by the amount held.

Explanation of the Transaction-Liquidity Cost Elements. When assets or commodity inventories are exchanged for consumption goods in period two, transaction charges are incurred. These include direct brokerage fees of course, but perhaps

[6]The basic constraint is $X_L - X_C \geqq 0$ but both sides were multiplied by the positive quantity $\tilde{P}_2 \tilde{d} \tilde{\lambda}_{XL}$ for later mathematical convenience.

[7]Storage, deterioration and obsolescence costs are assumed to be zero for assets.

more important are the non-pecuniary charges associated with finding a buyer for the assets owned or finding a seller to cover assets previously sold short: the time spent shopping for a better price and the uncertainty about whether to keep shopping. There is a second level of transaction costs too. This is associated with consumption goods planned for purchase, the uncertainty about their price and quality and thus the potential number of shopping stops required to obtain a good buy.[8] For example, if the individual were planning to consume commodities which varied widely in price across sellers, he would want to own a portfolio composed primarily of liquid assets that could be quickly exchanged for cash at a low cost whenever a bargain commodity price was discovered. Conversely, if his planned consumption bundle contained standard goods with little quality and price variation, he could afford to own assets that required some considerable time and effort to sell because he could be sure that the goods ultimately intended for purchase would be available at a predictable price whenever the assets were finally exchanged for cash.

The matrices denoted λ measure these transaction and liquidity costs. For an asset or commodity i, held in long position, $0 \leqslant \lambda_{iL} \leqslant 1$, while for an asset or commodity j sold short, $\lambda_{jS} \geqslant 1$. This parameterization is caused by the existence of money which is supposed to be perfectly liquid and can always be exchanged without paying brokerage fees. Implicitly, the λ for money is unity. Other assets possess some liquidity value but of a quality inferior to money's. They are "near-moneys" or liquidity substitutes.[9]

Because of their bulkiness and uniqueness, commodity inventories have even less liquidity value than assets. This is partly mitigated, however, for commodities that are consumed rather than exchanged in period two. For those commodities, transaction costs and opportunity costs of foregone liquidity are zero.

Solution to the Consumer's Maximization Problem

Each individual hopes to maximize his expected utility while conforming to constraints (1) through (5). This would be a simple Lagrangian maximization problem except that (2), (4), and (5) are random constraints that are temporally separated from the initial choices of consumption, assets and inventory. Fortunately, the consumer need only consult Kushner [8] to learn that an extremum of the problem must satisfy the conditions given in mathematical appendix A.[10]

Inter-commodity Conditions. For any pair of consumption commodities, say i and k, which are consumed in positive amounts, the optimality conditions can be solved to obtain

$$E(\tilde{U}_{1i}/P_{1i}) = E(\tilde{U}_{1k}/P_{1k}) \tag{6}$$

[8]An explicit statement of this mechanism was provided by Brunner and Meltzer [1].

[9]Recently, there have been attempts to measure the "nearness of near-money," i.e., to estimate the value of λ, for assets which were thought to possess a high liquidity value. See Chetty [2].

[10]I am indebted to John Long, Jr., for bringing the paper by Kushner to my attention and for pointing out its significance.

and

$$\tilde{U}_{2k}/\tilde{P}_{2k} = \tilde{U}_{2i}/\tilde{P}_{2i} \tag{7}$$

where $\tilde{U}_{tj}(t = 1, 2)$ is the marginal utility of consumption of commodity j in period t. These equations correspond to the consumer's optimum in the classical certainty case. With uncertainty, the classic condition for period-one consumption is simply replaced by its expectation, (6), while the certainty condition for period-two consumption is replaced by (7). At first, the intuitive meaning of (7) may not be obvious. It seems to, and actually does, imply that the classic certainty condition for period-two consumption cannot simply be replaced by *its* expectation.[11] Something more is required: The consumer may not know in period one what commodity prices, asset prices, or his own consumption choices in period two will be; but he does intend to make optimum choices once prices *are* known. This is exactly the idea depicted by (7): For any price realizations whatever, an optimum consumption allocation will be made. Notice how this corresponds to the assumption of dynamic programming that all future decisions will be optimal given the information available when they are made.

Inter-temporal Conditions. Turning from inter-commodity trade-offs within a given period to inter-temporal trade-offs in the same commodity, we can determine how nominal rates of return are related to rates of inflation. To obtain optimum conditions for money, assets, and commodity inventories in rate of return form, define the nominal rate of return for asset j as

$$\tilde{R}_j \equiv (\tilde{p}_{2j} - p_{1j})/p_{1j}$$

and the rate of commodity i's price inflation as

$$\tilde{I}_i \equiv (\tilde{P}_{2i} - P_{1i})/P_{1i}. \tag{8}$$

Let us assume that at least one commodity, denoted by subscript i, is consumed in positive quantity during both periods. Then the following inter-temporal conditions for money, assets, and commodity inventories must hold at an equilibrium point if it is to be optimum for each individual:

(money) $$E(\tilde{U}_{1i}) \geqslant E[\tilde{U}_{2i}/(1 + \tilde{I}_i)] \tag{9}$$

(asset, j, held long) $$E(\tilde{U}_{1i}) \geqslant E\{[\tilde{U}_{2i}(1 + \tilde{R}_j)\tilde{\lambda}_{ALj}]/(1 + \tilde{I}_i)\} \tag{10}$$

(asset, k, sold short) $$E(\tilde{U}_{1i}) \leqslant E\{[\tilde{U}_{2i}(1 + \tilde{R}_k)\tilde{\lambda}_{ASk}]/(1 + \tilde{I}_i)\} \tag{11}$$

[11] Of course, we are free to take expectations of both sides of (7) to obtain an equation that looks like the period-one condition (6). The result however, would be a necessary *but insufficient* requirement for an optimum.

