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ANALYZING CONVERTIBLE BONDS

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The convertible bond is a hybrid security which, while retaining most of the characteristics of straight debt, offers, in addition, the upside potential associated with the underlying common stock. As a *quid pro quo* for the upside potential the convertible bond is typically subordinated to other corporate debt and carries a lower coupon rate than would an otherwise equivalent straight bond.

The value of a convertible, like that of a straight bond, depends upon the coupon rate and maturity as well as the risk of default, which in turn reflects both the underlying asset risk of the issuing firm and the security provisions of the bond indenture. Also like a straight bond, the value of the convertible is influenced by the prevailing level of interest rates. However, the call provision which is standard in most straight bond indentures assumes greater importance in the case of the convertible because of its potential use in forcing conversion. Finally, the value of the convertible will depend upon the value of the conversion privilege: this is a function of the risk and capital structure of the firm, the payout policy of the firm, the call policy to be pursued by the firm, and the terms on which the bond may be converted into common stock as well as the current stock price.

The equilibrium value of a convertible bond is defined as that value which offers the potential of arbitrage profit neither to purchaser nor to short seller, given that the bondholder pursues an optimal strategy with respect to conversion and that the firm pursues an optimal policy with respect to calling the bonds. Given the multifarious factors determining the value of a convertible, the valuation problem is extremely complex, so that it is difficult without a formal model for the issuing firm to make an assessment of the feasible trade-offs between various characteristics of an issue: the conversion and call features,

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coupon rate and maturity, etc. The obverse side of this problem is faced by the investor choosing between convertibles and common stock.

This paper develops a model of the pricing of convertibles which can be used as an aid in making the above types of assessment. Since both the call and the conversion features are options exercisable by the firm and by the investor respectively, it is not surprising that the appropriate mode of analysis is to treat the convertible bond as a contingent claim and to value it using the option-pricing techniques pioneered by Black-Scholes and Merton.

Convertible bonds have been analyzed previously by Brennan and Schwartz and by Ingersoll. This paper differs from that earlier work both in allowing for the uncertainty inherent in interest rates and in taking acount of the possibility of senior debt in the firm's capital structure.

I. Call and Conversion Strategies

We shall consider a firm which has outstanding senior debt, convertible bonds and common stock. The total market value of the firm, V, is the sum of the market values of these three securities.

$$V = N_B\, B + N_C\, C + N_O\, S^{BC} \tag{1}$$

where B and N_B are market value of a straight bond and the number outstanding respectively; C and N_C refer similarly to the convertibles; N_O is the number of shares of common stock before conversion of the bonds; and S^{BC} is the stock price before conversion. Similarly, after conversion of the convertible, the market value of the firm is given by

$$V = N_B\, B + (N_O + \Delta N) S^{AC} \tag{2}$$

where ΔN is the number of shares issued as a result of conversion and S^{AC} is the stock price after conversion has taken place.

The conversion privilege is conventionally expressed either in terms of the price at which the bonds are convertible into common stock, the conversion price, or in terms of the number of common shares into which each bond is convertible, the conversion ratio. Assuming that the par value of the convertible is $1,000, these two measures are related by

Conversion Ratio = 1000/Conversion Price.

Writing the conversion ratio as q, it follows that the total number of new shares issued on conversion, ΔN, is

$$\Delta N = N_C \cdot q \tag{3}$$

and the fraction of the total shares owned by the holder of each convertible bond after conversion, Z, is

$$Z = q/(N_O + \Delta N). \tag{4}$$

In elementary textbooks the conversion value of a bond is defined as the conversion ratio, q, times the preconversion stock price:

$$\text{Conversion Value} = q \times S^{BC}.$$

The problem with the definition of conversion value is that it is not in general equal to the value of the shares into which each bond is convertible if all the bonds are converted: this value is given instead by $q \times S^{AC}$. Solving equations (1) and (2) for S^{BC} and S^{AC}, the two definitions of conversion value are:[1]

$$\text{Conversion Value (1)} = q \times S^{BC} = \frac{q}{N_O}[V - N_B B - N_C\, C].$$

$$\text{Conversion Value (2)} = q \times S^{AC} = \frac{q}{N_O+\Delta N}[V - N_B B] = Z[V - N_B\, B].$$

It may be seen that the two definitions are not equivalent, and that Conversion Value (1) depends on the actual value of the convertible bonds; in fact, the two definitions yield the same value only if the bonds are selling at their conversion value. We shall use Conversion Value (2) because it both represents the value of the bonds if they are all converted and it expresses the conversion value in terms of what is given or known, the values of the firm and the straight debt, and the conversion ratio.

