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The Journal of Business, Volume 52, Issue 1 (Jan., 1979), 63-93.

Stable URL:
http://links.jstor.org/sici?sici=0021-9398%28197901%2952%3A1%3C63%3AAISFTI%3E2.0.CO%3B2-4

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Alternative Investment Strategies for the Issuers of Equity Linked Life Insurance Policies with an Asset Value Guarantee*

In an earlier paper we derived an investment strategy for an insurance company which would eliminate the risks associated with the sale of equity linked life insurance policies with an asset value guarantee. In this paper we explore whether this riskless investment strategy has any practical utility, in view both of the impossibility of effecting continuous portfolio adjustment and of the costs which must be incurred in making discrete portfolio adjustments. By simulating the returns to issuers of these policies under different investment strategies, we find that discrete approximations to the riskless investment strategy do indeed reduce considerably the risk of extreme losses.

An equity linked life insurance policy with an asset value guarantee (ELPAVG) is an insurance policy whose benefit payable on death or at maturity consists of the greater of some guaranteed amount and the value of a reference portfolio which is defined by the deemed investment of a predetermined component of the policy premium in a portfolio of common stocks or mutual fund—the reference fund. In an earlier paper (Brennan and Schwartz 1976), it was demonstrated that the benefit payable under an ELPAVG could be decomposed into the known guaranteed amount and an immediately exercisable call option to purchase the reference portfolio for an exercise price equal to the guaranteed amount. The principles of the option pricing model[1] were then employed to derive the equilibrium premium for both a single-premium ELPAVG contract and a periodic-premium contract, and some numerical examples were presented. It was further noted that the hedging arguments, which are the core of most of the recent theory of option pricing,[2] could be em-

* This research was supported by a grant from the S. S. Huebner Foundation, The Wharton School, University of Pennsylvania. The authors are grateful for the suggestions of a referee.

1. For the genesis of modern option pricing theory, see Black and Scholes (1973) and Merton (1973); for an excellent survey, see Smith (1976).

2. Cox and Ross (1976) stress the general significance of the hedging arguments. For examples of option pricing theo-

(*Journal of Business*, 1979, vol. 52, no. 1)

0021-9398/79/5201-0003$02.42

ployed to derive an investment strategy for the insurance company which would *eliminate* the risks associated with the liability under its ELPAVG contracts. This is an important result, for ELPAVGs may pose a significant threat to the solvency of insurance companies since the risks of loss under different contracts are not independent but are commonly related to the overall performance of the reference fund.[3] Actuaries have responded to this threat by attempting to determine a level of reserves sufficient to reduce the probability of ruin to an acceptable level.[4] On the other hand, adoption of the riskless investment strategy described in Brennan and Schwartz (1976) in theory eliminates the need to hold any reserves except against mortality risk.

In practice, of course, complete elimination of the investment risk is unattainable. A major reason for this is the cost of transacting, neglected in our previous model, which precludes continuous adjustment of the amount invested in the reference fund as required under the riskless investment strategy. Corby goes so far as to say in reference to the riskless investment strategy that "here theory and practice are irreconcilable. Even given costless transfers two practical matters will lead to a mismatched position. First, the reaction time required and the 'size' of the market will make anything approaching instantaneous adjustment impossible. Secondly if the market is misspecified by the model then the position will anyway be mismatched" (1977, p. 273).

It is therefore of interest to inquire whether the riskless investment strategy has any practical utility, in view both of the impossibility of effecting continuous portfolio adjustment and of the costs which must be incurred in making discrete portfolio adjustments. In this paper we attempt to answer this question by simulating the returns to issuers of ELPAVGs under different investment strategies. In particular, we compare the results of following the naive strategy of actually investing in the reference fund the amount deemed to be so invested under the policy contract allowing for transactions costs, with the results of following discrete approximations to the riskless investment strategy. We find that such risk-reducing investment strategies do indeed reduce considerably the risk of extreme losses and therefore reduce the level of reserves which must be held by the insurance company to achieve a given probability of ruin.

In the next section we review briefly the theory of the equilibrium pricing of ELPAVGs and of the riskless investment strategy. In Sec-

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ries based on restrictions on investor tastes and general equilibrium considerations, rather than on hedging arguments, see Geske (1976) and Rubinstein (1976).

3. Corby notes in this connection: "The end of 1974 presented serious problems to life offices, but *except for a few rather special examples*, they survived" (1977, p. 265; emphasis added). This relates to the collapse of the London stock market in 1974.

4. See, for example, Di Paolo 1969. Corby (1977) reports that regulatory agencies in Canada and France prescribe minimum levels of reserves to be held by insurance companies against ELPAVG liabilities.

tion II the pricing assumption to allow for transactions costs is described, and the alternative investment strategies are defined. Section III presents the main simulation results for a basic single premium and periodic premium contract. Section IV presents some further results.

I. Pricing and the Riskless Investment Strategy without Transactions Costs

We shall use the following definitions: $x(t)$, the value of the reference portfolio at time t, which is unknown at the time the contract is initiated; $b(t)$, the benefit payable at time t in the event of death or contract maturity, whose value is also unknown at the time the contract is initiated; $g(t)$, the minimum guaranteed benefit payable at t, which is determined at the time the contract is initiated; $V_\tau[b(t)]$, the value at time τ of the right to receive the uncertain amount $b(t)$ at time t; r, the known constant riskless interest rate.

The benefit payable at time t may be written as

$$b(t) = g(t) + \max\,[x(t) - g(t), 0], \tag{1}$$

and the present value of this benefit at the time the contract is initiated, $\tau = 0$, is

$$V_0[b(t)] = g(t)e^{-rt} + W[x(0), t, g(t)], \tag{2}$$

where $W[x(\tau), t-\tau, g(t)]$ is the value at time τ at a European call option to purchase the reference portfolio at time t for the guaranteed amount $g(t)$, conditional on the current value of the reference portfolio, $x(\tau)$. It is assumed that between the dates of deemed investment in the reference portfolio the rate of growth of the reference portfolio, which is the rate of return on the reference fund, follows the stochastic differential equation

$$\frac{dx}{x} = \mu d\tau + \sigma dz, \tag{3}$$

where dz is a Gauss-Wiener process and $E(dz) = 0$; $E(dz^2) = d\tau$. The instantaneous expected rate of return on the reference fund is μ and may be either deterministic or stochastic; σ^2 is the instantaneous variance of the rate of return on the reference fund. If μ is constant, then (3) implies that, conditional on its current value, the value of the reference fund at any future point in time is lognormally distributed. Given (3), the hedging arguments developed by Black and Scholes (1973) and Merton (1973) imply that the value of the call option must satisfy the partial differential equation

$$\tfrac{1}{2}\sigma^2 x^2 W_{xx} + rxW_x - rW + W_\tau = 0 \tag{4}$$

where the subscripts denote partial derivatives. With the appropriate boundary conditions, (4) may be solved for the initial value of the call option, $W[x(0), t, g(t)]$, which appears in (2). In the case of the single-premium contract this reduces to the standard Black-Scholes equation for the value of a European call option; for the periodic-premium contract the solution must be obtained by numerical methods. In either case, the present value of the benefits payable at time t, $V_0[b(t)]$ may be evaluated.

To allow for mortality let $\alpha(\tau, t)$, $(t, \tau = 1, \ldots, T)$ denote the probability that the contract will mature in year t, given that the policyholder is alive in year τ.[5] The probability that the policyholder will die in year t is $\alpha(0, t)$, $(t = 1, \ldots, T-1)$, and

$$\alpha(0, T) = 1 - \sum_{t=1}^{T-1} \alpha(0, t).$$

Of course $\alpha(0, t)$ will depend upon the age of the policyholder at the time the contract is written as well as upon sex, race, and the other factors which affect mortality experience.

Assuming that there are no costs of running the business and that the insurance company is able to eliminate mortality risk by writing a large number of contracts, the equilibrium premium for the single-premium contract, Z, is equal to the weighted sum of the present values of the benefits payable in each year of the contract life:

$$Z = \sum_{t=1}^{T} \alpha(0, t)\, V_0[b(t)]. \tag{5}$$

For the periodic premium contract it is necessary to take account of the possibility that, on account of early death, not all of the premia will actually be paid. Then the equilibrium annual premium, z, is obtained by equating the present value of the expected premia to be paid to the expected present value of the benefits, Z:

$$z = Z/\sum_{t=1}^{T} \alpha(0, t) \sum_{k=0}^{t-1} e^{-rk}. \tag{6}$$

The riskless investment strategy is derived by noting that at time τ the insurance company is implicitly short $\alpha(\tau, t)$ call options on the reference portfolio of maturity t $(t = \tau+1, \ldots T)$. Following the Black-Scholes, Merton hedging arguments, the riskless investment strategy requires that an amount, $x(\tau)\, W_x[x(\tau), t-\tau, g(t)]$, be invested in the reference fund for each option of maturity t sold short; so the

5. We follow the convention that if death takes place in year t it occurs at the end of the year and before any of the following year's premia are paid. This discretization of the problem is on account of the discrete structure of the mortality tables used. If a continuous probability density were available, the summations over t would be replaced by the appropriate integrals.

aggregate amount invested in the reference fund under the riskless investment strategy, $H[x(\tau), \tau]$, is

$$H[x(\tau), \tau] = x(\tau) \sum_{t=\tau+1}^{T} \alpha(\tau, t)\, W_x[x(\tau), t-\tau, g(t)]. \tag{7}$$

The partial derivative $W_x(\cdot)$ may be evaluated directly for the single-premium contract following Black and Scholes (1973); for the periodic-premium contract it must be approximated by a finite difference derived from the numerical solution to the differential equation. It may be noted that $H(\cdot)$ is never greater than $x(\tau)$ since

$$\sum_{t=\tau+1}^{T} \alpha(\tau, t) = 1$$

and $W_x(\cdot) \leqslant 1$.[6] Thus the investment in the reference fund required under the riskless strategy never exceeds the amount deemed to be invested. If it were absolutely certain that the guarantee would not be effective the riskless investment strategy would require an investment in the reference fund equal to the deemed investment. However, since there is always some probability that the guarantee will be effective, the insurance company hedges against this under the riskless strategy by retaining some investment in riskless securities.

