Behavior-based price discrimination by a patient seller

by

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Abstract

We investigate a model in which one seller and one buyer trade in each of two periods. The buyer has demand for one unit of a non-durable object per period. The buyer's reservation value for the good is private information and is the same in both periods. The seller commits to prices in each of two periods. Prices in the second period may depend on the buyer's first-period behavior. Unlike the equal discount factor case studied in earlier papers, we show that when the seller is more patient than the buyer, second-period prices increase after a purchase. In particular, the optimal dynamic pricing scheme is not a repetition of the optimal static pricing scheme.

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1 Introduction

With reductions in the price of computing and data storage, it is not difficult for firms with millions of customers to keep track of past buying behavior of each customer. Grocery stores offer customers discount coupons that are tailored to their purchasing history. Netflix and Amazon use purchase history to send product suggestions and other marketing messages to their clients. It is eminently feasible for these firms to use historical information to offer customer-specific prices.[1]

We investigate the pricing problem of a firm when it sells a non-durable product to a buyer who wishes to purchase one unit of the good in each of two periods. The buyer's second-period demand remains one unit whether or not he buys in the first-period.[2] Because of the seller's ability to recognize past customers and recall their buying decisions, second-period prices can depend on the first-period decision of the buyer. The buyer's reservation value for the product is (i) known only to the buyer and (ii) remains the same in each period. The seller announces a price in each period and the buyer decides whether or not to buy. The buyer is strategic: he takes into account that his first-period purchase decision can affect second-period prices. The seller commits to second-period prices at the beginning of the first period.

Several authors have investigated models of behavior-based price discrimination, both with commitment by the seller and without. Hart and Tirole (1988), Villas-Boas (2004), and Acquisiti and Varian (2005) have shown that in a dynamic-pricing setting where the buyer's valuation is fixed and privately known, it is in the seller's interest to commit *not* to use in later periods information revealed by the buyer's earlier decisions. An optimal multiperiod pricing strategy for a seller who can commit is to simply offer the same take-it-or-leave-it price each period.[3]

Our contribution is to show that the optimality to commit not to use the information revealed via the buyer's purchase decision hinges upon the assumption of equal discount factors. If, instead, the seller is more patient than the buyer, then her optimal pricing strategy with commitment exploits information learned in the first period; in particular, the second-period price increases after a sale in the first period. The second-period price charged by a less patient seller is also history-dependent; in fact, the seller obtains pre-payment of the second-period unit in the first period. However, the less patient seller model is similar to the equal discount factors model in that the subset of types of buyers who trade is the same under both scenarios.

To our knowledge, the only other paper in which a less informed party commits

[1]Previous customers of an internet-based firm may provide a new shipping address and use another credit card to evade recognition. Such behavior would constrain the firm's ability to practice behavior-based price discrimination. However, there is anecdotal evidence of customer-behavior based pricing by internet firms (see Streitfeld 2007).

[2]The buyer either buys the product from the seller or obtains it from another (unmodeled) seller.

[3]A similar result was obtained in a different context by Baron and Besanko (1984), who showed that it is in the interest of a regulator to commit not to use in the second period information revealed in the first period about the marginal cost of a regulated firm.

to use information revealed in a dynamic game is Sobel and Takahashi (1983). We discuss this model in section 3.

If the seller and the buyer can borrow money at the same interest rate, then the assumption of equal discount factors is appropriate. But this assumption is not tenable when the seller is a large retail establishment and the buyer is an individual or a household, as in the examples presented above. Often, the seller is more patient than the buyer.

The optimal pricing strategy of a seller who *cannot* commit has been investigated in models with equal discount factors in Hart and Tirole (1988), Villas-Boas (2004), and Acquisiti and Varian (2005). See also Skreta (2006) and surveys by Armstrong (2006) and Fudenberg & Villas-Boas (2006). The pricing strategy is history dependent with the second-period price equal to the optimal take-it-or-leave-it price for the seller's updated distribution over buyer valuations. With unequal discount factors, the optimal pricing strategy is qualitatively similar. Therefore, except in an example where the no commitment case is introduced for comparison, we restrict our analysis to a seller who has the ability to commit.

In the next section, we describe the model and present the main result. The proof is in the appendix. We conclude with a comparison of the optimal contract in this model with the optimal contract for the sale of one unit.

2 The model and main result

A buyer wishes to buy one unit of an indivisible, non-durable good in each of two periods. The seller's belief about the buyer's valuation v is represented by c.d.f $F(v)$ which has positive density $f(v)$ when $v \in [a, 1]$, $1 > a \geq 0$. We assume that F is regular, i.e., $v - (1 - F(v))/f(v)$ is increasing. The buyer's valuation remains v in each period, and the seller's cost is 0. The seller posts prices and the buyer accepts or rejects. The buyer is strategic: he takes into account the impact of his first-period decision on second-period prices.