Conversion Strategy

Convertible bondholders will always find it optimal to convert if the value of the bond falls below its Conversion Value (2). Therefore, the value of the convertible can never fall below the level so that

[1]This involves the standard assumption that the value of the firm is independent of the capital structure.

(5) $$C \geq q\,S^{AC} = Z(V - N_B\,B).$$

On the other hand it will never be optimal to convert if the value of the bond exceeds its Conversion Value (2) for this involves a sure value loss. Therefore the condition for optimal conversion is that the value of the bond is equal to its conversion value:

(6) $$C = Z(V - N_B\,B).$$

Note that condition (6) does not define the optimal conversion strategy until the bond is valued and C is determined. Yet the bond value itself depends on the optimal conversion strategy: therefore the bond value and the optimal conversion strategy must be determined simultaneously.

Call Strategy

In determining its optimal call strategy for the convertible bonds the management of the firm is assumed to be concerned with maximizing the value of the original shares. The value of these shares is given from equation (1) by

(7) $$N_O S = V - N_B\,B - N_C C.$$

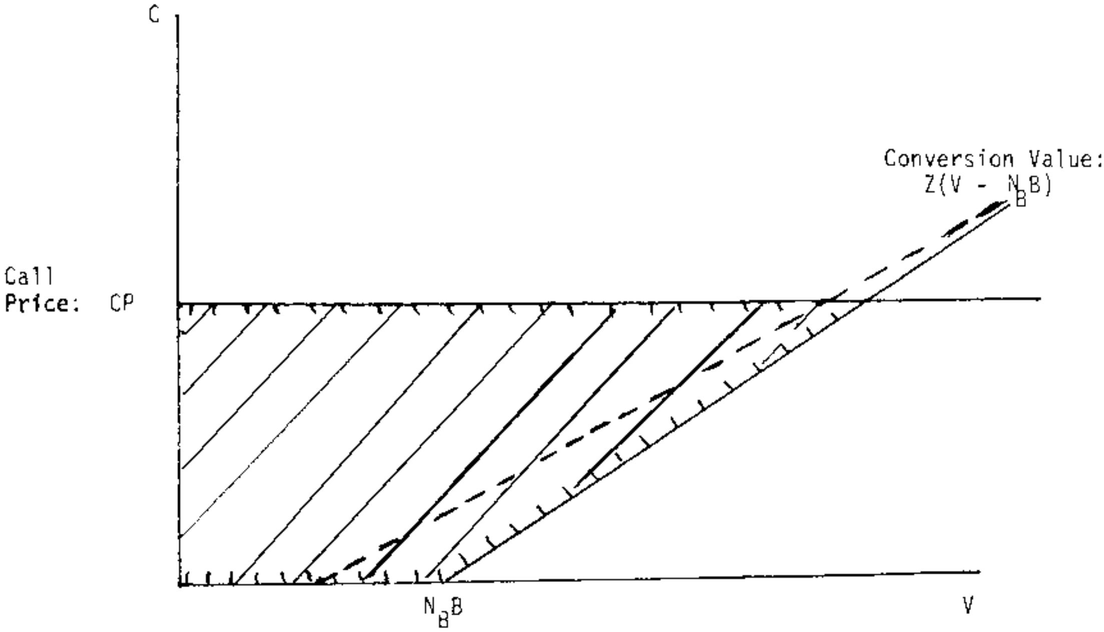
It is clear from this equation that, given the total value of the firm, V, the value of the shares is maximized by pursuing a call strategy which minimizes the value of the convertibles. Such a call strategy is incompatible with allowing the convertible to rise in value above the price at which it is currently callable, CP. Hence under an optimal call strategy

(8) $$C \leq CP.$$

On the other hand, it will not be optimal to call the convertible when its value is less than the call price, since this would confer a windfall gain on the convertible holders at the expense of the stockholders. Therefore, under the optimal strategy the convertible will be called when

(9) $$C = CP.$$

Again, equation (9) does not define the optimal call strategy until C is determined by solving the valuation problem which itself depends on the optimal call strategy. It follows that the valuation problem and the optimal call and conversion strategies must be solved for simultaneously.



Convertible Bond: Call and Conversion Conditions

Figure 1

Given that the senior debt is perfectly secured so that its value, B, is independent of the total value of the firm, V, the shaded region of Figure 1 represents the combinations of convertible bond and firm values that are consistent with the conversion value constraint (5) and the call policy constraint (8). At any time the bond is not convertible or is not callable, the relevant constraint in Figure 1 is removed. Finally, if the senior debt is not riskless, the conversion value line will no longer be straight but will look like the dotted line shown in the Figure.