II. Pricing with Transactions Costs and Alternative Investment Strategies

Pricing with Transactions Costs

Since transactions costs preclude the riskless investment strategy, it is no longer possible, when account is taken of these costs, to employ the hedging arguments developed by Black-Scholes and Merton to obtain the differential equation (4), from which the equilibrium premium was derived. In order to compare the profit outcomes under alternative investment strategies, it is necessary to make an *assumption* about the way in which an ELPAVG contract is priced when there are transactions costs. The assumption made should, however, be regarded as tentative: if the premium arrived at under this assumption leads to negative expected profits with the investment strategy chosen, then it should of course be increased so that the expected profits are no longer negative but are commensurate with the degree of risk borne by the insurance company under that investment strategy. The pricing assumption, therefore, provides no more than a benchmark, relative to which the profits and losses incurred under the different investment strategies may be assessed.

6. For the single-premium contract $W_x(\cdot)$ is equal to the value of a cumulative normal density function which can never exceed unity (see Brennan and Schwartz 1976); this is also the case for the periodic premium contract.

In defining the benefits payable under an ELPAVG contract to take account of transactions costs, we follow the convention that the benefit payable is still given by (1), so that the benefits are independent of the level of transactions costs. This means that, under the investment policy specified in the policy contract which defines the reference portfolio, the insurance company is responsible for the transactions costs which will be incurred in making the deemed investment in the reference portfolio and in liquidating this investment at contract maturity. The total premium charged is assumed to reflect these costs, and the particular pricing assumption adopted is that the contract is priced according to the methods of the previous section, with the difference that the costs of making the deemed investments in the reference portfolio are taken into account, together with the "present value"[7] of the costs of liquidating the investment in the reference portfolio at maturity.

The transactions costs are assumed to be a constant fraction, v, of the amount of the reference fund bought or sold; transactions in the riskless security are assumed to be costless. Then, taking account of transactions costs, the cost to the company, $c(t)$, of providing the benefit, $b(t)$, is given by

$$c(t) = (1 - v)^{-1}b(t). \tag{8}$$

This implicitly assumes that all sales of securities to finance the benefit payable are sales of the reference fund, so that if the value of the reference portfolio after transactions costs falls short of the guarantee the deficiency is made good by sale of other common stock held by the company.[8]

7. Transactions costs invalidate the formal concept of present value, so we use the term loosely here.

8. It should be noted that the implicit assumption we are making is only one of several possible. For example, if it is assumed that any deficiency between the net proceeds from the sale of the reference portfolio and the guaranteed amount is made good by sales of riskless securities which incur no transactions costs, then the cost to the company of providing the benefit will depend upon the investment strategy followed. If the company actually invests the deemed amount in the reference fund, the cost of providing the benefit will be

$$\begin{aligned} c(t) &= (1 + v)x(t) && \text{if } x(t) \geq g(t) \\ &= g + vx(t) && \text{if } x(t) \leq g(t) \end{aligned}$$

where the term vx represents the costs of selling the reference portfolio. On the other hand, if the company actually invests $(1 - v)^{-1}$ times the deemed investment in the reference fund, the value of risky securities held by the company at time t will be $(1 - v)^{-1}x(t)$ and the cost of providing the benefit will be given by

$$\begin{aligned} c(t) &= (1 - v)^{-1}x(t) && \text{if } x(t) \geq g(t) \\ &= (1 - v)^{-1}x(t) + g - x(t) && \text{if } x(t) \leq g(t). \end{aligned}$$

The cost of providing the benefit is always less under the first strategy than under the second. On the other hand, under the first strategy the company does not have enough invested in the reference fund to meet its liability in full if $x(t) > g(t)$. The simple

Considering first the single-premium contract, it is apparent that since the insurance company's liability is increased by the factor $(1 - \nu)^{-1}$ on account of transactions costs the premium allowing for transactions costs, $\hat{Z}$, must be $(1 - \nu)^{-1}$ times as large as the equilibrium premium without transactions costs, Z. In addition, the premium must be augmented by the transactions costs incurred in making the initial deemed investment in the reference portfolio. Denoting this deemed investment by D, the premium to be charged for a single-premium contract when there are transactions costs, $\hat{Z}$, is given by

$$\hat{Z} = (1 - \nu)^{-1}Z + \nu D. \tag{9}$$

The considerations are similar for the periodic-premium contract. The cost of providing the benefit payable is again given by (8). Hence the periodic premium allowing for transactions costs, $\hat{z}$, must be $(1 - \nu)^{-1}$ times as large as the equilibrium premium derived ignoring transactions costs, z. Additionally, each periodic premium must be increased by the amount of transactions costs incurred in making the deemed periodic investment in the reference portfolio, d. Hence, under our tentative pricing assumption, the premium to be charged for the periodic premium contract is

$$\hat{z} = (1 - \nu)^{-1}z + \nu d. \tag{10}$$

Alternative Investment Strategies

Two types of investment strategy are compared. The first, the naive investment strategy, ignores the risk associated with the provision of the guarantee and corresponds to the type of investment policy currently followed by issuers of ELPAVGs. The second, the risk-reducing investment strategies, approximate the riskless investment strategy by requiring at fixed adjustment intervals the full adjustment of the amount invested in the reference fund to the amount determined under the riskless investment strategy. A filter type of risk-reducing investment strategy was also tried. Under this strategy no adjustment of the amount invested in the reference fund took place until the actual amount differed from that required under the riskless strategy by a predetermined proportion, then full adjustment took place. The results of the filter strategy are reported in the Appendix.

If the value of the reference portfolio, $x(t)$, exceeds the guaranteed amount, $g(t)$, at contract expiration, the company's liability, $c(t)$, is equal to $(1 - \nu)^{-1}x(t)$, so that for the company's investment in the reference fund to be sufficient to meet this liability it is necessary that the amount the company actually invests in the reference fund be $(1 -$

assumption adopted in the text avoids these difficulties, and it should be remembered that the pricing assumption is to be regarded only as tentative so that approximations of this nature are unimportant.

$\nu)^{-1}$ times the amount deemed to be invested. Therefore, for the single-premium policy, the company is assumed under the naive strategy to make an initial investment $(1 - \nu)^{-1}D$ in the reference fund. Taking account of the transactions costs incurred in making this investment, the company is left with an amount $\hat{Z} - (1 + \nu)(1 - \nu)^{-1}D$, which is invested in the riskless security.

At the end of each year of contract life ($t = 1, \ldots, T$), the insurance company is assumed to enjoy its anticipated mortality experience on an average contract so that the benefit payable is $\alpha(0, t)b(t)$. This benefit is financed by selling off part of the investment in the reference fund, and the value of the investment in the reference fund is reduced by this amount plus transactions costs. Should the value of the investment in the reference fund fall to zero, the benefit is assumed to be paid by sales of the riskless security or by borrowing at the rate of interest earned on the riskless security. After paying the final benefit at maturity of the contract, $t = T$, the company is assumed to liquidate its remaining investment in the reference fund and the riskless security. Its resulting cash position represents the profit or loss on the contract.

The naive investment strategy is similar for the periodic-premium contract, whose premia are assumed to be payable monthly. Each month the company invests $(1 - \nu)^{-1}d$ in the reference fund from the premium received, the remainder of the premium being invested in the riskless security. At the end of each year ($t = 1, \ldots, T$) when the benefit $\alpha(0, t)b(t)$ is assumed payable, the incremental investment in the reference fund is $(1 - \nu)^{-1}d - \alpha(0, t)b(t)$. The profit or loss on the periodic premium contract is computed in the same way as for the single-premium contract.

Considering next the risk-reducing investment strategies, recall that the riskless investment strategy in the absence of transactions costs requires an investment in the reference fund at time τ of $H[x(\tau), \tau]$ given by (7). Recognizing again that the company's liability payable at time t, $c(t)$, is $(1 - \nu)^{-1}$ times the benefit, $b(t)$, it follows that the riskless investment strategy, taking account of the terminal transactions costs, requires an investment in the reference fund $\hat{H}[x(\tau), \tau]$, given by $\hat{H}[x(\tau), \tau] = (1 - \nu)^{-1} H[x(\tau), \tau]$.

For the single-premium contract, the initial investment in the reference fund under any of the risk-reducing investment strategies is $\hat{H}(D, 0)$ since the initial value of the reference portfolio is equal to the amount of the deemed investment, D. Taking account of the transactions costs incurred in making this initial investment in the reference fund, the amount of the premium left over to be invested in the riskless security initially is $\hat{Z} - (1 + \nu)\hat{H}(D, 0)$. A particular risk-reducing strategy of the class considered here is defined by the length of time between adjustments of the amount invested in the reference fund to the level required under the riskless investment strategy—the adjustment interval. At the end of each adjustment interval which does not coincide with a year end, the investment in the reference fund required under the riskless investment strategy, $\hat{H}[x(\tau), \tau]$, is calculated and compared with the actual amount invested in the reference fund. If the actual investment exceeds $\hat{H}[x(\tau), \tau]$ the excess is assumed to be sold, and the investment in the riskless security is increased by the net proceeds after transactions costs. If the actual investment falls short of $\hat{H}[x(\tau), \tau]$ the deficiency is made good by additional investment in the reference fund, and the investment in the riskless security is reduced by the corresponding amount plus transactions costs.