Let the price in the first period be p_1 and the price in the second period be p_2 [p_2^{n}] if there is [is not] a sale in the first period. The seller commits to prices $p_1,\ p_2,\ p_2^{\mathrm{n}}$ at the beginning of the first period. The discount factors of the seller and buyer are δ_s and δ_b, respectively.

A buyer with valuation v chooses among not buying at all, buying in the first period only, buying in both periods, and buying in the second period only:

$$\Pi_b(v; p_1, p_2, p_2^{\mathrm{n}}) = \max\{0, v - p_1, v - p_1 + \delta_b(v - p_2), \delta_b(v - p_2^{\mathrm{n}})\}.$$

We are interested in non-negative prices $p_1, p_2, p_2^{\mathrm{n}} \geq 0$ that maximize the seller's expected profits. The analysis depends on whether the seller is more patient than the buyer, $\delta_s > \delta_b$, or not, $\delta_s \leq \delta_b$.

Let v_c represent the valuation of a marginal first-period buyer. If $v_c \in (a, 1)$, then a buyer with valuation v_c is indifferent between buying or not buying in the first period. Therefore,

$$v_c - p_1 + \delta_b \max\{v_c - p_2, 0\} \quad = \quad \delta_b \max\{v_c - p_2^{\mathrm{n}}, 0\}. \tag{1}$$

It is easy to see that any buyer with valuation $v > v_c$ will buy in the first period, and any buyer with valuation $v < v_c$ will not buy in the first period.

Let

$$p^* \;=\; \frac{1 - F(p^*)}{f(p^*)} \tag{2}$$

if there exists such a $p^* \in [a, 1]$; otherwise, $p > \frac{1-F(p)}{f(p)}$ for all $p \in [a, 1]$ and define $p^* = a$.[4] The assumption that F is regular guarantees uniqueness of p^*. It is well known (see references in the Introduction) that when $\delta_s = \delta_b$ it is an optimal strategy for the seller to commit to prices $p_1 = p_2 = p_2^{\mathrm{n}} = p^*$. Our main result identifies optimal prices for all values of δ_s and δ_b.

Proposition: *The optimal commitment prices $p_1, p_2, p_2^{\mathrm{n}}$ are as follows.*

(i) If $\delta_s > \delta_b$, then $v_c = p_2 > p_1 > p_2^{\mathrm{n}}$ and $v_c = p_2 > p^* > p_2^{\mathrm{n}}$.

(ii) If $\delta_s = \delta_b = \delta$, then $p_1 \geq p^* = v_c = p_2^{\mathrm{n}}$ and $p_1 + \delta p_2 = (1 + \delta)p^*$.

(iii) If $\delta_s < \delta_b$, then $p_1 = (1 + \delta_b)p^*,\ p_2 = 0,$ and $p_2^{\mathrm{n}} = v_c = p^*$.

If $\delta_s \neq \delta_b$, then the optimal prices are unique.

When the seller and the buyer do not have the same discount factor, the second-period prices are history dependent. In particular, if the seller is more patient, $\delta_s > \delta_b$, equilibrium prices differ markedly from the equal discount factor case that earlier papers have focused on. If there is a purchase in the first period, then the second-period price increases and exceeds p^*, the optimal commitment price in the equal discount factors case. If there is no purchase in the first period, then second-period price decreases and falls below p^*. The first-period price p_1 may or may not exceed p^*. Because the second-period price is history dependent, the marginal first-period buyer has a positive surplus ($v_c > p_1$).

When the seller and the buyer have equal discount factors, there exists a continuum of equilibrium prices, all payoff equivalent for the seller and for the buyer. Earlier papers have identified one of these optimal prices, $p_1 = p_2 = p_2^{\mathrm{n}} = p^*$, which is the only history-independent optimal price in this case.

If the seller is less patient, $\delta_s < \delta_b$, the valuations of buyers who trade is the same as when $\delta_s = \delta_b$. The seller either sells one unit in each period (if the buyer's valuation exceeds $v_c = p^*$) or none (if $v < p^*$). However, $p_1 = p_2 = p_2^{\mathrm{n}} = p^*$ is not optimal. Instead, in the first period the seller obtains pre-payment for the second-period unit

[4] If $f(v)$ is bounded away from zero, $p < \frac{1-F(p)}{f(p)}$ for all $p \in [a, 1]$ is not possible.

from a buyer with valuation $v > p^*$. Because the buyer's discount factor is used to calculate the pre-payment, the seller is strictly better off with $p_1 = p^* + \delta_b p^*$ and $p_2 = 0$ rather than with $p_1 = p_2 = p^*$ while the buyer is indifferent between the two sets of prices.[5]

We illustrate the Proposition with an example.

Example: v is uniform on $[0, 1]$

For this distribution $p^* = 0.5$.