(inventory, k, held long)
$$E(\tilde{U}_{1i}) \gtreqless E\{(\tilde{\lambda}_{XLk}/\tilde{\lambda}_{XLi})\tilde{d}_k - \tilde{g}_k/\tilde{P}_{2k}]\tilde{U}_{2i}[(1+\tilde{I}_k)/(1+\tilde{I}_i)]\} \tag{12}$$

(inventory, j, sold short)
$$E(\tilde{U}_{1i}) \lesseqgtr E\{\tilde{U}_{2i}\tilde{d}_j\tilde{\lambda}_{XSj}[(1+\tilde{I}_j)/(1+\tilde{I}_i)]\} \tag{13}$$

The strict equality holds for a given expression whenever the asset or commodity is held or sold short in non-zero quantity.[12] For example, the equation form of (10) is a value-storing condition for assets. It determines how asset j is used to store the marginal utility of commodity i given up in period one in return for marginal utility *of the same commodity* expected to be gained in period two. The extremum condition for money, equation (9), differs from the conditions (10) for other assets only by its unitary nominal price in both periods and its perfect quality for transaction-liquidity, ($\lambda = 1$).

Expression (12) provides the equilibrium relation between commodity k's rate of inflation (which is its gross return from holding inventory), with its physical deterioration characteristics d_k, storage costs g_k, and transaction-liquidity quality λ_{XLk}, relative to the liquidity quality of the commodity i which is consumed in positive amounts during both periods. If the commodity held in inventory is also consumed in both periods, expression (12) reduces to

$$E(\tilde{U}_{1k}) \gtreqless E[(\tilde{d}_k - \tilde{g}_k/\tilde{P}_{2k})\tilde{U}_{2k}] \tag{14}$$

with the strict equality effective only when a positive inventory of commodity k is held. Since marginal utility is strictly positive, (14) implies that a consumer holding a positive inventory of commodity k would require $P_{2k}d_k > g_k$ on a weighted expectations basis. If the quantity left after physical deterioration $P_{2k}d_k$ did *not* exceed the costs of storage g_k, zero inventory would be optimal.

Conditions for short positions in assets and commodities have similar interpretations. For example, combining expressions (10) and (11), which are for long and short positions in assets, we obtain

$$E\{[\tilde{U}_{2i}(1+\tilde{R}_k)\tilde{\lambda}_{ASk}]/(1+\tilde{I}_i) \geqslant E\{[\tilde{U}_{2i}(1+\tilde{R}_k)\tilde{\lambda}_{ALk}]/(1+\tilde{I}_i)\}. \tag{15}$$

The only terms different on both sides of this inequality are the transaction-liquidity cost elements, $\tilde{\lambda}$'s; so the expression is readily seen to require, again on a weighted-average basis, that $\lambda_{ASk} \gtreqless \lambda_{ALk}$. Actually, this restriction was satisfied by the initial parameterization of these transaction fees. For short sales we required $\lambda_{AS} > 1$, while for assets held long, $\lambda_{AL} < 1$. Inequality (15) implies that arbitrage

[12]For the expression involving commodity inventories held long, (12), the strict equality is in affect only if (a) the commodity is held in non-zero quantity and (b) the commodity is also consumed out of inventory in non-zero amount during period two.

possibilities would enable the consumer to obtain infinite consumption if these restrictions on the transaction-liquidity elements were not satisfied.

Market Implications of the Consumer's Optimization Requirements

In order to elucidate further the consumer's optimum conditions, it is convenient to consider their perfect certainty counterparts, obtained simply by dropping the expectations operators and random variable notations. This permits algebraic manipulation to bring the results into more familiar and intuitively meaningful form. Perfect certainty carries a connotation of complete agreement among all investors about future prices; for if any two disagreed, and both were perfectly certain, they would engage in an infinite quantity of forward transactions. Thus, when examining the individual's optimum conditions under certainty, we are entitled to speak of them as market equilibrium conditions.

Commodity Prices. Under certainty, if commodity i is consumed in positive amounts during both periods, (9) and (12) can be combined to obtain

$$1 + I_i \leqslant (P_{1i} + g_i)/(P_{1i}d_i) \tag{16}$$

which shows that the rate of inflation is not exogenous but is fixed by investors' actions on period-one prices. The upper bound on inflation is higher for a good whose storage costs (g_i) and deterioration ($1/d_i$) are larger because its price must be allowed to appreciate more in order to compensate for those losses.

When a commodity is consumed from inventory holdings in period two and is produced for consumption as well, (i.e., when $X_{Ci} > 0$ and $C_{2i} > 0$), the equality in (16) is binding. Such a commodity cannot be very perishable. As we consider more perishable commodities (as d_i becomes smaller), the rate of inflation must grow indefinitely larger if the equality in (16) is binding, even up to the limiting case where the commodity is totally perishable and provides an infinite rate of inflation. This limiting case is not possible, of course, because a completely perishable commodity would never be held in inventory as a final product. It could, however, be stored in a transformed state as a durable factor of production which could then be used for producing the commodity at the desired time.

Since physically-perishable commodities can be stored in a transformed state as factors of production, we are led to some interesting speculations about production cycles and commodity price cycles. If the price of a commodity with zero storage costs fluctuates predictably, producers would sell from inventory in periods of high price and build up inventory in periods of low price. If the price of a commodity that deteriorates rapidly or is costly to store fluctuates predictably, producers could store factors, timing their use so the production process finishes in periods of high price. Alternatively, producers could use durable goods storage to smooth fluctuations in perishable goods; for if producers expected perishable commodities to have higher prices next period than now, they could produce durables now for storage, thereby releasing the same factors to produce perishable goods next period.

Profit opportunities exist whenever the price of any commodity fluctuates predictably, and the conditions preventing competitive producers from erasing the predictability of price changes are quite stringent: only when production must adhere to a cycle because of physical conditions, when demand does not follow the same cycle, when the product is not freely storable, and when it requires specific factors of production can a predictable sequence of prices evolve. Even in such cases, however, producers often invent new ways to profit by inter-temporal price movements. For example, lettuce is now grown in California for Ohio consumption because it can be grown throughout the year in California. The seasonal change in the Ohio lettuce price is limited to the cost of shipping a head from California.