II. The Model

The value of the convertible and the straight debt are assumed to depend upon the total value of the firm, V, and the current interest rate, r. V affects the convertible bond value through its influence on the probability of default (the asset backing of the bonds) and on the conversion value of the bonds; it affects the value of the straight debt only through the probability of default. The interest rate affects both bond values since it reflects the rate at which certain future returns are discounted. Both the value of the firm and the interest rate are assumed to follow random processes. Over a short interval of time the change in the interest rate, Δr, is assumed to be given approximately by

$$\tilde{\Delta r} = \alpha(\mu_r - r) + r\sigma_r\tilde{Z}_r \qquad , \alpha > 0 \tag{10}$$

where $\tilde{Z}_r$ is a normally distributed random variable with mean zero and variance of one. Equation (10) says that the change in the interest rate has a non-random component $\alpha(\mu_r - r)$ and a random component $r\sigma_r\tilde{Z}_r$. It is seen that the nonrandom component depends on the current value of r: if r is less than μ_r, then the nonrandom component is positive; if r is greater than μ_r, then it is negative. Thus the interest rate tends to revert to its long-run value, μ_r, at a rate which depends upon the speed of adjustment coefficient, α. A random change $r\sigma_r\tilde{Z}_r$ is superimposed on this reverting tendency. $r\sigma_r$ is the standard deviation of this random component of the change in the interest rate.

Similarly, the change in the value of the firm, ΔV, has a nonrandom and a random component and is given approximately by

$$\tilde{\Delta V} = [V\mu_V - Q(V,t)] + V\sigma_V\tilde{Z}_V. \tag{11}$$

μ_V is the expected total rate of return on the firm's value; $Q(V,t)$ is the total rate of cash distribution to the firm's security holders: coupon payments to the senior bondholders and to the holders of the convertible bonds, and dividend payments to the stockholders. Thus

$$Q(V,t) = I_B + I_{CB} + D(V,t) \tag{12}$$

where I_B and I_{CB} are the coupon payments to the two classes of bondholder, and $D(V,t)$ is the total dividend payment which may be any function of the firm value and time.

With the above assumptions it is shown in the Appendix that the value of the convertible bond satisfies the partial differential equation:

$$1/2C_{VV}V^2\sigma_V^2 + C_{Vr}Vr\rho\sigma_V\sigma_r + 1/2C_{rr}r^2\sigma_r^2 + C_r[\alpha(\mu_r - r) - \lambda r\sigma_r] \tag{13}$$

$$+ C_V[rV - Q(V,t)] - rC + cF + C_t = 0.$$

In this equation λ is the market price of interest rate risk: this is the reward to variability ratio of a portfolio whose rate of return is perfectly correlated with changes in the interest rate (e.g., any long-term bond). F is the par value of the convertible and c is its coupon rate. The subscripts on C

denote partial derivatives with respect to the respective arguments.

The convertible value also satisfies:

The Conversion Condition

(14) $$C(V,r,t) \geq Z(V - N_B B(V,r,t)).$$

This is equation (5) rewritten to show explicitly the dependence of both bond values on V, r, and t.

The Call Condition

(15) $$C(V,r,t) \leq CP(t)$$

where CP(t) is the price at which the convertible is callable at time t.