If $\delta_s > \delta_b$ *then equations (6) through (10) in the proof of the Proposition yield the following solution:*

$$\begin{aligned} v_c \;=\; p_2 &= \frac{2\delta_s^2 - \delta_b^2 - \delta_s\delta_b + 2\delta_s}{3\delta_s^2 - 2\delta_s\delta_b - \delta_b^2 + 4\delta_s} \\ p_2^{\mathrm{n}} &= \frac{\delta_b + (\delta_s - \delta_b)v_c}{2\delta_s} \\ p_1 &= v_c - \delta_b(v_c - p_2^{\mathrm{n}}) \end{aligned}$$

Let $\delta_s = 0.75$ *and* $\delta_b = 0.5$. *Inserting these values of* δ_s *and* δ_b *in the above equations yields* $p_1 = 0.483$, $v_c = p_2 = 0.542$, *and* $p_2^{\mathrm{n}} = 0.424$ *and the seller's expected profit is* 0.445. *If, instead, all prices are set equal to* $p^* = 0.5$, *then the seller's expected profit decreases to* $(1 + \delta_s)p^*(1 - F(p^*)) = 0.4375$.[6]

If $\delta_s < \delta_b$, *the unique optimal commitment prices are* $p_1 = 0.5(1+\delta_b)$, $p_2 = 0$, *and* $p_2^{\mathrm{n}} = v_c = 0.5$. *The seller's expected profit is* $p_1(1 - F(v_c)) = 0.5p_1$. *Thus, if* $\delta_s = 0.5$ *and* $\delta_b = 0.75$, *then* $p_1 = 0.875$, *and the seller's expected profit is* 0.4375.

It is instructive to compare these prices to optimal prices for a seller who does not have the ability to commit. In the no-commitment case denote the optimal prices by $\overline{p}_1, \overline{p}_2$, *and* $\overline{p}_2^{\mathrm{n}}$ *and the marginal buyer's valuation by* $\overline{v}_c$. *Then*[7]

$$\overline{v}_c = \frac{2 - \delta_b + 2\delta_s}{4 - 2\delta_b + 3\delta_s} \qquad \text{and} \qquad p_1 = \frac{(2 - \delta_b)\overline{v}_c}{2}.$$

Further, $\overline{p}_2^{\mathrm{n}} = 0.5\overline{v}_c$, $\overline{p}_2 = \overline{v}_c$, *and* $\overline{v}_c > p^* = 0.5$.

The table below summarizes the optimal prices and profits at the discount factors considered above. Not surprisingly, the seller does better when she is patient and/or can commit.

[5] Note that we assumed that prices are non-negative. If instead, one assumes that prices must be greater than a negative number $-k$, $k > 0$, then the equilibrium prices will be $p_1 = p^* + \delta_b p^* + \delta_b k$, $p_2 = -k$ yielding the seller expected profit of $[(1 + \delta_b)p^* + (\delta_b - \delta_s)k](1 - F(p^*))$, which increases linearly with k. Of course, as k increases it becomes more difficult for the seller to credibly commit to price $p_2 = -k$.

[6] With these discount factors, p_1 is less than $p^* = 0.5$. If instead, $\delta_s = 0.75$ and $\delta_b = 0.25$ then $p_1 = 0.507 > p^*$, and $v_c = p_2 = 0.559$, and $p_2^{\mathrm{n}} = 0.353$; the seller's expected profit increases to 0.463.

[7] When $\delta_s = \delta_b$, these formulae match those derived in Armstrong (2006, section 2.2) and Fudenberg and Villas-Boas (2006, section 2.1.3).

	Commitment			No commitment		
(δ_s, δ_b)	$(p_1, p_2, p_2^{\rm n})$	v_c	Π_s	$(\overline{p}_1, \overline{p}_2, \overline{p}_2^{\rm n})$	$\overline{v}_c$	$\overline{\Pi}_s$
$(0.75, 0.5)$	$(0.483,\ 0.542,\ 0.424)$	0.542	0.445	$(0.429, 0.571, 0.286)$	0.571	0.429
$(0.5, 0.75)$	$(0.875,\ 0,\ 0.5)$	0.5	0.438	$(0.352, 0.563, 0.281)$	0.563	0.317

$\triangle$

3 Comparison with pricing strategies for selling one unit

In our model, the buyer demands one unit in each of two periods. The buyer's value remains the same in the two periods. We compare our model to two other models in both of which the buyer's total demand is one unit. The first comparison point is a static one-period model (model A) and the second comparison point has two periods (model B). In both models, the seller makes offers and the buyer accepts or rejects; the seller commits to prices.

It is well known that when buyer and seller discount factors are equal, an optimal pricing strategy in our model with one-unit per period demand is a repetition of the optimal price p^* charged in model A. The question we ask is whether there is such a close connection when discount factors are not equal. We find that when the seller is more patient than the buyer, the optimal pricing policy in our model is neither a repetition of nor a version of the optimal pricing policy in either model A or model B.