At the end of each year, the expected benefits payable, $\alpha(0, t)b(t)$, are paid by reducing the investment in the riskless security by this amount. Since the year end always coincides with end of an adjustment interval for the adjustment intervals chosen, the procedure followed at the end of each adjustment interval, described above, is followed again. By assuming that the benefits are paid in the first instance out of the investment in the riskless security rather than the investment in the reference fund, unnecessary transactions costs are avoided. At the final maturity of the contract, $t = T$, the remaining investment in the reference fund and the riskless security is liquidated, and the net cash position after transactions costs represents the profit or loss on the contract.

The risk-reducing strategy for the periodic-premium contract is analogous to that for the single-premium contract, with the difference that the premium received each month is allocated to the reference fund if the level of investment in the reference fund falls short of that required under the riskless strategy, $\hat{H}[x(\tau), \tau]$, even if the month does not correspond to the end of an adjustment interval.

III. Simulation Results for Two Basic Contracts

The performance of the risk-reducing investment strategies was compared with that of the naive investment strategy by simulating the results for a basic single-premium contract and a basic periodic-premium contract. For each contract 1,000 simulations were run, and the distribution of profits earned by the insurance company under the different investment strategies was analyzed. The parameters of the basic contracts considered are given in table 1 and are representative of the contracts currently written by insurance companies: Note in particular that the guarantee is limited to the sum of the investments made in the reference portfolio up to the time of death or contract maturity. This investment component of the premium is \$1,000 in the case of the single-premium contract and \$8.33 per month for the periodic-premium contract whose premia are paid monthly.

TABLE 1 Parameters of Basic Contracts

	Single-Premium Contract	Periodic-Premium Contract
Age of purchaser (yr)	35*	35*
Term of contract (yr)	10	10
Investment component of premium ($)	1,000	100/year paid monthly
Guarantee ($)	1,000	Sum of investments deemed made to date

* The mortality experience is determined from the Canadian Assured Lives Select 1958–64 for males, actuarial table compiled by Canadian Institute of Actuaries, Ottawa.

TABLE 2 Environmental Parameters

Variance rate on the reference fund (σ^2)	.04 per annum
Mean instantaneous rate of return on reference fund (μ)	.08 per annum
Riskless interest rate	.06 per annum
Transactions costs	.02

The environmental parameters assumed are given in table 2. Besides the level of transactions costs, the critical environmental parameters are the mean and variance of the rate of return on the reference fund. The instantaneous-risk premium on the reference fund is $(\mu + \frac{1}{2}\sigma^2 - r)$, and this quantity represents the instantaneous reward to bearing the investment risk of the reference fund. Since under the naive strategy the company is bearing the part of this investment risk represented by the guarantee, it is to be expected that under the naive strategy the mean profit of the insurance company will be highly dependent on this risk premium; insofar as the risk-reducing investment strategies are successful in eliminating this risk they will also eliminate this mean profit. Further, the additional transactions costs incurred under the risk-reducing strategies can be expected to result in mean losses. The mean and variance of the rate of return on the reference fund were chosen to correspond to the returns reported by Ibbotson and Sinquefield (1976) for the Standard and Poor's Index for the period 1926–74.[9] The effect of varying these environmental parameters as well as the term of the policy was also explored by simulation, and the results are reported below.

The frequency distributions of profits realized under the naive strategy and under the risk-reducing strategies with adjustment inter-

9. Ibbotson and Sinquefield (1976) report the arithmetic mean m and variance s^2 of the annual returns on the S & P Index 1926–74 as 0.1086 and 0.05067. Assuming a lognormal distribution for the returns, the parameters of the lognormal distribution are given by

$$\sigma^2 = \ln(1 + s^2/(1 + m)^2) = .0392$$

and

$$\mu = \ln(1 + m) - 1/2\sigma^2 = .0835.$$

vals of 1, 3, 6, and 12 months are plotted in figures 1 and 2 for the single-premium contract and in figures 3 and 4 for the periodic-premium contract. As anticipated, the distribution of profits under the naive investment strategy exhibits extreme negative skewness. On most occasions the guarantee is not effective, and the insurance company profits by the amount charged for the guarantee—the excess of the premium over the amount deemed to be invested in the reference portfolio, allowing for transactions costs. However, on those occasions on which the value of the reference portfolio falls short of the guarantee, the company runs the risks of extreme losses, which the risk-reducing strategy is intended to eliminate. By comparison, the distribution of profits for the risk-reducing strategies is much more symmetric, and as the adjustment interval is reduced the distribution becomes more concentrated.

Further information on the results for this single-premium contract is presented in the first five lines of table 3 and for the periodic-premium contract in the first five lines of table 4. The naive strategy yields positive average profits as expected, while the risk-reducing strategy

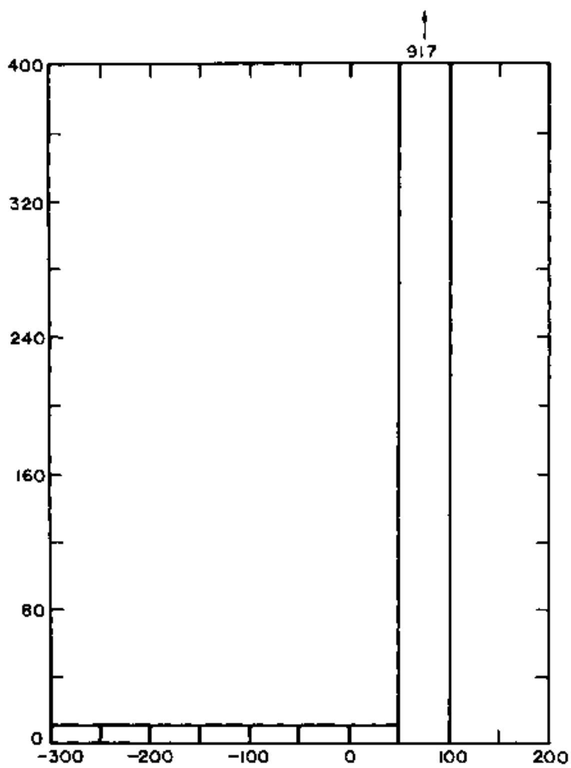

FIG. 1.—Basic example single-premium contract frequency distribution of profits under the naive investment strategy.

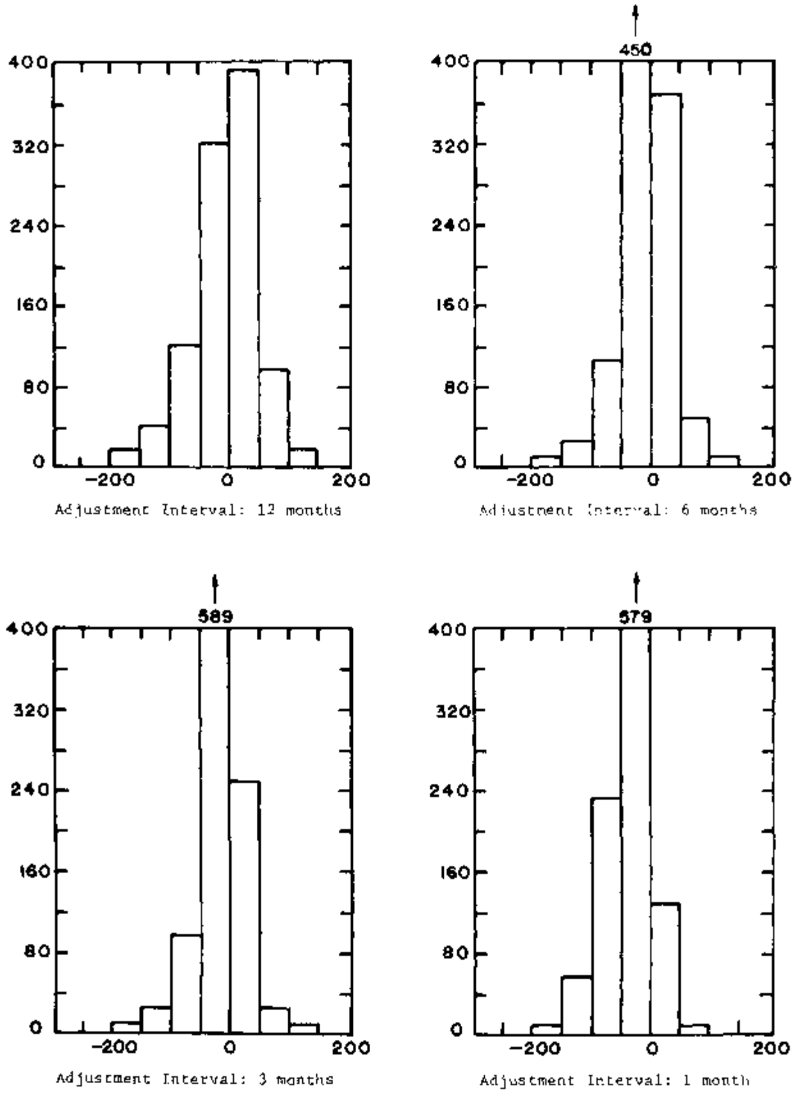

FIG. 2.—Basic example single-premium contract frequency distribution of profits under the risk-reducing investment strategies.