Uncertainty causes only a few complications for inter-temporal commodity pricing and only one complication is important: namely, risk premia will be demanded as part of the anticipated inflation rate. These premia will depend on the comovements of commodity i's price with its (uncertain) deterioration and storage costs and its marginal utility in period 2, (see expression 12). For most risk averters, variation in the marginal utility of a particular commodity will depend on the variation in all other commodity prices and in all asset prices. It has become well-accepted, given a predominance of risk averters, that the anticipated rate of return on an asset will be positively related to its covariation with other assets' rates of return, at least to a second level of approximation. Not surprisingly, when asset and commodity prices are determined jointly, identical relations are found for commodity rates of return, (i.e., rates of inflation).

For a commodity, k, sold short, the perfect certainty requirement for optimality is

$$1 + I_k \geqslant 1/(d_k \lambda_{XSk}). \tag{17}$$

Note that the short seller could obtain infinitely large gain in utility if this condition were not true: He could borrow a unit of commodity k in period one, sell it for P_{1k} and hold cash until period two when he would cover the short sale by purchasing a deteriorated unit for a net price of $P_{2k} d_k \lambda_{XSk}$, including transaction costs. The (certain) gain of $P_{1k} - P_{2k} d_k \lambda_{XSk}$ per unit would prompt him to trade as often as possible.

The long and short commodity optimum conditions can be combined to provide

$$1 + g_k/P_{1k} \geqslant 1/\lambda_{XSk}. \tag{18}$$

This is always satisfied for commodities with positive costs of storage $g_k > 0$ and reasonable transaction fees for short sales, $\lambda_{XSk} > 1$.

Assets Returns. For positive long positions in assets under perfect certainty, (10) becomes

$$1 + R_j = (U_{1i}/U_{2i})(1 + I_k)/\lambda_{ALj}. \tag{19}$$

Aside from the transaction-liquidity cost term λ_{Aj}, this appears to be a restatement of Irving Fisher's [3] famous relation between nominal interest rates, real interest rates, and the rate of inflation: U_{1i}/U_{2i} is the marginal rate of time preference for commodity i, i.e., unity plus the real rate of interest.

Of course, as equation (19) indicates, there is no such thing as "the" real rate of interest or as "the" rate of inflation. As Fisher later recognized, "There are, theoretically, just as many rates of interest expressed in terms of goods as there are kinds of goods diverging from one another in value" [4, p. 42]. With the introduction of uncertainty, this problem of multiple real interest and inflation rates becomes even more acute; for as Peles has recently noted, "Since different investors expect to spend their wealth on different goods and services and because they have different elasticities, it follows that they attribute different degrees of risk to the same asset, even if they maintain homogenous expectations regarding the probability distributions of returns on this asset" [12, p. 644]. The risky environment counterpart to (19), expression (10), depicts Peles' idea. Even if all investors happened to agree on the distributions of asset returns and inflation rates, (R_j's and I_i's), they would have differing risk-return tradeoffs whenever their marginal utilities of consumption (or their access to liquid asset markets) differed.

Even though such non-uniqueness offers potentially severe measurement difficulties in econometric analysis of nominal interest rates and inflation, on a theoretical level it is overshadowed by a different problem which arises because (19) is only one member of a system of equations. Note that (19) is really not a reduced form because it relates one rate of return R_j to another return I_k. The latter return is also an endogenous variable which can be and should be obtained as a function of the exogenous elements of the system.[13] As will be seen now, the full solution brings rather surprising implications.

One of the equations of the system that has been neglected so far is the non-stochastic equivalent of (9), the first-order condition for money:

$$(U_{1i}/U_{2i})(1 + I_i) \geqq 1. \tag{20}$$

Fisher seems to have overlooked the fact that his basic relation between nominal interest rates, real rates, and expected inflation rates also applies to the asset money, whose nominal rate is fixed at zero. Correction of this very important omission, by expression (20), shows that inflation rates are not free to adjust to any level but are fixed, through the first-order condition on money balances, into a relation with the marginal rate of time preference. In fact, if we assume that a non-zero quantity of money is held, (20) becomes a strict equality and it shows that the rate of commodity i's price inflation must be just offset in equilibrium by commodity i's marginal time preference rate; or, $U_{1i}/U_{2i} = 1/(1 + I_i)$.

Combining this latter condition with (19) gives $R_j = 1/\lambda_{ALj} - 1$; the nominal rate of interest need only compensate for transactions costs and deficiencies in the asset's liquidity quality when money is held and uncertainty is absent. Again, this is a

[13] In the absence of a supply side, the exogenous variables are the marginal rates of time preference, U_1/U_2, the physical rates of commodity inventory deterioration, d, the costs of storage, g, and the transaction -liquidity attributes of assets and commodities λ_A and λ_X.

well understood proposition: If money exists and offers no superior non-pecuniary service (i.e., if $\lambda_{ALj} = 1$), the nominal rate of interest on nominally riskless bonds must be zero. The only new feature of its appearance here is its explicit recognition as a logical implication of Fisher's real-nominal-inflation rate formula.

Now let us add uncertainty back to the system while still assuming that money is held in positive amounts. This will provide an expression relating the *expected* return on an asset to the *expected* rate of inflation with explicit identification of the impact of risk. Define an expected rate of return $\bar{R}_j$, on asset j by the standardization formula

$$\tilde{R}_j \equiv \bar{R}_j + \sigma_j \tilde{Z}_j \tag{21}$$

where $\sigma_j > 0$ is a dispersion parameter and $\tilde{Z}_j$ is a random variable with zero mean and unit dispersion. Furthermore, assume for simplicity that the transaction-liquidity element is non-stochastic. Then the stochastic conditions (9) and (10) are combined to obtain

$$\bar{R}_j = (1/\lambda_{ALj} - 1) - \sigma_j E[\tilde{U}_{2i}\tilde{Z}_j/(1+\tilde{I}_i)]/E[\tilde{U}_{2i}/(1+\tilde{I}_i)] \tag{22}$$

which depicts the expected rate of return on an asset as a function of liquidity and *risk alone* and not as a direct function of *expected* inflation. The expected return will approach zero as the dispersion σ_j of the probability distribution of $\tilde{R}_j$ approaches zero. Whenever the investor is a risk averter, his expected return $\bar{R}_j$ will tend to be positive[14] and will tend to be positively affected by $\tilde{R}_j$'s correlation with other asset returns. This corresponds to the Sharpe-Lintner [9, 13] capital asset pricing model that is now so familiar. The differences arise solely from the new assumptions here: no riskless asset but a risky rate of commodity price change.[15]