Model A: One-unit total demand, one period

Because there is only one period, there is no discounting. The seller posts a price, the buyer takes his purchase decision, and there are no further moves by either player. The optimal pricing strategy in this model is the take-it-or-leave price offer of p^* defined in equation (2) (see Myerson 1981 and Riley & Zeckhauser 1983).

In our model, offering p^* in each period is an optimal strategy when $\delta_s = \delta_b$. When $\delta_s < \delta_b$, the optimal strategy in our model is to sell both units as a bundle at a price of $(1+\delta_b)p^*$. As the buyer's valuation for the bundle is $(1+\delta_b)v$, the optimal strategy in our model duplicates the optimal strategy in model A applied to the bundle. In either case -- $\delta_s = \delta_b$ or $\delta_s < \delta_b$ -- the seller in our model commits not to use any information that is revealed in the first period.

A patient seller in our model, however, will commit to use information revealed in the first-period transaction. When $\delta_s > \delta_b$ we have $p_1 \neq p^*$ and $p_2 > p^* > p_2^{\rm n}$: it is not optimal for the seller to commit to charge p^* each period. Nor is it optimal to bundle the two units together. The buyers separate into three intervals based on their valuations: high, intermediate and low. High valuation buyers ($v > v_c$) buy one unit in each period, and low valuation buyers ($v < p_2^{\rm n}$) do not buy in any period.

A buyer with intermediate valuation $v \in (p_2^{\mathrm{n}}, v_c)$ buys in the second period but not in the first. Thus, bundling the two units together cannot replicate the effect of optimal prices p_1, p_2, p_2^{n}. The seller maximizes her profit by committing to use the information revealed by the buyer's first-period decision.

Model B: One-unit total demand, two periods

Because our model has two periods, we also compare it with model B in which the buyer's total demand is one unit and the seller makes offers in each of two periods. Of course, the seller makes an offer in the second period only if there is no sale in the first period.

Theorem 1 of Sobel and Takahashi (1983) provides the optimal pricing strategy for model B. They show that if $\delta_s \leq \delta_b$, then offering p^* in each period remains optimal. Thus, when $\delta_s \leq \delta_b$, our model's close connection with model A is maintained with model B.

When $\delta_s > \delta_b$, Sobel and Takahashi show that the seller does not offer p^* in each period: model B optimal prices $(p_1^{\mathrm{ST}}, p_2^{\mathrm{ST}})$ satisfy $p_1^{\mathrm{ST}} > p_2^{\mathrm{ST}}$. Buyers separate into three intervals with high valuation buyers purchasing in the first period at price p_1^{ST} and intermediate valuation buyers purchasing in the second period at price p_2^{ST}. Low valuation buyers do not buy. We argue next that the optimal prices $(p_1^{\mathrm{ST}}, p_2^{\mathrm{ST}})$ are not in any way related to the optimal prices $p_1, p_2, p_2^{\mathrm{n}}$ in our model.

If the two units in our model are bundled and offered at prices $((1+\delta_b)p_1^{\mathrm{ST}}, (1+\delta_b)p_2^{\mathrm{ST}})$ then intermediate-value buyers do not have the option of buying only one unit in the second period. Thus, bundling the two units in both periods and offering suitably re-scaled Sobel-Takahashi prices does not replicate the equilibrium in part (i) of the Proposition.

An alternative bundling strategy for the seller is to attempt to sell a bundle of two units in the first period, and if she is unsuccessful, then offer only one unit in the second period. This allows (intermediate-value) buyers to purchase nothing in the first period and one unit in the second period. The re-scaled Sobel-Takahashi prices, translated into our model, that support this alternate bundling strategy are $\hat{p}_1 = (1+\delta_b)p_1^{\mathrm{ST}}, \hat{p}_2 = 0$, and $\hat{p}_2^{\mathrm{n}} = p_2^{\mathrm{ST}}$. Does this version of model B pricing strategy yield the same expected profits as the prices in part (i) of the Proposition? The answer is no. Recall that optimal prices in our model are unique. Therefore, as $p^* > 0$ and optimal prices satisfy $p_2 > p^* > 0$, prices $\hat{p}_1, \hat{p}_2 = 0, \hat{p}_2^{\mathrm{n}}$ are not optimal in our model.

When a patient seller can commit to second-period prices, the optimal pricing policy in the one-unit per period demand model is neither a repetition of nor a version of the optimal pricing policy in the one-unit total demand model. Contrast this with the following observation made by Armstrong (2006) after surveying prior work (which assumes equal discount factors) on behavior-based price discrimination with commitment: "...it is a standard result in principal-agent theory that when the agent's private information does not change over time, the optimal dynamic incentive scheme repeats the optimal static incentive scheme." This theme is also expressed in chapter 8 of Laffont and Martimort (2002) where it is stated that in repeated adverse selection models in which private information is constant "the optimal long-term contract is obtained in a straightforward manner as a *replica of the one-shot optimal contract...*". We have shown that this link between optimal long-term contracts and optimal short-term contracts in repeated adverse selection models is severed if one drops the assumption of equal discount factors. It is not optimal for a less informed but more patient party to commit to not use in later periods information that is revealed in an earlier period.