The Maturity Condition

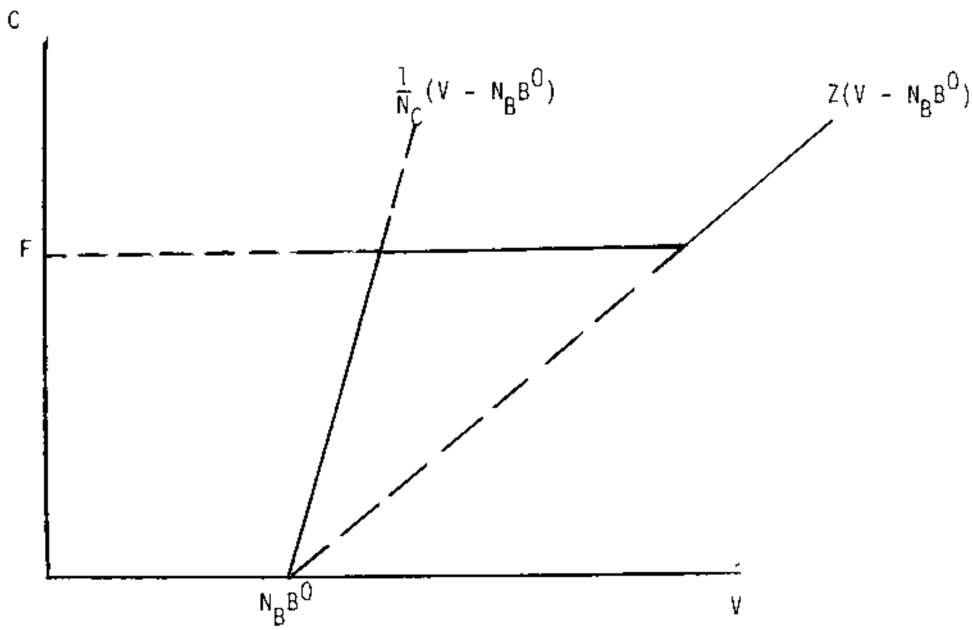
The convertible is assumed to mature at time T, prior to the maturity of the senior debt. Its value at maturity is given by

(16) $$C(V,r,T) \begin{cases} = Z(V - N_B B(V,r,T)) & \text{if } Z(V - N_B B(V,r,T)) \geq F \\ = F & \text{if } Z(V - N_B B(V,r,t)) \leq F \leq \frac{1}{N_C}(V - N_B B^O) \\ = \frac{1}{N_C}(V - N_B B^O) & \text{if } \quad F \geq \frac{1}{N_C}(V - N_B B^O) \geq 0 \\ = 0 & \text{if } \quad V < N_B B^O. \end{cases}$$

This condition states that the convertible holders receive the conversion value if it exceeds the par value of the bond; then they receive the par value provided that this does not exceed the value of the firm less the par value of the senior debt (B^O). If this condition is not satisfied, the firm goes bankrupt and the convertible bondholders are paid after the senior bondholders. The maturity condition is illustrated in Figure 2 on the assumption that the senior debt is selling at par.

The Bankruptcy Condition

It is assumed that the convertible bond indenture is written so that the bondholders will receive a fraction k of the par value when the firm goes bankrupt. This implies that the firm will go bankrupt if its value falls to the sum of the par value of the straight debt and k times the par value of the convertibles:



Convertible Value at Maturity

Figure 2

(17) $$C(V,r,t) = kF \qquad \text{if} \quad V = N_B B^0 + kN_C F.$$

III. Solving for the Convertible Bond Value

The convertible bond value is obtained by solving numerically the partial differential equation (13) subject to the conditions imposed by conversion (14), call (15), maturity (16), and bankruptcy (17). Note, however, that both the conversion and maturity conditions involve the value of the senior debt at the time of conversion or maturity, $B(V,r,t)$, where t is equal to the time of maturity or conversion. Denote by $B^*(V,r,t)$ the value of the senior debt given that the convertibles are no longer outstanding. Then it can be shown by methods similar to those described in the Appendix that $B^*(V,r,t)$ satisfies the partial differential equation (13) with $Q(V,t)$ replaced by $Q^*(V,t)$ where

(18) $$Q^*(V,t) = I_B + D^*(V,t).$$

Equation (18) gives recognition to the fact that when the convertibles have been eliminated from the firm's capital structure by conversion or maturity, the aggregate payout on the firm's securities will no longer include interest on the convertibles, and the aggregate dividend payment as a function of firm value will reflect the changed capital structure of the firm.[2]

In addition, the value of the senior debt will satisfy the maturity value condition

$$(19) \qquad B^*(V,r,T') = B^O \qquad \text{if } B^O \le \frac{1}{N_B} V$$
$$= \frac{1}{N_B} V \qquad \text{if } B^O > \frac{1}{N_B} V$$

where T' is the maturity date of the senior debt. The bankruptcy condition is assumed to be of the form

$$(20) \qquad B^*(V,r,t) = hB^O \qquad \text{if} \quad V = N_B hB^O .$$

Thus, the senior bondholders are assumed to receive a fraction h of the par value in the event of bankruptcy.

The first stage in valuing the convertibles is to solve the differential equation subject to (19) and (20) for $B^*(V,r,t)$. Substituting the resulting values of $B^*(V,r,t)$ in the conversion and maturity conditions (15) and (16), the partial differential equation is then solved for $C(V,r,t)$, the value of the convertible.