Only a second-order relationship exists between *expected* rates of inflation and nominal interest rates; that is, if expected inflation affects nominal interest rates at all, the effect must be transmitted through risk premia. If, for example, the inflation rate I_i were non-stochastic or if it were uncorrelated with asset returns and with other commodity returns,[16] it would cancel from equation (22)[17] and thus not affect the expected nominal rate of return of asset j.

[14] An intuitive argument to support this notes that U_{2i} is a declining function of consumption, (for a risk averter), while negative values of Z are associated with low consumption levels and vice versa. The combination produces a negative tendency for $E(\tilde{U}_{2i}\tilde{Z}_j)$. In fact, in the one asset case with no inflation, $E(\tilde{U}_{2i}\tilde{Z}_j)$ can be proven to be negative for any probability distribution whose expectation exists. However, in a multiple asset case, the possibility of negative correlations among I_i, R_j, and other assets and commodity returns which are transmitted through U_{2i}, causes the sign of R_j to be ambiguous as a general rule.

[15] Proof of the correspondence between equation (22) and the Sharpe-Lintner equilibrium model is contained in appendix B.

[16] Of course, these are not very likely circumstances.

[17] Because, e.g., if I_i and $U_{2i}Z_j$ were uncorrelated.

$$E[\tilde{U}_{2i}\tilde{Z}_j/(1+I_i)] = E(\tilde{U}_{2i}\tilde{Z}_j)E[1/(1+I_i)].$$

The denominator in (22) would factor similarly if U_{2i} and I_i were uncorrelated. Thus the terms $E[1/(1+\tilde{I}_i)]$ would cancel.

In the more general case, when all asset and commodity returns are permitted to be stochastic and statistically dependent, expected nominal rates of return will be inversely related to expected rates of inflation for some assets and positively related for other assets, the sign being determined (roughly speaking) for each asset by the correlation between its nominal return and the rate of inflation relative to the correlation between a composite asset and the rate of inflation.

To illustrate, assume that both money and asset j are held, and (for simplicity) assume the asset can be traded without cost and is perfectly liquid, ($\lambda_{ALj} = 1$). Then by elimination of $E(\tilde{U}_{1i})$ from equations (9) and (10), we obtain the condition

$$E(\tilde{U}_{2i}\tilde{R}_j\tilde{\rho}_i) = 0 \tag{23}$$

where $\rho_i \equiv 1/(1 + I_i)$ is defined to simplify notation. If we define the expected "force" of inflation, $\bar{\rho}_i$ by another standardization formula as $\tilde{\rho}_i = \bar{\rho}_i + \sigma_i \tilde{Z}_i$, where again $\sigma_i > 0$ is a dispersion parameter and Z_i is a standardized variate, (zero mean and unit dispersion), the relation between the expected nominal rate of return on this asset and the expected force of inflation may be obtained by differentiating (23), then setting $d\sigma_i = 0$ to maintain a constant inflation risk level and setting $dU_{2i} = 0$ to keep the individual on the same indifference curve. This gives

$$\partial\bar{R}_j/\partial\bar{\rho}_i \Big|_{\substack{d\sigma_i=0 \\ dU_{2i}=0}} = -E(\tilde{U}_{2i}\tilde{R}_j)/E(\tilde{U}_{2i}\tilde{\rho}_i). \tag{24}$$

Since $U_{2i} > 0$ if there are no bliss points and since $\rho_i > 0$ if prices cannot fall below zero (free disposal), we have $\partial\bar{R}_j/\partial\bar{\rho}_i \gtreqless 0$ as $E(\tilde{U}_{2i}\tilde{R}_j) \lesseqgtr 0$ or as

$$\bar{R}_j \lesseqgtr -\sigma_i E(\tilde{U}_{2i}\tilde{Z}_j)/E(\tilde{U}_{2i}). \tag{25}$$

Since (25) does not contain a term involving inflation, the co-movement of expected nominal rates and expected inflation rates ($\partial\bar{R}/\partial\bar{\rho}$) will have the same sign no matter what value is taken by the expected inflation rate. Thus, if we record an asset's expected return at a position of *certain* zero inflation and then add a stochastic rate of inflation, the expected return will move in the same direction no matter whether the (new) *expected* inflation rate is positive or is negative![18] This

[18]Proof: Starting from $E(\tilde{U}_{2i}\tilde{R}_j\tilde{\rho}_i) = 0$, we seek the partial derivative $\partial\bar{R}_j/\partial\sigma_i|\sigma_i = 0$; i.e., the movement of $\bar{R}_j$ when we change from a position of non-stochastic inflation, $\sigma_i = 0$, to a position of stochastic inflation. It is natural to maintain $dU_{2i} = 0$, and $d\rho_i = 0$ since we are only interested in the effect of inflation dispersion. With these conditions, the total derivative of the equation above can be solved to obtain,

$$\frac{\partial\bar{R}_j}{\partial\sigma_i}\Big|_{\sigma_i = 0} = -E(\tilde{U}_{2i}\tilde{R}_j\tilde{Z}_i)/E(\tilde{U}_{2i}\tilde{\rho}_i)$$

Again relying on free disposal and the absence of bliss points to make the denominator positive, the sign of $\partial\bar{R}_j/\partial\sigma_i$ is determined by $-E(\tilde{U}_{2i}\tilde{R}_j\tilde{Z}_i)$ which is unaffected by $\bar{\rho}_i$. Specifically $\partial\bar{R}_j/\partial\sigma_i|_{\sigma_i=0} \gtreqless 0$ as $\bar{R}_j \gtreqless -\sigma_j E(\tilde{U}_{2i}\tilde{Z}_i\tilde{Z}_j)/E(\tilde{U}_{2i}\tilde{Z}_i)$.

is, of course, an implication of inflation working through risk premia alone. The uncertainty about inflation can arise by the rate deviating about either a positive or a negative mean and the market price of such uncertainty will be the same.