yields small mean losses which increase in absolute value as the frequency of adjustment is increased, reflecting the consequent increase in transactions costs. On the other hand, the risk-reducing strategies reduce the standard deviation of the profit outcomes by more than 50% and effect a dramatic reduction in the risk of extreme losses. Further, recalling that the riskless investment strategy without transactions costs would yield a mean profit of zero, it is apparent that the excess

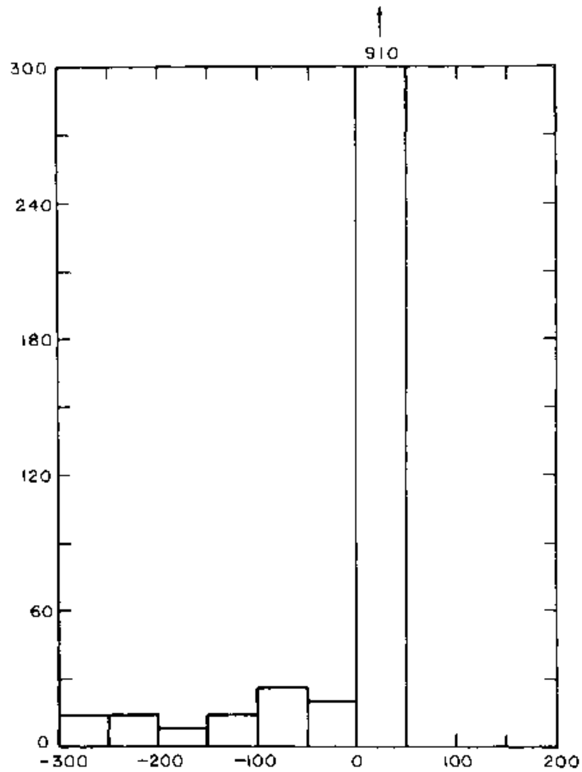

FIG. 3.—Basic example periodic-premium contract frequency distribution of profits under the naive investment strategy.

transactions costs generated by following the risk-reducing investment strategies do not have a large effect on the mean profit realization. For example, the mean loss for the periodic-premium contract with an adjustment interval of 3 months is \$10.32; this mean loss would be eliminated by increasing the monthly premium from \$8.98 by a mere 6.3¢. This amount, invested in the riskless security each month, would compound to \$10.32.

Suppose that the monthly premium for the periodic-premium contract were raised by 6.3¢ to \$9.04. The mean profit under the naive strategy would then be \$34.82, and zero under the 3-month adjustment strategy. On the other hand, the naive strategy would have a 5% chance of resulting in a loss of more than \$91.47 and a 1% chance of a loss of more than \$282.04. The corresponding figures for the 3-month strategy would be only \$38.96 and \$70.70. It seems that the 3-month adjustment strategy could well appear preferable to the naive strategy to an insurance company concerned with the risk of ruin. Of course, the strategy chosen will reflect the attitudes of the insurance company

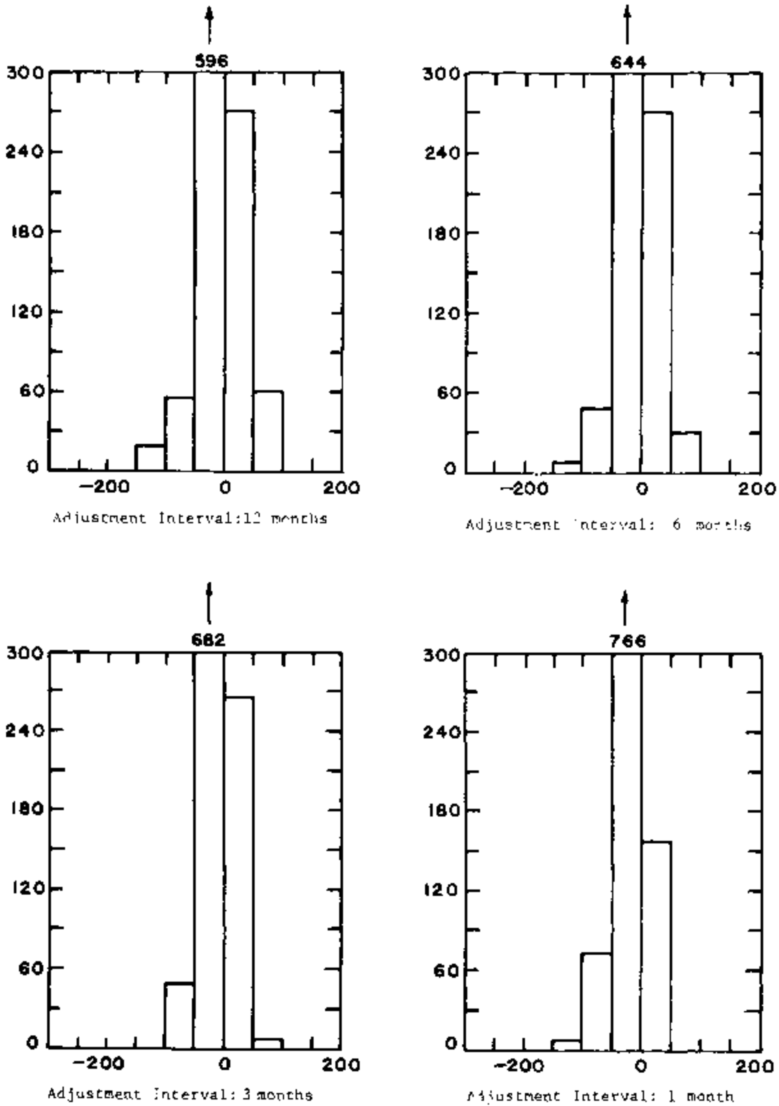

FIG. 4.—Basic example periodic-premium contract frequency distribution of profits under the risk-reducing investment strategies.

toward risk, and the risk-reducing strategies should be regarded only as providing a range of alternatives to the naive strategy which has been the only one considered heretofore. In the next section we shall consider the possibility of further extending the range of possible investment strategies by combining the five basic strategies considered in this section. Meanwhile, we consider the sensitivity of our results to the parameter values chosen.

Sensitivity Analysis

The remainder of tables 3 and 4 report the results of varying one parameter at a time from its value in the basic examples. The particular parameter varied is specified, and the sensitivity of the results may be assessed by comparing each line with the corresponding line for the basic example.

The return on the riskless security (r). Holding constant the distribution of return on the reference fund, a change in the riskless rate implies a change in the reward for bearing the investment risk of the reference fund. Since under the naive strategy the insurance company is bearing a part of this risk by provision of the guarantee, it is to be expected that a reduction in the riskless rate, by raising the reward for bearing investment risk, will increase the mean profit under the naive strategy, and indeed this is found to be the case. In fact, a change in the riskless interest rate, *ceteris paribus*, results in a very simple change in the distribution of profits under the naive investment strategy. A reduction in the riskless rate raises the equilibrium premium on the contract,[10] and under the naive investment strategy the whole of this increase in the premium is invested at the riskless interest rate. Although this now compounds at a lower rate, the net effect is an increase in the insurance company's terminal investment in riskless securities. Since, given the parameters of the distribution of the return on the reference fund, the company's liability under the guarantee is unaffected by the change in the riskless rate, a change in the riskless rate affects only the mean of the profit distribution under the naive strategy, leaving the other central moments unaltered.

In comparison with the naive investment strategy, the mean profit of the risk-reducing strategies is relatively insensitive to the riskless interest rate and bears no systematic relationship to it. This is to be expected, as insofar as the risk-reducing strategies eliminate the investment risk associated with the reference fund they eliminate the reward for bearing that risk. The major effect of an increase in the riskless rate is to reduce the standard deviation of the profits under the risk-reducing strategies. This appears to be because the closer the riskless rate is to the expected rate of return on the reference fund, the less is the expected profit or loss from being overhedged or underhedged. Since the risk-reducing strategies result in random overhedging and underhedging, this reduces the variability of the profit outcomes. Support for this surmise is gained from the fact that the effect is much less pronounced for short adjustment intervals where the errors in hedging are smaller.

10. It is well known that a reduction in the riskless rate raises the value of a European put contract. Ignoring transactions costs, the premium paid in excess of the deemed investment is the price of a put contract on the reference portfolio (see Brennan and Schwartz 1976).