4 Appendix: Proof of Proposition

If $p_2^{\mathrm{n}} > v_c$ then a buyer who does not buy in the first period will not buy in the second period; reducing p_2^{n} to equal v_c gives the same expected profit to the seller. Therefore, without loss of optimality, we restrict attention to $p_2^{\mathrm{n}} \leq v_c$.

The following lemma is useful in proving the Proposition.

Lemma A: *In any optimal solution:*

(i) $1 > v_c > 0$.

(ii) $p_1 > 0$, $p_2^{\mathrm{n}} > 0$.

Proof: (i) Suppose that prices p_1, p_2 and p_2^{n} are such that $v_c = 1$. Then with probability one there is no sale in the first period. Therefore, in the second period $p_2^{\mathrm{n}} = p^*$ is optimal and the seller's expected profit is $\delta_s p^*(1 - F(p^*))$. The seller can earn a higher expected profit of $(1 + \delta_s)p^*(1 - F(p^*))$ by choosing $p_1 = p_2 = p_2^{\mathrm{n}} = p^*$. Thus, $v_c < 1$ in any optimal solution.

Next we show that $v_c > 0$. Clearly, $v_c \geq a$. If $a > 0$ then $v_c \geq a > 0$. Therefore, consider the case $a = 0$. If $v_c = 0$ then, as $p_2, p_2^{\mathrm{n}} \geq 0$, we must have $p_1 = 0$; with probability one there is sale in the first period and in the second period $p_2 = p^*$ is optimal. Thus, if $v_c = 0$ then the seller's expected profit is at most $\delta_s p^*(1 - F(p^*))$. Selecting $p_1 = p_2 = p_2^{\mathrm{n}} = p^*$ yields a higher expected profit of $(1 + \delta_s)p^*(1 - F(p^*))$. Thus, if $a = 0$ then $v_c > 0$.

(ii) We first prove that $p_2^{\mathrm{n}} > 0$. Suppose that there is no sale in the first period. Clearly, $p_2^{\mathrm{n}} < a$ cannot be optimal as $p_2^{\mathrm{n}} = a$ does not decrease the probability of second-period sale. Thus, if $a > 0$ we are done. Next, suppose that $a = 0$. By (i), $v_c > 0$. Then $p_2^{\mathrm{n}} = 0$ yields zero expected profit whereas a price $p_2^{\mathrm{n}} \in (0, v_c)$ yields positive expected profit.

From (i), $v_c > 0$. Therefore, if $p_1 \geq v_c$ then $p_1 > 0$. Suppose, instead, that $v_c > p_1$ and $p_2^{\mathrm{n}} \geq p_1$. Then

$$v_c - p_1 \ > \ \delta_b \max\{v_c - p_2^{\mathrm{n}}, 0\}$$

and (1) implies that the marginal buyer is strictly better off buying in the first period at price p_1. Contradiction. Thus, if $v_c > p_1$ then $p_1 > p_2^{\mathrm{n}}$. As $p_2^{\mathrm{n}} > 0$ we have $p_1 > 0$. ∎

Proof of Proposition: The analysis divides into two cases, depending on whether or not the marginal first-period buyer buys in the second period as well. We first derive KKT conditions for each of these two cases and then apply them to prove parts (i), (ii), and (iii) of the Proposition.

Case A: $p_2 \leq v_c$.

Because $p_2 \leq v_c$ and, without loss of optimality $p_2^{\mathrm{n}} \leq v_c$, a buyer with value v_c is indifferent between buying in both periods or buying only in the second period.

Thus, (1) becomes

$$\begin{aligned} v_c - p_1 + \delta_b(v_c - p_2) &= \delta_b(v_c - p_2^{\mathrm{n}}) \\ \Longrightarrow \qquad v_c &= p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}). \end{aligned} \tag{3}$$

The seller's profit function is

$$\Pi_s(p_1, p_2, p_2^{\mathrm{n}}) = [1 - F(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))][p_1 + \delta_s p_2] + [F(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}})) - F(p_2^{\mathrm{n}})]\delta_s p_2^{\mathrm{n}}$$

subject to the conditions

$$\begin{aligned} p_2 &\leq p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}) = v_c \\ p_2^{\mathrm{n}} &\leq p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}) = v_c. \end{aligned}$$