IV. Numerical Example

To illustrate the operation of the model a convertible was valued for the particular set of parameter values given in Table 1. The parameters of the interest rate process were estimated by regression using monthly data on U.S. Treasury Bill yields for 1970-79. The estimate of μ_r was 6 percent and this was adjusted to 8 percent to reflect contemporary capital market conditions.

The standard deviation of firm value was taken as 20 percent per year. It has been found that most common stocks on the New York Stock Exchange have annual standard deviations of return in the range of 10-50 percent. Bearing in mind that most of these stocks are levered by debt issues, the 20 percent value assumed here seems representative. The correlation between random changes in the interest rate and in firm values of -0.01 was based on T Bills and the CRSP market index. For simplicity the firm was assumed to have no senior debt outstanding; the firm payment $Q(V,t)$, includes only dividends and the coupons on

[2] In (18) we are implicitly assuming that at maturity the convertible is replaced by equity: other assumptions are possible.

the convertible. The dividend payout parameters were based on the assumption that the firm's earnings follow a random walk, that the firm's overall cost of capital is 10 percent and that both the payout ratio and the corporate tax rate are 50 percent.

The convertible bond is assumed to be a \$6m issue with an 8 percent coupon and a 10-year maturity. It is callable after 5 years at 108 with the call price declining in equal annual decrements to 100 at maturity. Each \$1000 bond is convertible into 18.52 shares of common stock so that the conversion price is \$54.

The stock price at the time the issue is made is taken as \$44.02 and the short-term riskless rate as 15 percent. Given the assumed process for interest rates and the assumption that the pure expectations theory of the term structure holds so that $\lambda = 0$, this is consistent with a yield to maturity on a default-free bond of 11.46 percent. The convertible bondholders are assumed to recover 2/3 of the par value of their investment in the event of bankruptcy.

The differential equation was solved first assuming that the bond was neither convertible nor callable. The yield to maturity on this straight corporate bond was 11.52 percent reflecting a .06 percent default premium over the corresponding riskless rate. Finally the convertible bond was valued and each \$1000 bond was found to have a market value at time of issue of \$997.

Figure 3 shows the value of each \$1000 bond at time of issue as a function of the firm value and the stock price: the latter is derived by subtracting the aggregate market value of the bonds from the firm value and dividing by the number of shares. The left-hand end of the curve represents bankruptcy when the value of the firm is \$4m; this is equal to the bankruptcy value of the bonds, \$6m x 2/3. The convertible value is a monotonically increasing function of the stock price, asymptotically approaching the conversion value: note that for the particular parameter values chosen it is not optimal to convert the bond immediately for any firm value within the range considered.

Figure 4 shows the bond value as a function of the firm value and the stock price at the time the bond first becomes callable for the same short-term riskless rate. Note that the call condition now keeps the bond value below the call price while the conversion condition keeps it above the conversion value. For the interest rate chosen the bond is called optimally when the conversion value is equal to the call price; for lower interest rates the bond may be called optimally at lower firm values (see Figure 8).

Figure 5 shows the effect of the interest rate on the bond value when the firm value is \$50 at time of issue. As the interest rate rises, the bond value asymptotically approaches the conversion value: it never reaches it, however,