TABLE 3 Simulation Results for Single-Premium Contract

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	Percentile Loss 5 (5)	Percentile Loss 10 (6)	Probability of Loss >$150 (7)	Probability of Loss >$200 (8)	Mean Transaction Costs ($) (9)
			Basic Example. Premium = $1,082.70*					
N	59.87	82.27	−416.88	−108.14	69.17	4.2	3.3	75.61
12	− 7.07	54.70	−177.38	−109.63	− 74.50	2.2	.8	82.96
6	− 9.71	41.80	−132.73	− 86.10	− 59.52	.7	.3	86.75
3	−16.91	38.63	−142.57	− 81.09	− 59.95	.9	.2	92.48
1	−34.91	37.82	−143.62	−103.90	− 87.00	.7	.0	104.71
			r = 0.04. Premium = $1,122.22*					
N	99.96	82.27	−371.76	− 63.03	114.28	3.5	2.6	75.61
12	−10.10	60.43	−188.67	−119.45	− 88.33	2.9	.8	84.29
6	−10.65	44.72	−146.05	− 89.55	− 66.76	.8	.3	89.01
3	−17.53	40.41	−153.87	− 88.64	− 61.20	1.1	.2	96.08
1	−37.12	38.77	−142.40	−105.38	− 89.74	.7	.0	111.74
			r = 0.08. Premium = $1,060.72*					
N	22.83	82.27	−448.91	−140.17	37.14	4.8	4.0	75.61
12	− 7.03	49.34	−179.18	− 95.42	− 65.56	1.5	.8	81.72
6	−10.28	38.81	−127.98	− 83.97	− 54.19	.6	.3	84.68
3	−17.06	36.49	−136.40	− 77.52	− 54.24	.8	.2	89.21
1	−32.71	36.09	−139.30	− 98.53	− 81.36	.7	.0	98.46
			σ^2 = 0.02. Premium = $1,053.98*					
N	19.23	31.78	−196.84	21.31	22.80	1.1	1.0	70.54
12	− 2.67	27.32	−102.11	− 45.82	− 31.60	.3	.0	74.81
6	− 5.20	22.66	− 88.11	− 43.58	− 27.33	.0	.0	76.87
3	− 9.61	22.73	− 95.85	− 49.81	− 34.24	.0	.0	79.74
1	−19.37	26.39	−110.82	− 72.81	− 51.77	.2	.0	86.46
			σ^2 = 0.06. Premium = $1,110.59*					
N	87.65	120.96	−507.61	−221.08	− 14.39	6.7	5.4	81.09
12	− 8.11	77.59	−238.85	−151.37	−110.64	5.3	2.5	90.49
6	−10.78	57.00	−163.80	−110.72	− 80.81	1.9	.5	95.52
3	−18.84	51.86	−169.68	−106.62	− 73.01	1.7	.4	102.70
1	−41.53	45.50	−148.66	−116.69	− 99.25	1.0	.0	116.96
			μ = 0.06. Premium = $1,082.70*					
N	36.87	113.92	−507.15	−254.68	− 85.95	7.7	6.6	65.64
12	− 9.08	59.00	−189.68	−125.56	− 84.78	2.8	.9	72.90
6	−13.62	42.28	−138.19	− 91.54	− 66.00	.7	.2	77.63
3	−22.47	38.16	−142.84	− 90.61	− 65.13	.7	.3	84.37
1	−44.15	36.72	−142.79	−111.36	− 96.31	.8	.0	99.52

TABLE 3 *(Continued)*

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	Percentile Loss 5 (5)	Percentile Loss 10 (6)	Probability of Loss >$150 (7)	Probability of Loss >$200 (8)	Mean Transaction Costs ($) (9)
			μ = 0.10. Premium = $1,082.70*					
N	65.48	56.10	−306.51	64.59	73.26	2.0	1.5	87.78
12	− 3.88	51.96	−165.82	− 95.65	− 67.51	1.6	.7	95.18
6	− 4.45	41.76	−127.46	− 72.93	− 49.82	.5	.2	98.25
3	− 9.76	39.66	−137.60	− 74.16	− 52.74	.5	.0	102.92
1	−24.70	39.05	−131.97	− 96.87	− 75.49	.4	.0	112.43
			ν = 0.01. Premium = $1,062.28*					
N	55.22	82.25	−416.52	−107.78	69.52	4.2	3.3	37.42
12	− 2.03	52.83	−171.30	−102.59	− 66.93	1.6	.6	41.06
6	− 1.99	39.62	−113.62	− 72.37	− 48.56	.4	.3	42.94
3	− 5.24	34.63	−111.29	− 61.47	− 41.57	.2	.0	45.78
1	−13.67	27.93	− 87.98	− 60.83	− 47.83	.0	.0	51.83
			ν = 0.03. Premium = $1,103.12*					
N	53.74	82.27	−418.01	−109.27	68.03	4.2	3.3	114.58
12	−13.33	56.76	−189.13	−119.53	− 84.82	2.8	.9	125.72
6	−18.72	44.41	−153.16	− 99.84	− 72.50	1.2	.4	131.47
3	−29.94	43.50	−175.61	−102.45	− 79.49	2.1	.6	140.15
1	−57.71	49.31	−199.78	−152.18	−126.52	5.6	1.0	158.68
			Term = 5 years. Premium = $1,098.11*					
N	44.63	88.83	−349.90	−176.83	− 64.86	6.0	4.0	54.27
12	− 5.28	56.82	−175.54	−107.56	− 72.92	2.5	.5	56.11
6	−10.33	44.50	−149.04	− 94.20	− 68.44	1.0	.0	61.11
3	−15.62	31.25	−104.40	− 75.85	− 55.98	.2	.0	67.08
1	−33.07	28.11	−119.95	− 88.18	− 71.23	.4	.0	81.86
			Term = 15 years. Premium = $1,069.60*					
N	56.28	67.42	−293.41	44.22	61.82	2.7	2.2	111.74
12	15.57	103.23	−184.70	−111.61	− 78.45	2.0	.6	121.54
6	13.77	94.58	−145.45	− 96.72	− 71.47	.8	.3	123.82
3	2.32	82.90	−142.07	−101.30	− 79.08	.9	.0	127.15
1	−12.83	92.50	−168.95	−125.78	−104.26	1.8	.0	138.71

NOTE.—Col. 1 = length of interval between portfolio adjustments for risk-reducing strategies; N indicates naive strategy. Cols. 2 and 3 = mean and standard deviation of profit achieved under each strategy; profit is measured by the insurance company's net cash position at contract maturity. Cols. 4–6 = level of loss which is exceeded in 1%, 5%, and 10% of the simulations. Cols. 7 and 8 = fraction of the simulations resulting in losses exceeding $150 and $200, respectively. Col. 9 = mean of the sum of the dollar transactions costs incurred under each strategy.

Premium charged allowing for transactions costs; the particular parameter varied from the basic data given in tables 1 and 2.

TABLE 4 **Simulation Results for the Periodic-Premium Contract**

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	5 (5)	10 (6)	Probability of Loss >$150 (7)	>$200 (8)	Mean Transaction costs ($) (9)
				Basic Example. Premium = $8.98*				
N	24.50	64.14	−292.36	−101.79	23.32	3.8	3.2	35.91
12	− 7.59	35.20	−118.72	− 62.38	−42.77	.3	.0	38.53
6	− 8.22	27.33	− 82.72	− 51.88	−36.38	.2	.2	42.62
3	−10.32	22.44	− 81.02	− 49.28	−35.05	.2	.0	48.69
1	−17.63	21.68	− 89.03	− 58.76	−44.68	.0	.0	66.26
				r = 0.04. Premium = $9.19*				
N	49.80	64.14	−267.05	− 76.48	48.62	3.3	2.4	35.91
12	− 9.49	38.78	−128.21	− 72.10	−50.54	.6	.2	39.20
6	− 9.09	29.41	− 90.97	− 56.90	−40.12	.2	.2	43.68
3	−10.77	23.86	− 84.15	− 51.97	−38.09	.2	.0	50.16
1	−18.64	22.50	− 93.03	− 61.24	−47.24	.2	.0	68.63
				r = 0.08. Premium = $8.84*				
N	5.65	64.14	−311.20	−120.6	4.47	4.5	3.3	35.91
12	− 6.84	32.15	−115.55	− 53.3	−37.41	.3	.0	37.93
6	− 8.17	25.50	− 74.31	− 45.2	−33.53	.2	.2	41.68
3	−10.51	21.07	− 78.32	− 46.79	−32.51	.2	.0	47.38
1	−17.20	20.76	− 88.97	− 57.30	−42.27	.0	.0	64.25
				σ^2 = 0.02. Premium = $8.78*				
N	4.66	29.37	−159.06	8.00	9.68	1.4	.8	34.03
12	− 9.08	18.10	− 68.45	− 30.87	−23.97	.0	.0	36.31
6	− 9.25	14.70	− 58.26	− 29.56	−21.10	.0	.0	39.43
3	−10.04	12.83	− 55.38	− 31.95	−21.36	.0	.0	44.25
1	−13.29	13.68	− 64.85	− 37.83	−28.63	.0	.0	59.39
				σ^2 = 0.06. Premium = $9.17*				
N	42.89	89.62	−366.98	−164.50	−32.68	5.7	4.3	37.93
12	− 4.38	47.93	−157.36	− 80.13	−56.44	1.3	.5	40.69
6	− 5.13	37.87	−109.56	− 69.74	−45.09	.4	.2	45.22
3	− 7.67	31.42	−101.76	− 59.99	−45.40	.3	.2	51.99
1	−16.98	28.41	−104.39	− 66.42	−53.77	.3	.0	70.99
				μ = 0.06. Premium = $8.98*				
N	12.86	82.28	−357.24	−177.66	−68.80	6.0	4.5	32.15
12	− 4.33	38.16	−119.71	− 65.18	−43.50	.5	.2	34.10
6	− 6.78	30.83	− 96.56	− 55.86	−37.49	.3	.2	38.52
3	−10.40	25.25	− 86.54	− 52.07	−38.64	.3	.2	45.24
1	−20.00	22.35	− 93.19	− 62.08	−47.96	.0	.0	64.06