Therefore, the seller maximizes the Lagrangian

$$L(p_1, p_2, p_2^{\mathrm{n}}, \lambda_1, \lambda_2) = \Pi_s(p_1, p_2, p_2^{\mathrm{n}}) - \lambda_1[p_2 - (p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))] - \lambda_2[p_2^{\mathrm{n}} - (p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))].$$

All prices and Lagrange multipliers are non-negative, and by Lemma A(ii), $p_1 > 0$ and $p_2^{\mathrm{n}} > 0$ at an optimal solution. Therefore, the KKT conditions are

$$\frac{\partial L}{\partial p_1} = \frac{\partial L}{\partial p_2^{\mathrm{n}}} = 0, \qquad p_2 \frac{\partial L}{\partial p_2} = \lambda_1 \frac{\partial L}{\partial \lambda_1} = \lambda_2 \frac{\partial L}{\partial \lambda_2} = 0. \tag{4}$$

Differentiating L with respect to p_1 and p_2^{n} and then substituting from (3) we get

$$\begin{aligned} \frac{\partial L}{\partial p_1} &= [1 - F(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))] - f(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))[p_1 + \delta_s(p_2 - p_2^{\mathrm{n}})] + \lambda_1 + \lambda_2 \\ &= [1 - F(v_c)] - f(v_c)[v_c + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}})] + \lambda_1 + \lambda_2 = 0 \\ \Longrightarrow \quad [1 - F(v_c)] &= f(v_c)[v_c + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}})] - \lambda_1 - \lambda_2 \qquad (5) \\ \frac{\partial L}{\partial p_2^{\mathrm{n}}} &= \delta_b\{f(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))[p_1 + \delta_s(p_2 - p_2^{\mathrm{n}})] - \lambda_1 - \lambda_2\} \\ &\quad + \delta_s[F(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}})) - F(p_2^{\mathrm{n}})] - \delta_s f(p_2^{\mathrm{n}})p_2^{\mathrm{n}} - \lambda_2 \\ &= \delta_b\{f(v_c))[v_c + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}})] - \lambda_1 - \lambda_2\} \\ &\quad + \delta_s[F(v_c)) - F(p_2^{\mathrm{n}})] - \delta_s f(p_2^{\mathrm{n}})p_2^{\mathrm{n}} - \lambda_2 \\ \text{[using (5)]} \qquad &= -(\delta_s - \delta_b)[1 - F(v_c)] + \delta_s[1 - F(p_2^{\mathrm{n}})] - \delta_s p_2^{\mathrm{n}} f(p_2^{\mathrm{n}}) - \lambda_2 = 0. \end{aligned}$$

Further,

$$\begin{aligned} \frac{\partial L}{\partial p_2} &= \delta_s[1 - F(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))] - \delta_b f(p_1 + \delta_b(p_2 - p_2^{\mathrm{n}}))[p_1 + \delta_s(p_2 - p_2^{\mathrm{n}})] \\ &\quad - (1 - \delta_b)\lambda_1 + \delta_b\lambda_2 \\ &= \delta_s[1 - F(v_c)] - \delta_b\{f(v_c)[v_c + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}})] + \lambda_1 + \lambda_2\} - \lambda_1 \\ \text{[using (5)]} \qquad &= (\delta_s - \delta_b)[1 - F(v_c)] - \lambda_1 \end{aligned}$$

Thus, (4) becomes

$$\begin{aligned} f(v_c)[v_c+(\delta_s-\delta_b)(p_2-p_2^{\mathrm{n}})]-[1-F(v_c)] &= \lambda_1+\lambda_2 && (6)\\ -(\delta_s-\delta_b)[1-F(v_c)]+\delta_s[1-F(p_2^{\mathrm{n}})]-\delta_s p_2^{\mathrm{n}} f(p_2^{\mathrm{n}}) &= \lambda_2 && (7)\\ p_2\Big\{(\delta_s-\delta_b)[1-F(v_c)]-\lambda_1\Big\} &= 0 && (8)\\ \lambda_1[p_2-v_c] &= 0 && (9)\\ \lambda_2[p_2^{\mathrm{n}}-v_c] &= 0 && (10)\end{aligned}$$

Case B: $p_2 > v_c$.

As $p_2 > v_c$ and $p_2^{\mathrm{n}} \le v_c$, equation (1) becomes

$$\begin{aligned} v_c-p_1 &= \delta_b(v_c-p_2^{\mathrm{n}})\\ \Longrightarrow \qquad v_c &= \frac{p_1-\delta_b p_2^{\mathrm{n}}}{1-\delta_b} && (11)\end{aligned}$$

The seller's profit function becomes

$$\hat{\Pi}_s(p_1,p_2,p_2^{\mathrm{n}}) = \Big[1-F\Big(\frac{p_1-\delta_b p_2^{\mathrm{n}}}{1-\delta_b}\Big)\Big]p_1+[1-F(p_2)]\delta_s p_2+\Big[F\Big(\frac{p_1-\delta_b p_2^{\mathrm{n}}}{1-\delta_b}\Big)-F(p_2^{\mathrm{n}})\Big]\delta_s p_2^{\mathrm{n}}.$$