TABLE 1

PARAMETERS OF NUMERICAL EXAMPLE

1. The Interest Rate Process

$$\Delta \tilde{r} = \alpha(\mu_r - r) + r\sigma_r \tilde{Z}_r$$

α = .13 per year

μ_r = 8% per year

σ_r = 26% per year

2. Firm Characteristics

$$\Delta \tilde{V} = [V\mu_V - Q(V,t)] + V\sigma_V \tilde{Z}_V$$

$$Q(V,t) = d_1 V + d_2 + I_{CB}$$

d_1 = .05 d_2 = −.12

σ_V = 20% per year

ρ = −0.01

I_{CB} = $.48m per year

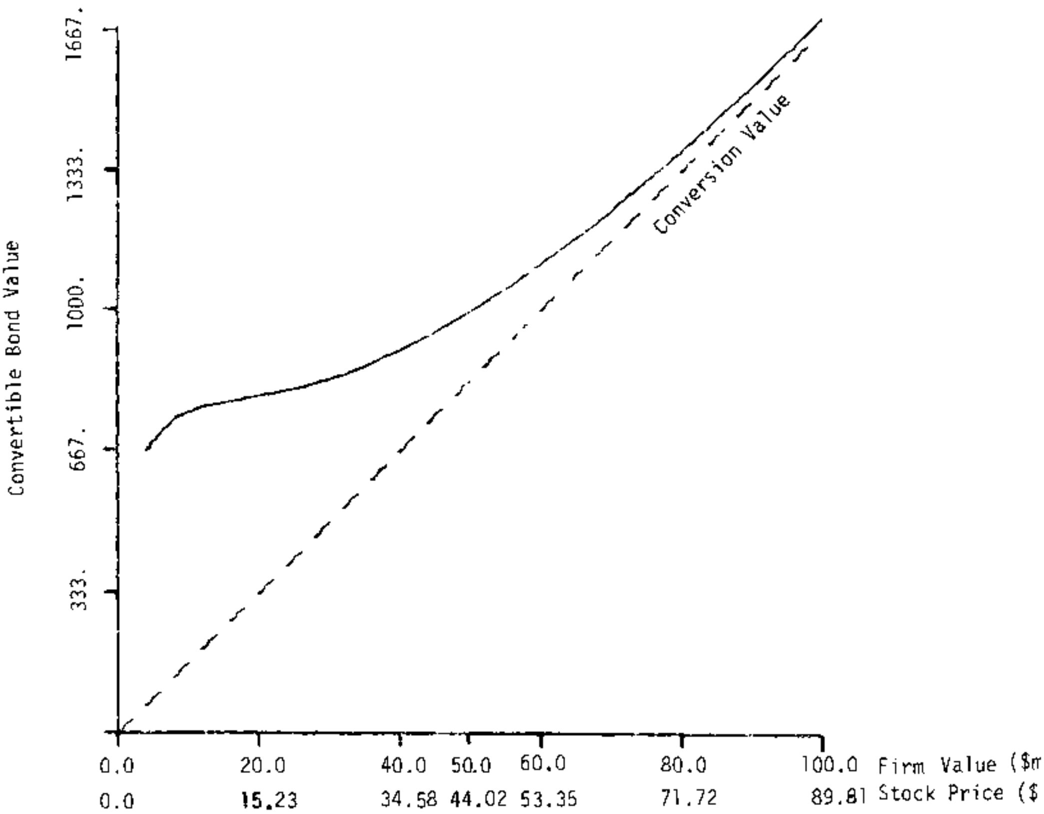
1 million shares of common stock outstanding. No senior debt.

3. Bond Characteristics

Issue size:	$6m
Coupon Rate:	8%
Conversion Price:	$54
Maturity:	10 years
Callable after 5 years	
Recovery in Bankruptcy:	2/3 of par value.

4. Environmental Parameters

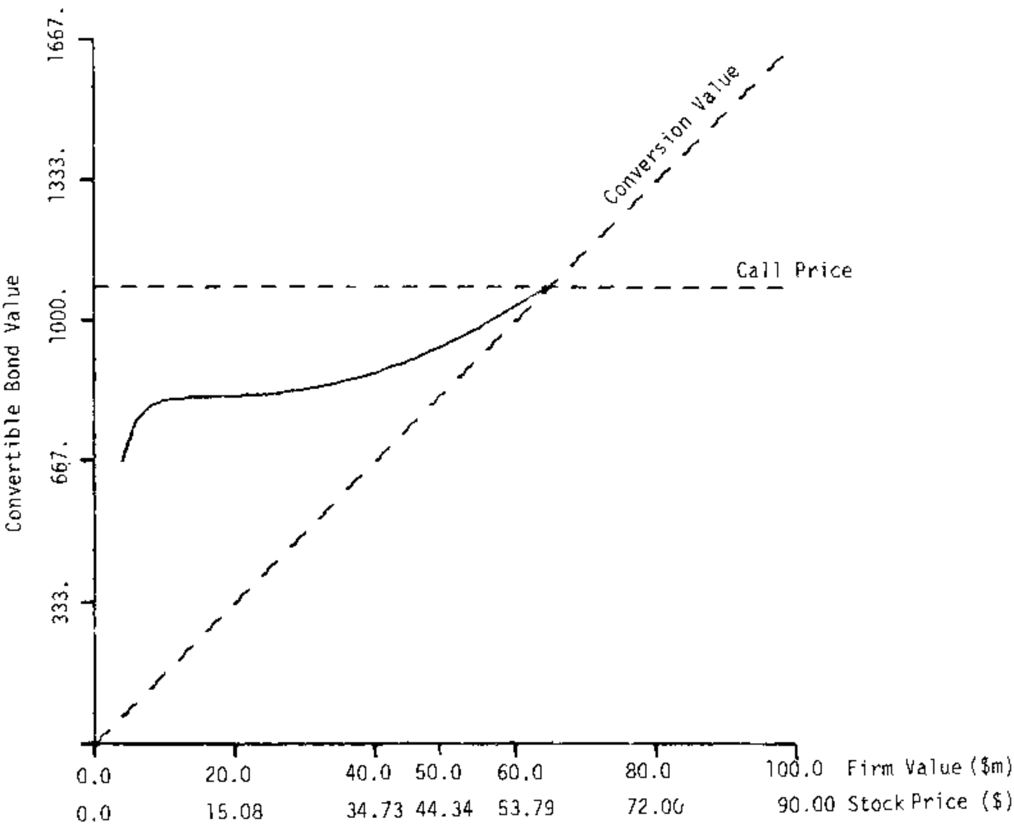
Stock Price:	$44.02
Short-term Riskless Rate:	15%
10-year Riskless Rate:	11.46%
10-year Corporate Bond Rate:	11.52%