Since this is a rather difficult point, it would seem helpful to examine a specific utility function, the quadratic, which has familiar qualities in the non-inflation case. In appendix B, a condition corresponding to the Sharpe-Lintner model is derived with a quadratic utility function and the aid of two assumptions:

(a) There exists a composite consumption good whose price can be used as a proxy for all commodity prices. (Its inflation rate I is denoted without a subscript, and $\rho = 1/(1 + I)$.)
(b) A riskless (in real terms) asset exists. If this is asset k, we require $[(1 + \tilde{R}_k)\tilde{\rho}] = (1 + R_F)\bar{\rho}$ = a constant.

Given these conditions, it is shown in Appendix B that

$$E(\tilde{R}_j\tilde{\rho}) = R_F\bar{\rho} + \beta_j(r_j)\,E[\tilde{R}_m\tilde{\rho} - R_F\bar{\rho}] \tag{26}$$

where R_m is the so-called "market" portfolio return, actually a value-weighted average of returns on all assets held by the investor, $\beta(\tilde{r}_j)$ is a risk coefficient *function* of r_j defined in "real" terms as

$$\beta[\tilde{r}_j] \equiv \mathrm{Cov}[\tilde{r}_j, \tilde{r}_m]/\mathrm{Var}[r_m]$$

and $\tilde{r}_j = \tilde{\rho}(1 + \tilde{R}_j) - \bar{\rho}(1 + R_F)$ is an "excess return" in real terms.

By using the standard definitions of covariance, (26) can be altered to

$$\bar{R}_j = R_F + (\bar{R}_m - R_F)\,\beta(\tilde{r}_j) - [\mathrm{Cov}(\tilde{R}_j, \tilde{\rho}) - \mathrm{Cov}(\tilde{R}_m, \tilde{\rho})\,\beta(\tilde{r}_j)]/\bar{\rho} \tag{27}$$

Since the terms before brackets have exactly the same form as the Sharpe-Lintner model; (27) indicates how additional risk premia, (to compensate for uncertainty in the inflation rate), are added to the demanded nominal rate of return. This is a potentially testable equation, provided we can use the common technique of measuring expected returns by actual observed returns plus stochastic errors.

Equation (27) may appear to be quite easily differentiable to yield an expression relating expected nominal interest rates to the rate of anticipated inflation.[19] Unfortunately, this is only an illusion because the risk coefficient $\beta(\tilde{r}_j)$ is not unchanging with respect to $\bar{\rho}$ and, as a consequence, the algebra necessary to find $\partial\bar{R}_j/\partial\bar{\rho}$ is exceedingly tedious. Nevertheless, it can be shown that

$$\partial\bar{R}_j/\partial\bar{\rho} \gtreqless 0 \quad \text{as} \quad \mathrm{Cov}(\tilde{R}_j, \tilde{\rho}) \gtreqless \bar{r}_m\;[\beta(\tilde{r}_j) - \bar{\rho}\beta(\tilde{R}_j)]\,.^{20} \tag{28}$$

Thus, the more an asset's nominal return co-moves with the rate of inflation, the

[19] Note that $\bar{\rho}$ and the expected inflation rate are negatively related. Since $\rho \equiv 1/(1 + I)$, $\bar{\rho} = E[1/(1 + I)]$ and $\partial\bar{\rho}/\partial\bar{I} = -E(\rho^2)$.

[20] Note again that $\beta(X)$ is a function of X, not $\beta \cdot X$. Thus, e.g., $\beta(R_j) = \mathrm{Cov}[\tilde{R}_j, \tilde{r}_m]/\mathrm{Var}(\tilde{r}_m)$.

more *likely* we are to observe the Fisher effect—positive co-movement between *expected* asset returns and *expected* rates of inflation. However, it is neither necessary nor sufficient to have a positive covariation between nominal interest rates and inflation rates in order to obtain a positive relation between their means. The covariation need only exceed a function which measures the difference between a risk coefficient for real rates of return $\beta(\tilde{r}_j)$ and a risk coefficient for the relation of the nominal return on the asset to the real return on the market, $\beta(\tilde{R}_j)$.[21] Note again that any such positive co-movement between the expected rate of asset return and expected inflation is due strictly to risk and would vanish entirely if events become certain.

III. SUMMARY

When consumer-investors are presumed to make simultaneous choices of assets and consumption in an economy containing money, the relation between nominal interest rates and expected rates of commodity price inflation is no longer the simple Fisherian equation. In fact, there will be no direct relation between nominal interest rates and the rate of anticipated inflation. If also there is neither uncertainty nor transaction costs, the nominal rate of interest will be zero or money will not be held. If there is uncertainty about future asset and commodity prices, however, money may be held even when nominal expected rates of return on assets are positive. Similarly, if assets can be traded only by paying transaction fees or if they cannot be traded as quickly as cash, money may be held even if there are positive returns on assets and perfect certainty.

Because different assets may have different co-movements between their nominal returns and the rate of inflation, a stochastic component in commodity prices will cause some nominal expected asset returns to be higher and other nominal expected returns to be lower than when commodity prices are non-stochastic. For the same reason of differing risk premia, expected nominal returns will vary positively with the expected inflation rate for some assets and negatively for other assets. Holding everything else equal, a positive relation is more likely the greater the covariance between observed nominal asset returns and observed inflation rates.