Substituting $v_c = \frac{p_1-\delta_b p_2^{\mathrm{n}}}{1-\delta_b}$ into the inequality $p_2^{\mathrm{n}} \le v_c$ yields $p_2^{\mathrm{n}} \le p_1$. Thus, the Lagrangian to be maximized is

$$\hat{L}(p_1,p_2,p_2^{\mathrm{n}}) = \hat{\Pi}_s(p_1,p_2,p_2^{\mathrm{n}})-\hat{\lambda}_1(v_c-p_2)-\hat{\lambda}_2(p_2^{\mathrm{n}}-p_1).$$

By Lemma A(ii), $p_1 > 0$ and $p_2^{\mathrm{n}} > 0$. By assumption, $p_2 > v_c$ which together with Lemma A(i) implies $p_2 > 0$. Therefore, the KKT conditions are

$$\begin{aligned} [1-F(v_c)]-f(v_c)[v_c+\frac{(\delta_b-\delta_s)}{(1-\delta_b)}p_2^{\mathrm{n}}]-\hat{\lambda}_1+\hat{\lambda}_2 &= 0 && (12)\\ \delta_s[1-F(p_2)]-\delta_s f(p_2)p_2+(1-\delta_b)\hat{\lambda}_1 &= 0 && (13)\\ \frac{\delta_b f(v_c)}{1-\delta_b}(p_1-\delta_s p_2^{\mathrm{n}})-f(p_2^{\mathrm{n}})\delta_s p_2^{\mathrm{n}}+[F(v_c)-F(p_2^{\mathrm{n}})]\delta_s+\hat{\lambda}_1\delta_b-\hat{\lambda}_2 &= 0 && (14)\\ \hat{\lambda}_1[v_c-p_2] &= 0 && (15)\\ \hat{\lambda}_2[p_2^{\mathrm{n}}-p_1] &= 0. && (16)\end{aligned}$$

Because $v_c < p_2$, by (15) $\hat{\lambda}_1 = 0$. Then by (13), $p_2 = p^*$.

(i) The seller is more patient than the buyer, i.e., $\delta_s > \delta_b$.

We first show that Case B cannot hold at any optimal solution. Noting that $\hat{\lambda}_1 = 0$, we can write (12) as

$$f(v_c)v_c-[1-F(v_c)] = \frac{(\delta_s-\delta_b)}{(1-\delta_b)}p_2^{\mathrm{n}} f(v_c)+\hat{\lambda}_2$$

As $p_2^n > 0$ by Lemma A(ii), $\delta_s > \delta_b$, and $\hat{\lambda}_2 \geq 0$, the fact that F is regular implies that $v_c > p^*$. But $p^* = p_2$, which contradicts our assumption that $p_2 > v_c$. Thus, at any optimal solution, $p_2 \leq v_c$.

Next, we apply the KKT conditions for Case A. If $p_2 = 0$ then, as $v_c > 0$ by Lemma A(i), (9) implies $\lambda_1 = 0$. Next, as $v_c < 1$, again by Lemma A(i), we have $F(v_c) < 1$. But then $\lambda_1 = 0$ and $F(v_c) < 1$ together with (8) imply that $\frac{\partial L}{\partial p_2} > 0$. Thus, $p_2 = 0$ cannot be optimal. Therefore, $p_2 > 0$. Then (8) becomes

$$(\delta_s - \delta_b)[1 - F(v_c)] - \lambda_1 = 0. \tag{17}$$

As $F(v_c) < 1$, we have $\lambda_1 > 0$, which together with (9) implies that $p_2 = v_c$. Thus, as $p_2^{\mathrm{n}} \leq v_c$, we have $p_2 \geq p_2^{\mathrm{n}}$.

Substituting from (17) into (7) yields

$$\begin{aligned} \delta_s[1 - F(p_2^{\mathrm{n}}) - f(p_2^{\mathrm{n}})p_2^{\mathrm{n}}] &= \lambda_1 + \lambda_2 \qquad (18) \\ &= v_c f(v_c) - [1 - F(v_c)] + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}})f(v_c) \\ &= v_c f(v_c) - [1 - F(v_c)] + (\delta_s - \delta_b)(v_c - p_2^{\mathrm{n}})f(v_c) \end{aligned}$$

where the second equality follows from (6). As $\lambda_1 > 0$, $\lambda_2 \geq 0$ and F is regular, $p_2^{\mathrm{n}} < p^*$. Further, if $v_c = p_2^{\mathrm{n}}$ then $v_c > p^*$ which contradicts $p^* > p_2^{\mathrm{n}}$. Thus, $v_c > p_2^{\mathrm{n}}$ and hence, $\lambda_2 = 0$.