TABLE 4 *(Continued)*

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	5 (5)	10 (6)	Probability of Loss >$150 (7)	>$200 (8)	Mean Transaction costs ($) (9)
			μ = 0.10. Premium = $8.98*					
N	31.90	48.33	−231.14	− 17.30	41.22	2.4	2.0	40.25
12	−10.04*	32.09	−117.96	− 58.01	−41.85	.3	.0	43.39
6	− 8.71	24.87	− 78.93	− 50.20	−33.11	.2	.2	47.06
3	− 9.21	21.38	− 74.25	− 44.51	−30.93	.0	.0	52.50
1	−13.66	20.54	− 88.35	− 51.03	−37.78	.0	.0	68.95
			ν = 0.01. Premium = $8.81*					
N	24.05	64.10	−292.63	−102.02	22.99	3.8	3.2	17.77
12	− 8.10*	33.80	−124.44	− 59.44	−41.69	.3	.0	19.07
6	− 7.45	25.92	− 80.87	− 49.10	−34.36	.2	.2	21.09
3	− 7.89	20.75	− 73.48	− 42.28	−30.09	.2	.0	24.10
1	−11.23	17.54	− 75.94	− 43.30	−30.98	.0	.0	32.80
			ν = 0.03. Premium = $9.15*					
N	24.66	64.18	−292.36	−101.77	23.36	3.8	3.2	54.43
12	− 7.49	36.92	−122.07	− 62.86	−44.93	.5	.0	58.39
6	− 9.40	29.03	− 88.28	− 56.18	−38.43	.2	.2	64.60
3	−13.16	24.49	− 88.77	− 56.77	−40.10	.3	.0	73.79
1	−24.45	26.57	−112.68	− 76.74	−59.33	.2	.0	100.41
			Term = 5 years. Premium = $9.03*					
N	6.00	29.67	−118.85	− 68.1	−28.32	.6	.2	14.04
12	− 4.79	20.01	− 62.34	− 34.60	−26.21	.0	.0	13.08
6	− 5.82*	16.14	− 57.44	− 28.3	−21.95	.0	.0	15.40
3	− 7.28	12.74	− 43.14	− 27.34	−19.68	.0	.0	18.51
1	−10.69	9.92	− 42.25	− 27.24	−22.34	.0	.0	27.06
			Term = 15 years. Premium = $8.95*					
N	58.10	74.45	−382.71	− 18.69	68.65	2.7	2.0	72.03
12	23.57	69.48	−149.57	− 74.73	−46.54	1.0	.9	80.13
6	23.13	67.17	−150.78	− 77.09	−48.93	1.1	.4	86.11
3	20.75	68.55	−145.01	− 77.85	−51.71	.9	.4	94.97
1	10.43	75.76	−161.00	−100.50	−75.95	1.4	.3	121.25

NOTE.—See footnotes to table 3.

The variance rate on the reference fund (σ^2). One effect of an increase in the variance rate on the reference fund is an increase in the risk premium ($\mu + \frac{1}{2}\sigma^2 - r$). Just as a reduction in the riskless rate, this tends to increase the mean profit under the naive strategy. However, this increase in the mean profit is accompanied by a very great increase in the probability of extreme losses.

The extent of risk reduction attainable under the risk-reducing strategies increases with the risk of the reference fund, while at the same time the risk associated with any particular adjustment interval is positively related to the risk of the reference fund. The mean profit under the risk-reducing strategies is relatively insensitive to the variance, increasing slightly with the variance rate for the periodic contract and decreasing slightly for the single-premium contract.

Mean instantaneous return on the reference fund (μ). An increase in the mean instantaneous return on the reference fund, like a reduction in the riskless rate, implies an increase in the reward to bearing investment risk and consequently an increase in the mean profit under the naive strategy. Notice, however, that the mean return does not affect the premium charged for the contract, so that the increased mean profit is solely the result of a more favorable probability distribution for the value of the reference portfolio, which reduces the probability that the guarantee will be effective. This shift in the probability distribution also reduces the standard deviation of profits and the risk of incurring extreme losses under the naive strategy, although even when $\mu = 0.10$ there is a 1% probability that the insurance company will lose more than \$306 on the single-premium contract and \$231 on the periodic-premium contract.

For the risk-reducing investment strategies, we find that the mean profit is again relatively insensitive to the value of μ. This is as it should be, since under the riskless investment strategy without transactions costs profits are independent of μ. Such effects as are observed are attributable to imperfect hedging and the dependence of the level of transactions costs on μ.

The level of transactions costs (v). Under the naive investment strategy, the distribution of profits is virtually independent of the transactions-cost assumption, because the contract premium is set to take account of transactions costs. The slight dependence on transactions costs arises because the premium is set on the assumption that the deemed investment is made in the reference portfolio, whereas under the naive strategy the actual amount invested in the reference fund is $(1 - \nu)^{-1}$ times the deemed amount.

It is encouraging to note the modest impact of different transactions-cost assumptions on the mean profits of the risk-reducing strategies. To take an extreme example, the mean loss with a monthly adjustment interval for the single-premium contract is \$57.71. This loss could be recouped by increasing the contract premium from \$1,103.12 to \$1,134.79 and investing the difference in the riskless security. The reason for this insensitivity is that the risk-reducing investment strategies require surprisingly small increases in the amount of transactions over that required under the naive strategy, at least for the

TABLE 5 **Ratios of Mean Transactions under Risk-reducing Strategies to Mean Transactions under Naive Strategy for Basic Examples**

	Adjustment Interval (Months)			
Contract	12	6	3	1
Single premium	1.11	1.18	1.28	1.52
Periodic premium	1.07	1.19	1.36	1.84

adjustment intervals considered. Considering the basic examples, we may infer from the mean transactions costs the ratios of the mean amount transacted under the different strategies. These are shown in table 5, and the evidence of this table suggests that the advantages of the risk-reducing strategies are not likely to be eliminated by any reasonable assumption about transactions costs.

The contract term (T). The probability that a 35-year-old male will survive for 10 years is .9873. Hence, in considering the effect of the contract term, we may neglect the possibility of early death without much loss of accuracy and assume that the contract expires only at maturity. For the single-premium contract, the assumption of lognormality implies that the logarithm of one plus the return on the reference fund up to maturity is distributed normally with mean μT and standard deviation $\sigma\sqrt{T}$. Since the guarantee is effective only if this logarithm is zero, increasing the term decreases the expected losses under the guarantee, so that we observe a decrease in the risk of loss under the naive strategy as the term is extended. It is perhaps for this reason that the Canadian Supervisor of Insurance has restricted the offer of guarantees to policies with a minimum term of 10 years. There appears to be no warrant for such a restriction if the risk-reducing strategies are followed, since the risk of extreme losses on a 5-year contract under a risk-reducing strategy is less than the risk of extreme losses on a 10-year single-premium contract if the naive strategy is followed.

For the periodic-premium contract, the risk of very large losses under the naive strategy is increased as the contract term is increased, due to the increased level of the guarantee and the increased amount invested in the reference portfolio. The risk-reducing strategies are successful in reducing the risks of the most extreme losses—more than $150—but the 5% loss level is higher under all of the risk-reducing strategies than it is under the naive strategy for a 15-year contract.

The results of the simulations for the two basic contracts reported in this section suggest that significant reductions in risk may be accomplished by adopting a risk-reducing investment strategy. Further, the cost in terms of additional transactions required under the risk-reducing strategies is modest. Moreover, all of the simulations reported relate to a single contract. This overstates the transactions costs which would actually be incurred by an insurance company with a large and changing portfolio of ELPAVGs which was following a risk-reducing investment strategy, since the transactions required under the risk-reducing strategy for one ELPAVG contract may be offset to some degree by transactions required under the same strategy for another contract. To consider the simplest example of this possibility, suppose that the required portfolio adjustment for one contract called for a reduction in the amount invested in the reference fund and that a new single-premium contract had just been written. Ordinarily, this new contract would call for some new investment in the reference fund; however, given the required decrease in investment in the reference fund under the first contract, these purchases and sales may be offset with a consequent saving of transactions costs. Similarly, the benefits payable under one contract may be paid directly out of the premia received under another contract since only the aggregate investment in the reference fund is of importance. Therefore, the transactions costs calculated in the simulations should be regarded as providing an upper bound on the transactions costs likely to be incurred in practice.

IV. Further Results

In this section we report some further simulation results relating to the risk-reducing strategies. These concern the risk of a contract with an increasing guarantee when the risk-reducing strategy is employed: the effect of errors in the specification of the variance rate for the return on the reference fund and the effect of combining the risk-reducing strategies with the naive strategy to form a 'portfolio' of strategies.

A Contract with an Increasing Guarantee

The effectiveness of the risk-reducing strategies described in the previous section suggests that, if such strategies are employed, an insurance company may be able to offer a much more significant guarantee without incurring undue risk. The two basic contracts considered above offer a guarantee equal in amount to the investment component of the premia paid up to death or maturity, so that the insurance company is in effect guaranteeing that the rate of return on this investment would not be negative. However, in the event that the benefit paid is equal to the guaranteed amount, the policyholder has experienced a substantial opportunity loss which is measured by the return he could have earned by investing the deemed investment in the reference portfolio in riskless securities. A more attractive guarantee therefore would consist of a minimum benefit equal to what would have been received had the deemed investment in the reference portfolio been invested in riskless securities.