The rate of price inflation of a given commodity is determined endogenously and simultaneously with asset returns. A commodity rate of inflation will be positively related to its perishability and the costs of keeping it in inventory. Extremely perishable commodities will not be held in inventory because it is cheaper

[21] One frequently hears that observed rates of return on bonds are negatively correlated with observed rates of inflation, the reason being that actual rates of inflation which are higher or lower than anticipated cause revisions in the expected rate of inflation for the subsequent period. If the Fisher relation is valid, an upward revision in the expected rate of inflation brings about an increase in expected nominal yields. For bonds, this implies a reduction in the market price. But a reduction in market price is the same thing as a negative observed rate of return for that period, aside from the coupon payment. Hence, the result is that $\mathrm{Cov}(R_j, I) < 0$. Note from (28), however, that assets which actually display $\mathrm{Cov}(R_j, I) < 0$, and thus $\mathrm{Cov}(R_j, \rho) > 0$, are more likely than other assets to display a *negative* relation between *expected* rates of return and *expected* rates of inflation.

to store them in a transformed state, as factors of production, or to time their production to match the temporal pattern in demand.

Numerous restrictions are required on transaction costs, inventory storage costs, and physical deterioration of inventory in order to preclude infinitely-profitable arbitrage opportunities. For example, if short sales of commodity inventories are permitted, the rate of inflation must be at least as large as physical deterioration and transaction costs.

These conclusions have been provided by a two-period model of consumer choice. Assets, commodity inventories, money, and consumption are selected in period one. In period two, the assets are sold (and transaction fees are incurred); commodity inventories, (which are allowed to deteriorate in quality), are either sold or consumed, and the resulting cash is used to purchase other commodities for period-two consumption. The model allows uncertainty about nearly everything.

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APPENDIX A: MATHEMATIC OF THE CONSUMER'S MAXIMIZATION PROBLEM

The consumer's formal mathematical problem is to maximize expected utility,

$$\max E\,[U(C_1, \tilde{C}_2 + \tilde{d}\tilde{X}_C)]$$

subject to the following constraints:

$$P_1' C_1 + M + p_1' (A_L - A_S) + P_1' (X_L - X_S) - W \leqslant 0 \tag{A1}$$

$$-\,[P_1' C_1 + M + p_1' (A_L - A_S) + P_1' (X_L - X_S) - W] \leqslant 0 \tag{A2}$$

$$\tilde{p}_2'(\tilde{\lambda}_{AL} A_L - \tilde{\lambda}_{AS} A_S) + M + \tilde{P}_2' \tilde{d}\,[\tilde{\lambda}_{XL}(X_L - \tilde{X}_C) - \tilde{\lambda}_{XS} X_S] - \tilde{g}' X_L - \tilde{P}_2' \tilde{C}_2 \leqslant 0 \tag{A3}$$

$$-\,\{\tilde{p}_2'(\tilde{\lambda}_{AL} A_L - \tilde{\lambda}_{AS} A_S) + M + \tilde{P}_2' \tilde{d}\,[\tilde{\lambda}_{XL}(X_L - \tilde{X}_C) - \tilde{\lambda}_{XS} X_S] - \tilde{g}' X_L - \tilde{P}_2' \tilde{C}_2\} \leqslant 0 \tag{A4}$$

$$\tilde{P}_2' \tilde{d} \tilde{\lambda}_{XL} (\tilde{X}_C - X_L) \leqslant 0 \tag{A5}$$

$$M, X_L, X_S, \tilde{X}_C, A_L, A_S, C_1, \tilde{C}_2 \geqslant 0 \tag{A6}$$

where the symbols denote:

Variables

A = Asset quantity; η-element vector
X = Commodity Inventory; N-element vector
$1 - d$ = Physical inventory depreciation; $N \times N$ diagonal matrix
g = Inventory Storage Costs; N-element vector
$1 - \lambda$ = Transaction-Liquidity costs; $N \times N$ or $\eta \times \eta$ diagonal matrices
p = Asset Prices; η-element vectors
P = Commodity Prices; N-element vectors
W = Initial Wealth; scalar
C = Consumption; N-element vectors
M = Nominal Money Balances; scalar

Subscripts

A Assets
X Commodities
L Long Positions
S Short Positions
1 Period One
2 Period Two

In period one, consumption is denoted by the vector C_1. In period 2, consumption can occur either from commodities purchased at period two, the vector $\tilde{C}_2$, or from commodities held over in inventory from period one, the vector $\tilde{X}_C$, corrected for physical deterioration by premultiplication by $\tilde{d}$.

Note that inequality constraints (A1) and (A2) combined give the period-one financial *equality* constraint; expenditures on assets, inventory, and consumption equal initial wealth. Similarly, (A3) and (A4) combined give the period-two financial equality constraint.

Associated with each constraint (A1) through (A5) is a random Lagrange multiplier which is denoted $\tilde{\mathbf{J}}_k$ for constraint (Ak).

The "stochastic calculus" conditions for an extremum of this problem are as follows, (where U_{tk} denotes the partial derivative of the utility function with respect to the k^{th} element of the t^{th} vector argument; e.g., U_{1i} is the marginal utility of commodity i in period one):

For Variable	*The First Order Condition is*	
C_{1i}	$E[\tilde{U}_{1i} + P_{1i}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2)] \leqslant 0; i = 1, \ldots, N$	(A7)
$\tilde{C}_{2i}$	$\tilde{U}_{2i} - \tilde{P}_{2i}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4) \leqslant 0; i = 1, \ldots, N$	(A8)
$\tilde{X}_{Ci}$	$\tilde{U}_{2i} - \tilde{P}_{2i}\tilde{\lambda}_{XLi}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_5) \leqslant 0; i = 1, \ldots, N$	(A9)
M	$[E(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + (\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] \leqslant 0$	(A10)
A_{Lj}	$E[p_{1j}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + \tilde{p}_{2j}\tilde{\lambda}_{ALj}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] \leqslant 0; j = 1, \ldots, \eta$	(A11)
A_{Sk}	$E[p_{1k}(\tilde{\mathbf{J}}_2 - \tilde{\mathbf{J}}_1) + \tilde{p}_{2k}\tilde{\lambda}_{ASk}(\tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_3)] \leqslant 0; k = 1, \ldots, \eta$	(A12)
X_{Li}	$E[P_{1i}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + \tilde{P}_{2i}\tilde{d}_i\tilde{\lambda}_{XLi}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_5) - \tilde{g}_i(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] \leqslant 0; i = 1, \ldots, N$	(A13)
X_{Si}	$E[P_{1i}(\tilde{\mathbf{J}}_2 - \tilde{\mathbf{J}}_1) + \tilde{P}_{2i}\tilde{d}_i\tilde{\lambda}_{XSi}(\tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_3)] \leqslant 0; i = 1, \ldots, N$	(A14)