Substituting $\lambda_1 = (\delta_s - \delta_b)[1 - F(v_c)]$ and $\lambda_2 = 0$ in (6), and noting that $p_2 = v_c$, $p_2^{\mathrm{n}} > 0$, $\delta_s > \delta_b$ yields

$$\begin{aligned} (1 + \delta_s - \delta_b)\frac{1 - F(v_c)}{f(v_c)} &= v_c + (\delta_s - \delta_b)(p_2 - p_2^{\mathrm{n}}) \\ &= v_c + (\delta_s - \delta_b)(v_c - p_2^{\mathrm{n}}) \\ &< (1 + \delta_s - \delta_b)v_c \\ \implies \qquad v_c &> p^*. \end{aligned}$$

Thus, we have proved that

$$v_c \;=\; p_2 \;>\; p^* \;>\; p_2^n.$$

Further, (18) is satisfied by unique p_2^{n}, v_c. Uniqueness of p_1 follows from (1). Next, $v_c = p_2 > p_2^n$ and (3) imply that

$$v_c \;=\; p_2 \;>\; p_1 \;>\; p_2^n.$$

The KKT conditions are satisfied at a unique set of interior prices. Therefore, this solution is either a global maximum or a global minimum for the Lagrangian. As the expected profit at these prices is positive and greater than zero expected profits at the solution $p_1 = p_2 = p_2^{\mathrm{n}} = \lambda_1 = \lambda_2 = 0$, this solution represents a global maximum.

(ii) The seller and the buyer are equally patient, i.e., $\delta_s = \delta_b$.

An identical argument to that used to for $\delta_s > \delta_b$ eliminates Case B for $\delta_s = \delta_b$. Let $\delta \equiv \delta_b = \delta_s$. It is straightforward to check that and p^1, p^2 such that $p^1 \geq p^*$, $p^1 + \delta p_2 = (1+\delta)p^*$, and $p_2^{\mathrm{n}} = v_c = p^*$, and $\lambda_1 = \lambda_2 = 0$ satisfy (6) through (10), the KKT conditions for Case A.

(iii) The seller is less patient than the buyer, i.e., $\delta_s < \delta_b$.

First consider Case A. As $\delta_s < \delta_b$, $\lambda_1 \geq 0$, and $F(v_c) < 1$, equation (17) is not satisfied. Therefore, $\partial L / \partial p_2 < 0$ and $p_2 = 0$. As $v_c > 0$, (9) implies that $\lambda_1 = 0$. The remaining KKT conditions, (6) and (7), have as a unique solution $v_c = p_2^{\mathrm{n}} = p^*$, $p_1 = (1+\delta_b)p^*$, $\lambda_2 = (\delta_b - \delta_s)p^* f(p^*) = (\delta_b - \delta_s)(1 - F(p^*))$, $\lambda_1 = p_2 = 0$. In essence, the price is p^* in each period but the seller gets all of the buyer-discounted revenue in the first period. The marginal buyer buys in both periods.

The seller's expected profit at these prices are $(1+\delta_b)p^*(1-F(p^*))$. It remains to show that prices in Case B ($p_2 > v_c$) do not yield greater expected profits. Note that optimal prices within Case B must satisfy $p_2 = p^*$. Therefore, $v_c < p^*$. Further, from (11), $p_1 = (1-\delta_b)v_c + \delta_b p_2^{\mathrm{n}}$. As $p_2^{\mathrm{n}} \leq v_c$, optimal prices within Case B must satisfy

$$p_2 = p^* > v_c \geq p_1 \geq p_2^{\mathrm{n}}.$$

The expected profits at these prices are

$$\begin{aligned}
& (1-F(v_c))p_1 + \delta_s p_2(1-F(p_2)) + \delta_s(F(v_c) - F(p_2^{\mathrm{n}}))p_2^{\mathrm{n}} \\
= \; & (1-F(v_c))p_1 + \delta_s p^*(1-F(p^*)) + \delta_s(F(v_c) - F(p_2^{\mathrm{n}}))p_2^{\mathrm{n}} \\
< \; & (1-F(v_c))p_1 + \delta_b p^*(1-F(p^*)) + \delta_s(F(v_c) - F(p_2^{\mathrm{n}}))p_2^{\mathrm{n}} \\
\leq \; & (1-F(v_c))v_c + \delta_b p^*(1-F(p^*)) + \delta_s(F(v_c) - F(p_2^{\mathrm{n}}))p_2^{\mathrm{n}} \\
< \; & (1+\delta_b)p^*(1-F(p^*))
\end{aligned}$$

where the last inequality follows from the fact that when selling one unit to a buyer, a take-it-or-leave-it price of p^* is the optimal mechanism (which yields greater expected profit than setting a price of v_c in the first period and, if there is no sale, dropping prices to p_2^{n} in the second period). ∎

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