Bond Value at Time of Issue

r = 15%

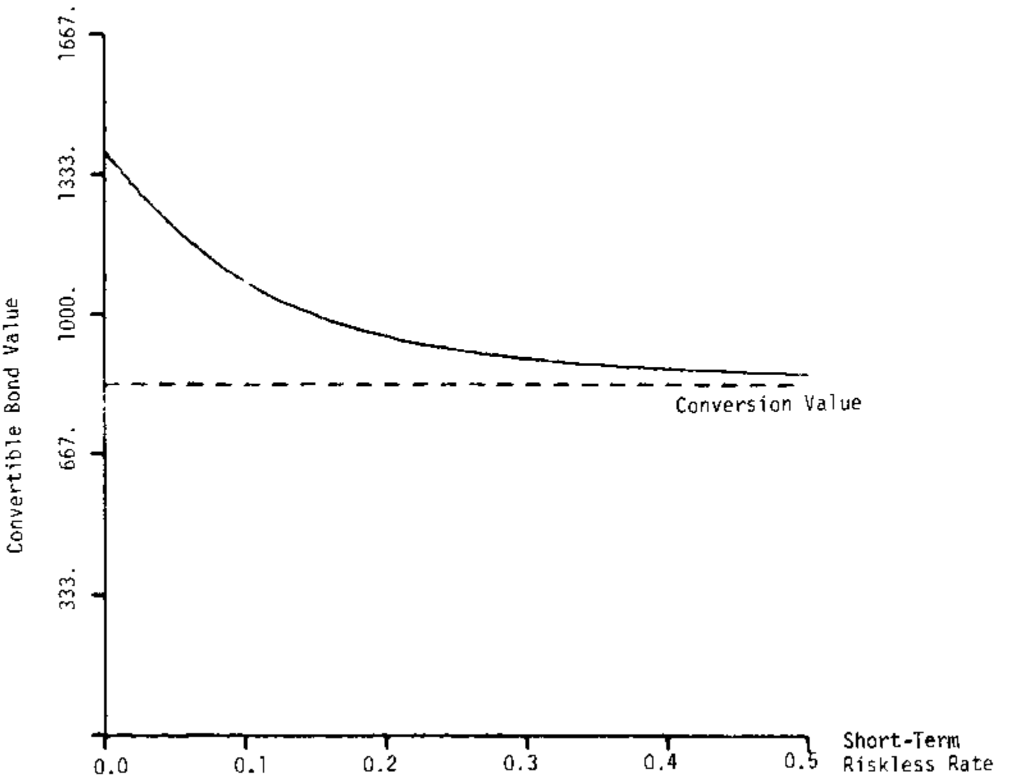
Figure 3



Bond Value at Time of First Call

r = 15%

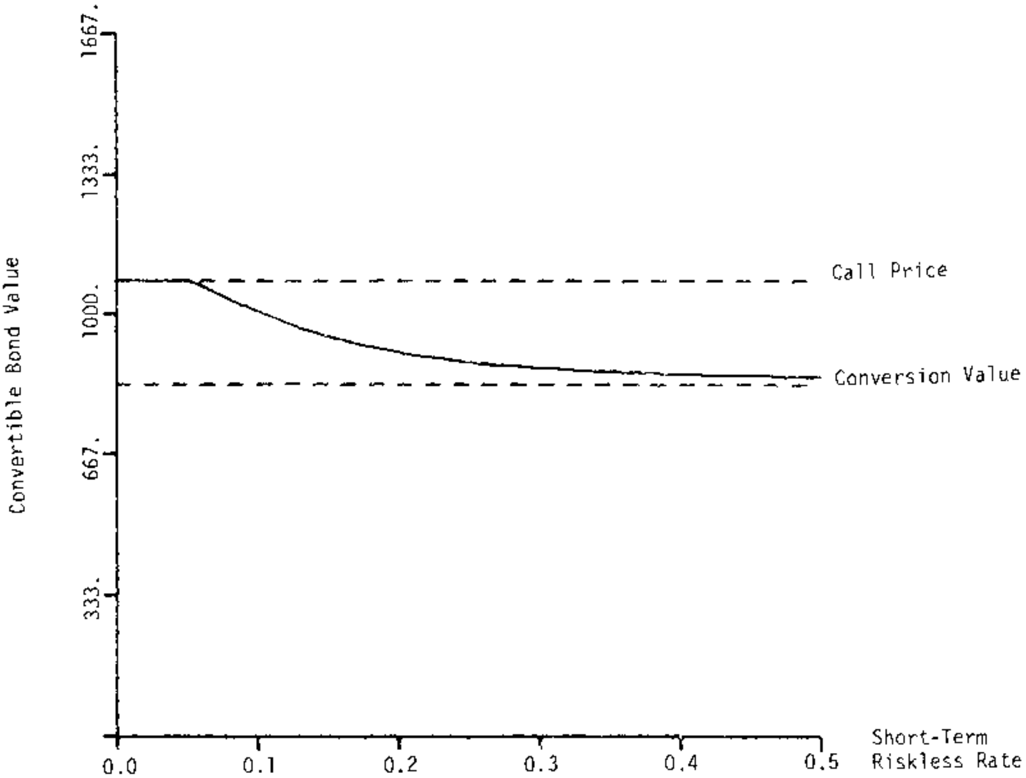
Figure 4