Corresponding to the basic seven inequality conditions for the seven choice variable vectors, the following equality conditions are required:

$$C_{1i}E[\tilde{U}_{1i} + P_{1i}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2)] = 0; i = 1, \ldots, N \tag{A15}$$

$$\tilde{C}_{2i}[\tilde{U}_{2i} - \tilde{P}_{2i}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] = 0; i = 1, \ldots, N \tag{A16}$$

$$\tilde{X}_{Ci}[\tilde{U}_{2i} - \tilde{P}_{2i}\tilde{\lambda}_{XLi}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_5)] = 0; i = 1, \ldots, N \tag{A17}$$

$$M E[(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + (\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] = 0 \tag{A18}$$

$$A_{Lj}E[p_{1j}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + \tilde{p}_{2j}\tilde{\lambda}_{ALj}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] = 0; j = 1, \ldots, \eta \tag{A19}$$

$$A_{Sk}\,E[p_{1k}(\tilde{\mathbf{J}}_2 - \tilde{\mathbf{J}}_1) + \tilde{p}_{2k}\tilde{\lambda}_{ASk}(\tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_3)] = 0; k = 1, \ldots, \eta \tag{A20}$$

$$X_{Li}\,E[P_{1i}(\tilde{\mathbf{J}}_1 - \tilde{\mathbf{J}}_2) + \tilde{P}_{2i}\tilde{d}_i\tilde{\lambda}_{XLi}(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_5) - \tilde{g}_i(\tilde{\mathbf{J}}_3 - \tilde{\mathbf{J}}_4)] = 0; i = 1, \ldots, N \tag{A21}$$

$$X_{Si}\,E[P_{1i}(\tilde{\mathbf{J}}_2 - \tilde{\mathbf{J}}_1) + \tilde{P}_{2i}\tilde{d}_i\tilde{\lambda}_{XSi}(\tilde{\mathbf{J}}_4 - \tilde{\mathbf{J}}_3)] = 0; i = 1, \ldots, N \tag{A22}$$

In addition, there are, of course, the original constraints (A1) through (A6); these being the first partial derivatives with respect to the random Lagrange multipliers.

To obtain conditions (A7) through (A22), Kushner's [8] basic method has been adapted to accommodate inequality constraints. Kuhn-Tucker [7] methodology was applied by adding slack variables to the stochastic inequality constraints. Equations (A15) through (A22) are, of course, the familiar Kuhn-Tucker conditions for non-negative variables. If the commodity or asset is held in positive quantity, these conditions require the corresponding expressions (A7) through (A14) to hold as equalities.

It is well known that Kuhn-Tucker conditions are necessary and sufficient for a maximum when the objective function is concave and differentiable and the constraints are linear. In the present application, this is equivalent to requiring risk aversion everywhere in the consumption space.

To obtain results that are more amenable to interpretation, we must eliminate the Lagrange multipliers (J's). For example, if money is held in non-zero quantity, and if C_{1i} and C_{2i} represent non-zero consumption of commodity i, (A7), (A8), and (A10) can be combined to provide

$$E(\tilde{U}_{1i}/P_{1i}) = E(\tilde{U}_{2i}/\tilde{P}_{2i})$$

which indicates that the expected ratio of marginal utility to price must be equal for commodity i in the two periods. Equations in the text are all obtained from (A7) through (A22) in this way.

APPENDIX B: THE CORRESPONDENCE BETWEEN SHARPE-LINTNER EQUILIBRIUM AND FIRST-ORDER EXTREMUM CONDITIONS FOR AN INVESTOR'S ASSET HOLDINGS (AS DEPICTED BY EXPRESSION 10)

To show the correspondence, we adopt the simple expedient of eliminating differences in assumptions between the present model and a Sharpe-Lintner [9, 13] environment. The crucial assumptions to be added to the present model are: (a) a riskless asset exists; (b) the commodity price inflation is zero (or equivalently, is any constant percentage for all commodities); (c) utility functions are quadratic; (d) where the inflation rate is non-stochastic there is no money or else money pays a fixed positive nominal rate of interest; (e) there are no transaction costs, inventory storage costs, or liquidity differences among assets and money; and (f) assets, money, and commodities are owned in positive amounts.

The Sharpe-Lintner Model with Inflation

First, we derive the equivalent condition to the Sharpe-Lintner model when inflation is stochastic but when an asset exists whose return is riskless in real terms. If this is asset k, we must have

$$(1 + \tilde{R}_k)\,\tilde{\rho}_i = \text{constant} \equiv (1 + R_F)\,\bar{\rho}_i \tag{B1}$$

where $\bar{\rho}_i = E[1/(1 + \tilde{I}_i)]$ and the constant R_F is chosen to satisfy the definition. For notational simplicity, we also define an *excess real return* on asset j as

$$\tilde{r}_j \equiv (1 + \tilde{R}_j)\tilde{\rho}_i - (1 + R_F)\bar{\rho}_i. \tag{B2}$$