In table 6 we report the simulation results for a single-premium contract which is identical with the basic example in all respects save

TABLE 6 Simulation Results for Contract Whose Guarantee Increases at 6% Riskless Interest Compared with Basic Example

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	Percentile Loss 5 (5)	Percentile Loss 10 (6)	Probability of Loss >$150 (7)	Probability of Loss >$200 (8)	Mean Transaction Costs ($) (9)
			Basic Example. Premium = $1,082.70					
N	− 54.87	82.27	−416.88	−108.14	69.17	4.2	3.3	75.61
12	− 7.07	54.70	−177.38	−109.63	− 74.50	2.2	.8	82.96
6	− 9.71	41.80	−132.73	− 86.10	− 59.52	.7	.3	86.75
3	− 16.91	38.63	−142.57	− 81.09	− 59.95	.9	.2	92.48
1	− 34.91	37.82	−143.62	−103.90	− 87.00	.7	.0	104.71
			Increasing Guarantee $g(t)$ = $1,000.$e^{rt}$. Premium = $1,292.02					
N	249.40	341.61	−851.14	−542.68	−334.73	15.6	13.5	75.59
12	− 13.98	138.76	−414.81	−270.67	−187.64	14.7	8.9	86.54
6	− 27.26	100.08	−292.69	−199.70	−161.15	11.6	5.0	99.68
3	− 45.99	76.71	−282.38	−189.02	−149.69	10.0	4.0	115.40
1	−100.91	68.03	−303.98	−227.22	−188.33	21.6	8.3	155.43

that the guarantee is obtained by compounding the $1,000 deemed investment in the reference portfolio at the 6% riskless interest rate. The premium for this increasing-guarantee contract is some $210 higher than for the level-guarantee contract, reflecting the value of the improved guarantee which ensures a minimum benefit payable of $1,822 in 10 years instead of $1,000 for the basic contract. The mean profit realized by the insurance company under the naive strategy is considerably greater for the increasing-guarantee contract than for the basic contract, reflecting the increased risk borne. The increased risk is shown by the standard deviation of profit of $341.61 versus $82.27 for the basic example and by the rise in the 5% loss level from $108.14 to $542.68. It seems reasonable to suppose that the degree of risk posed by the increasing-guarantee contract under the naive investment strategy would be unacceptable to most insurance companies.

Consider however the results of the increasing-guarantee contract under the risk-reducing strategy with a 3-month adjustment interval. The standard deviation is less than for the basic example under the naive strategy, and the 1% loss level is $282.38 compared with $416.88 for the basic example under the naive strategy. Inspection of table 6 reveals that if the premium for the increasing-guarantee contract were raised by the present value of $50 or $27.90 (that is, by 2.2%), not only would the 1% loss level be about half of that for the basic example with the naive strategy but the probability of losses exceeding $150 would also be less.

While no contracts with guarantees of this type are known to the authors, the above results indicate that contracts with such attractive guarantees could be sold without incurring more risk than is borne by insurance companies who issue orthodox ELPAVG contracts and follow a naive investment strategy. The risk borne by the insurance company is a function not only of the type of guarantee offered but also of the premium charged and the type of investment strategy followed.

Misspecification of the Stochastic Process

We have seen that the profit distribution under the risk-reducing strategies is largely insensitive to the assumed mean of the distribution of return on the reference fund. However, all of the foregoing examples have implicitly assumed that the insurance company correctly assesses the variance rate on the reference fund[11] and that the stochastic process of fund return is correctly described by equation (3). To test the sensitivity of the results to the instantaneous dispersion parameter of the stochastic process, two further sets of simulations were carried out. For these, a single-premium contract was priced, and the risk-reducing investment strategy was determined on the basis of the assessed variance rate of 4% per annum. For the two sets of simulations, the actual variance of the lognormal distribution from which returns on the reference fund were drawn was set at 2% and at 6% per annum. Thus, in the first case the true variance rate is overestimated by 100% while in the second case it is underestimated by 50%. The results of these simulations are compared in table 7 with the basic example single-premium contract in which the actual variance rate is equal to the assessed variance rate of 4%.

When the variance rate is overestimated the contract premium is set too high, so that positive mean profits are achieved with the risk-reducing strategies as well as with the naive strategy. Because of the low actual variance rate, the standard deviation of profits under the naive strategy is small, and only modest reductions are achieved with the risk-reducing strategies. Similarly, the probability of extreme losses is low under the naive strategy so that the scope for improvement is limited. Nevertheless, the risk-reducing strategy does cut the 1% loss level from \$144.51 under the naive strategy to only \$12.96 for a 3-month adjustment interval.

Errors in overestimating the variance rate are errors in the insurance company's favor since this causes too high a premium to be charged,[12]

11. It is interesting to note that the composition of the reference fund is typically at the discretion of the insurance company. This means that the insurance company can adjust the composition of the reference fund to achieve the variance on which the premium was based. It also means that the insurance company is free to unilaterally alter the value of the contract it has sold by changing the variance rate on the reference fund.

12. In table 3, the premium is shown as \$1,053.98 when $\sigma^2 = 0.02$. The insurance company, assessing $\sigma^2 = 0.04$, sets the premium at \$1,082.70, resulting in an "overcharge" of \$28.72.

TABLE 7 Effect of Misspecifying Variance Rate: Single-Premium Contract, Premium = $1,082.70

Adjustment Interval (Months) (1)	Mean Profit ($) (2)	Standard Deviation (3)	Percentile Loss 1 (4)	Percentile Loss 5 (5)	Percentile Loss 10 (6)	Probability of Loss >$150 (7)	Probability of Loss >$200 (8)	Mean Transaction Costs ($) (9)
			Basic Example. $\sigma^2 = 0.04$					
N	54.87	82.27	−416.88	−108.14	69.17	4.2	3.3	75.61
12	− 7.07	54.70	−177.38	−109.63	− 74.50	2.2	.8	82.96
6	− 9.71	41.80	−132.73	− 86.10	− 59.52	.7	.3	86.75
3	−16.91	38.63	−142.57	− 81.09	− 59.95	.9	.2	92.48
1	−34.91	37.82	−143.62	−103.90	− 87.00	.7	.0	104.71
			Variance Overestimated. $\sigma^2 = 0.02$					
N	71.56	31.78	−144.51	73.64	75.13	1.0	.6	70.54
12	33.60	35.97	− 40.53	− 20.51	− 6.18	.0	.0	75.28
6	31.30	27.16	− 17.79	− 6.89	1.25	.0	.0	77.78
3	26.48	21.12	− 12.96	− 3.60	2.96	.0	.0	81.33
1	14.45	14.07	− 13.74	− 6.13	− 1.52	.0	.0	89.67
			Variance Underestimated. $\sigma^2 = 0.06$					
N	36.84	120.96	−558.43	−271.90	− 65.21	7.5	6.7	81.09
12	−47.35	85.56	−353.08	−219.18	−161.59	11.6	6.5	90.71
6	−49.89	68.40	−259.97	−169.46	−136.46	7.8	3.6	95.64
3	−58.76	68.24	−270.22	−171.62	−139.95	8.0	3.4	102.89
1	−82.17	71.02	−263.05	−214.36	−182.37	17.7	6.8	117.13

NOTE.—Assessed variance rate = 0.04 per annum.

and since an overestimated variance rate will imply risks less than anticipated, the insurance company is less likely to be concerned about this type of error than about the possibility of underestimating the variance rate. When this occurs the contract will be underpriced, so it is not surprising to find that mean losses are incurred under the risk-reducing strategies. The key issue of course is the effect of the risk-reducing strategies on the probability of incurring extreme losses, and we note that the 3-month adjustment interval strategy reduces the 1% loss level from $558.43 to $270.22 and the 5% loss level from $271.90 to $171.62. On the other hand, the risk-reducing strategy does not reduce the probability of losses exceeding $150, although the probability of losses exceeding $200 is approximately halved with an adjustment interval of 3 months.

In summary, despite the gross errors assumed in the variance estimates, the risk-reducing strategies retain their value, albeit to a diminished extent. Moreover, we have portrayed these strategies in a most unfavorable light by assuming a fixed incorrect variance estimate. In reality, Bayesian methods could be employed to update the variance

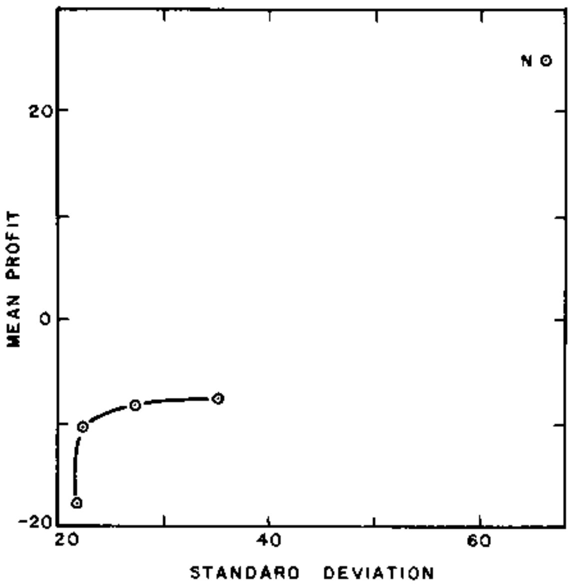

FIG. 5.—Basic example periodic-premium contract: mean and standard deviation of profit of alternative investment strategies.

estimate on the basis of experience, and it is to be expected that this would improve the results reported here.

Portfolio Selection of Strategies

In figure 5 the mean and standard deviation of profits for the basic example periodic-premium contract are plotted for the naive strategy and the four risk-reducing strategies considered. The solid line drawn through the points corresponding to the risk-reducing strategies represents risk-reducing strategies with intermediate adjustment intervals. Instead of considering strategies with intermediate adjustment intervals we could also consider combining strategies with different adjustment intervals (and the naive strategy) into portfolios of strategies. For example, we could follow a 12-month adjustment strategy with respect to a fraction q of the liability under a contract and follow a 6-month adjustment strategy with respect to the remaining $(1 - q)$ of the liability. The profit realized under this combined strategy will then be a weighted average of the profits realized under the two basic strategies. There is of course no reason to restrict the portfolio of strategies to a combination of two basic strategies: in general a mixed strategy may be defined by a set of weights q_i $(i = 1, \ldots, 5)$ where q_i is the weight accorded basic strategy i. A set of weights may then be chosen to maximize the expected utility of the profit outcome. In the absence of knowledge of the appropriate utility function we have fallen back on

TABLE 8 **Correlations of Profits Realized under Different Strategies for Basic Example Periodic-Premium Contract**

	Adjustment Interval (Months)				
	Naive	12	6	3	1
Naive	1.0	.02	−.03	−.12	−.09
12	.02	1.0	.74	.45	−.04
6	−.03	.74	1.0	.66	.15
3	−.12	.45	.66	1.0	.49
1	−.09	−.04	.15	.49	1.0

standard mean-variance analysis and determine the set of weights which minimizes the variance or standard deviation of profits for a given expected profit.