When there are no trading costs and the riskless asset has perfect liquidity, the investor's first-order condition (10) can be written in terms of these newly-defined variables as $E(\tilde{U}_{1i}) = (1 + R_F)\bar{\rho}\ E(\tilde{U}_{2i})$ for the riskless real asset and $E(\tilde{U}_{1i}) = E[\tilde{U}_{2i}(1 + \tilde{R}_j)\tilde{\rho}_i]$ for any arbitrary asset j. Combining these two conditions gives

$$E[\tilde{U}_{2i}\tilde{r}_j] = 0. \tag{B3}$$

When individuals have quadratic utility functions, (a sufficient condition for the Sharpe-Lintner result), marginal utility for period-two consumption takes the form

$$U_{2i} = b_{2i} - 2a_{2i}b_{2i}C_{2i}$$

where a_{2i} and b_{2i} are positive constants. Thus (B3) can be simplified to

$$\bar{r}_j = 2a_{2i}E[\tilde{C}_{2i}\tilde{r}_j]. \tag{B4}$$

Given the absence of transaction-liquidity costs and inventory costs the period-two constraint on consumption is

$$C_{2i} = \frac{1}{\tilde{P}_{2i}}\left\{p_2'A + M + \tilde{P}_2'\tilde{d}(X - \tilde{X}_C) - \sum_{k \neq i}\tilde{P}_{2k}\tilde{C}_{2k}\right\},$$

where A and X denote *net* asset and commodity positions, respectively. Thus, the term on the right of B4 involves covariation between the real excess return for asset j, the returns on all other assets, and the inflation rates on all commodities.

Because of the presence of $\Sigma\,\tilde{P}_{2k}\tilde{C}_{2k}$ on the right of this equation, I have been unable to simplify (B4) without the aid of one further assumption; namely, a composite consumption good exists. If this good is denoted without an i subscript, period two consumption is

$$\tilde{C}_2 = \tilde{\rho}[\tilde{p}_2' A + M + \tilde{P}_2 \tilde{d}(X - \tilde{X}_C)] P_1$$

and thus

$$\bar{r}_j = 2aE\{\tilde{r}_j\tilde{\rho}[\tilde{p}_2' A + M + \tilde{P}_2 \tilde{d}(X - \tilde{X}_c)]\} P_1 \qquad \text{(B5)}$$

Now note that $\tilde{p}_2' A + M + P_2 \tilde{d}(X - \tilde{X}_C)$ can be written as a weighted average rate of return on the investor's total portfolio of assets, money, and inventory as follows:

$$\tilde{p}_2' A + M + \tilde{P}_2 \tilde{d}(X - \tilde{X}_C) = \sum_{k=0}^{N+1} p_{1k}(1 + \tilde{R}_k)A_k \equiv (W - P_1 C_1)(1 + \tilde{R}_m)$$

where $A_0 = M$, $(p_{10} = 1,\ R_0 = 0)$; $A_{N+1} = X - \tilde{X}_C$; $p_{1,N+1} = P_1$, $\tilde{R}_{N+1} = (1 + \tilde{I})\tilde{d}$; $W - P_1 C_1 = \Sigma_{k=0}^{N+1} p_{1k} A_k$ and $\tilde{R}_m$ is the average return. Thus, (B5) becomes

$$\bar{r}_j = \frac{2a(W - P_1 C_1)}{P_1} E[\tilde{r}_j \tilde{\rho}(1 + \tilde{R}_m)]. \qquad \text{(B6)}$$

Multiplying both sides of B6 by $p_{1j} A_j$ and summing over j gives

$$\bar{r}_m = \frac{2a(W - P_1 C_1)}{P_1} E[\tilde{r}_m \tilde{\rho}(1 + \tilde{R}_m)]$$

which can be used to eliminate $2a(W - P_1 C_1)/P_1$ from B6 and thus to obtain:

$$\bar{r}_j E[\tilde{r}_m \tilde{\rho}(1 + \tilde{R}_m)] = \bar{r}_m E[\tilde{r}_j \tilde{\rho}(1 + \tilde{R}_m)].$$

Recalling the definition of the real excess return, this becomes,

$$\bar{r}_j E[\tilde{r}_m^2 + \tilde{\rho}(1 + R_F)\tilde{r}_m] = \bar{r}_m E[\tilde{r}_j \tilde{r}_m + \tilde{\rho}(1 + R_F)\tilde{r}_j]$$

so that the terms involving R_F cancel and we obtain

$$\bar{r}_j E(\tilde{r}_m^2) = \bar{r}_m E(\tilde{r}_j \tilde{r}_m)$$

or, after subtracting $\bar{r}_j \cdot \bar{r}_m^2$ from both sides,

$$\bar{r}_j = \bar{r}_m \mathrm{Cov}(\tilde{r}_j, \tilde{r}_m)/\mathrm{Var}(\tilde{r}_m)$$

This can be written in nominal terms as

$$E(\tilde{R}_j \tilde{\rho}) = R_F \bar{\rho} + \beta_j E[\tilde{R}_m \tilde{\rho} - R_F \bar{\rho}] \tag{B7}$$

where $\beta_j = \text{Cov}[(1 + \tilde{R}_j)\tilde{\rho}, (1 + \tilde{R}_m)\tilde{\rho}]/\text{Var}[(1 + \tilde{R}_m)\tilde{\rho}]$.

The Case of Non-Stochastic Inflation

When the inflation rate is non-stochastic, $\rho = \bar{\rho}$ and all the inflation terms cancel from (B-7). This gives

$$\bar{R}_j = R_F + (\bar{R}_m - R_F)[\text{Cov}(\tilde{R}_j, \tilde{R}_m)/\text{Var}(\tilde{R}_m)] \tag{B8}$$

Equation (B-8) is identical in form to the Sharpe-Lintner equilibrium condition. There is a critical difference, however. The return R_m in (B8) is only an average on the *investor's* portfolio and R_F is a return viewed as risk-free *by the investor*. With an additional assumption that all investors hold the same probability beliefs, Sharpe and Lintner showed that (B8) is also a market-clearing condition where R_m is the average return on all risky assets (which are held in the same proportions by all investors) and R_F is, of course, the return to a unique riskless asset.

It is also easy to show that equation (B8) can be obtained without the assumption of the existence of a composite good when all commodity prices are non-stochastic. In the stochastic commodity price case, the composite good seems to play a role similar to the composite (market) asset in that it permits the system of optimum conditions to be reduced to a set of rather simple linear equations.