Strategy combinations were obtained for the basic example periodic-premium contract by solving the quadratic programming problem

minimize

$$\sum_{i=1}^{5}\sum_{j=1}^{5} q_i q_j \sigma_{ij} - \lambda \sum_{i=1}^{5} q_i \bar{p}_i$$

subject to

$$\sum_{i=1}^{5} q_i = 1$$

$$q_i \geqslant 0 \qquad (i = 1, \ldots, 5)$$

for different values of λ, where σ_{ij} is the covariance of profits under strategies i and j and $\bar{p}_i$ is the mean profit under strategy i.

The covariances of profits realized under the different strategies were calculated, and the resulting correlation matrix is shown in table 8. Profits realized under the naive strategy are almost uncorrelated with the profits under the risk-reducing strategies, and the correlation between different risk-reducing strategies declines rapidly as the difference between the adjustment intervals is increased.

The results of the strategy combinations obtained are plotted in figure 6 and presented in full in table 9. The first five lines of this table correspond to the five basic strategies, and the remainder of the table shows the characteristics of the strategy combinations obtained for different values of λ. It is clear from figure 6 that the portfolio of strategies significantly dominates the basic risk-reducing strategies in a mean-variance framework. Further, by considering combinations of the risk-reducing strategies and the naive strategy the gulf between the risk-reducing strategies and the naive strategy is eliminated, suggesting

TABLE 9 Basic Example Periodic-Premium Contract: Portfolio Approach Compared with Basic Strategies

λ	q_n	q_{12}	q_6	q_3	q_1	Mean Profit ($)	Standard Deviation	Percentile Loss 1	Percentile Loss 5	Percentile Loss 10	Probability of Loss >$150	Probability of Loss >$200	Mean Transaction Costs ($)
...	1.0	...	...	...	...	24.50	64.14	−292.36	−101.79	23.32	3.8	3.2	35.91
...	...	1.0	...	...	...	− 7.59	35.20	−118.72	− 62.38	−46.77	.3	.0	38.53
...	...	...	1.0	...	...	− 8.22	27.33	− 82.72	− 51.88	−36.38	.2	.2	42.62
...	...	...	...	1.0	...	−10.32	22.44	− 81.02	− 49.28	−35.05	.2	.0	48.69
...	...	...	...	...	1.0	−17.63	21.68	− 89.03	− 58.76	−44.68	.0	.0	66.26
.0	.089	.125	.115	.149	.523	−10.47	16.65	− 63.03	− 41.67	−32.15	.0	.0	54.82
1.0	.093	.123	.116	.156	.512	−10.25	16.65	− 62.58	− 40.91	−31.90	.0	.0	54.54
4.0	.106	.118	.118	.178	.481	− 9.59	16.70	− 61.23	− 40.31	−31.23	.0	.0	53.92
9.0	.127	.108	.122	.214	.429	− 8.48	16.92	− 60.07	− 40.21	−30.31	.0	.0	52.77
16.0	.157	.096	.128	.264	.355	− 6.92	17.48	− 63.03	− 41.68	−28.84	.0	.0	51.17
25.0	.195	.080	.135	.329	.261	− 4.92	18.62	− 72.32	− 40.99	−26.81	.0	.0	49.15
36.0	.242	.060	.144	.409	.145	− 2.48	20.52	− 86.87	− 42.01	−23.81	.0	.0	46.66
49.0	.298	.036	.154	.503	.009	0.41	23.32	− 98.15	− 43.30	−19.87	.2	.0	43.74
64.0	.351	.031	.169	.450	.0	2.33	25.53	−112.75	− 39.71	−17.65	.3	.0	42.91
81.0	.410	.025	.185	.379	.0	4.42	28.34	−132.92	− 43.10	−14.55	.4	.2	42.02
100.0	.477	.020	.204	.300	.0	6.76	31.85	−154.87	− 45.48	−10.68	1.1	.3	41.20
121.0	.550	.013	.225	.212	.0	9.34	36.05	−173.45	− 55.27	− 5.76	2.0	.4	40.16
144.0	.630	.006	.247	.116	.0	12.16	40.91	−193.79	− 58.48	− 0.53	2.4	1.0	39.03
169.0	.718	.0	.27	.012	.0	15.23	46.42	−221.20	− 67.55	6.11	2.9	1.9	37.88
200.0	.820	.0	.180	.0	.0	18.62	52.72	−249.30	− 81.64	15.05	3.3	2.3	37.12
225.0	.903	.0	.097	.0	.0	21.32	57.90	−267.80	− 94.20	15.90	3.6	2.4	36.56
250.0	.986	.0	.014	.0	.0	24.02	63.20	−288.11	− 99.89	23.02	3.7	3.1	36.00
275.0	1.0	.0	.0	.0	.0	24.50	64.14	−292.36	−101.79	23.32	3.8	3.2	35.91

NOTE.— q_i = fraction of portfolio allocated to risk-reducing strategy with i month adjustment interval; q_n = fraction of portfolio allocated to naive strategy.

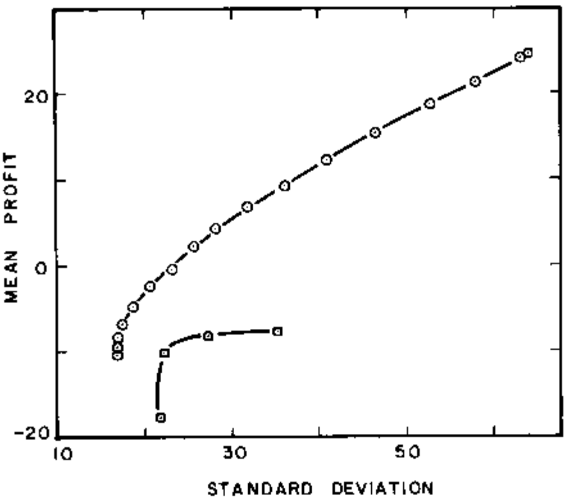

FIG. 6.—Basic example periodic-premium contract mean and standard deviation of profits attainable under the portfolio approach compared with the basic risk-reducing strategies.

that for a wide range of utility functions it will be optimal to combine the naive strategy with at least some of the risk-reducing strategies.

As an example of the power of the strategy combinations, compare the strategy resulting from $\lambda = 121$ with the naive strategy. While the mean profit is reduced from \$24.50 to \$9.34, the 1% loss is cut from \$292 to \$173 and the 5% loss from \$102 to \$55, the probability of losses exceeding \$150 is almost halved, and the probability of losses exceeding \$200 is reduced to minuscule proportions.

Thus the portfolio approach provides further evidence of the practical usefulness of investment strategies derived from the theoretical construct—the riskless investment strategy—and, by extending the range of risk-return alternatives available to insurance company managements, increases the likelihood that they will find it appropriate to pursue strategies of the general type described in this paper.

Appendix: Filter Strategies

As an alternative to the fixed revision interval strategies considered in the body of the paper, we also employed a filter strategy for the basic example periodic-premium contract. Under this strategy portfolio revisions occurred only when the difference between the actual amount invested in the reference fund and the amount required under the riskless strategy at the end of a month exceeded a predetermined fraction—the filter size. In addition, the monthly premia received were allocated to the reference fund if the actual investment in the fund fell short of the riskless requirement.

The results for different filter sizes are shown in table A1.

TABLE A1 **Basic Example Periodic-Premium Contract: Results of Filter Strategies**

Filter (%)	Revisions per Year (N)	Standard Deviation Revisions per Year	Mean Profit ($)	Standard Deviation	Percentile Loss			Probability of Loss		Mean Transaction Costs ($)
					1	5	10	>$150	>$200	
0	12	.0	−17.63	21.68	−89.03	−58.76	−44.68	.0	.0	66.26
1	9.15	.96	−17.00	20.76	−88.31	−57.33	−43.38	.0	.0	65.29
2	7.45	1.13	−16.60	19.72	−91.91	−55.62	−41.10	.0	.0	63.94
3	6.26	1.09	−16.21	19.18	−86.19	−54.11	−40.67	.2	.0	62.76
4	5.37	1.02	−16.42	19.52	−81.65	−52.18	−40.38	.0	.0	61.70
5	4.71	.95	−16.55	21.06	−77.92	−55.08	−44.10	.0	.0	60.78
6	4.21	.87	−16.84	23.67	−88.30	−60.56	−46.41	.0	.0	60.09
7	3.80	.81	−18.23	26.76	−89.65	−68.29	−53.15	.0	.0	59.19
8	3.45	.75	−17.91	29.49	−98.28	−70.92	−57.44	.3	.0	58.37
9	3.18	.71	−18.52	32.78	−108.99	−76.84	−59.08	.4	.2	57.83
10	2.95	.67	−19.33	34.30	−123.49	−80.62	−61.51	.5	.0	57.35

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