Bond Value at Time of Issue as a Function of the Interest Rate

V = $50m

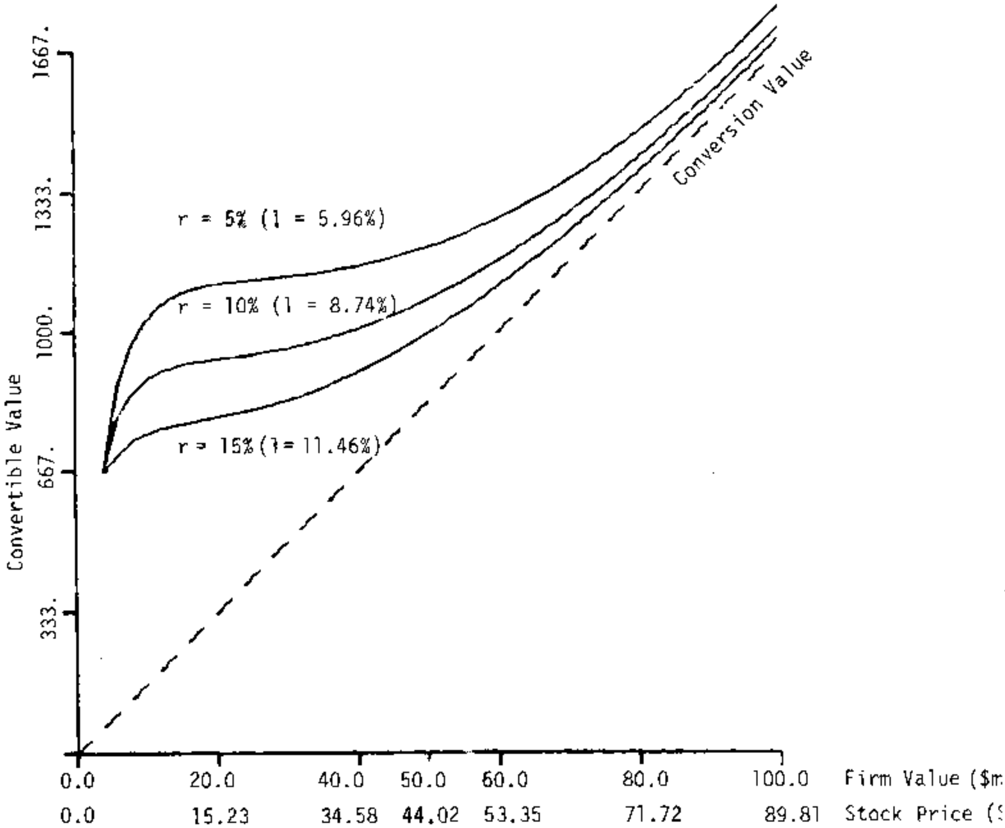
Figure 5



Bond Value at First Call Date as a Function of the Interest Rate.

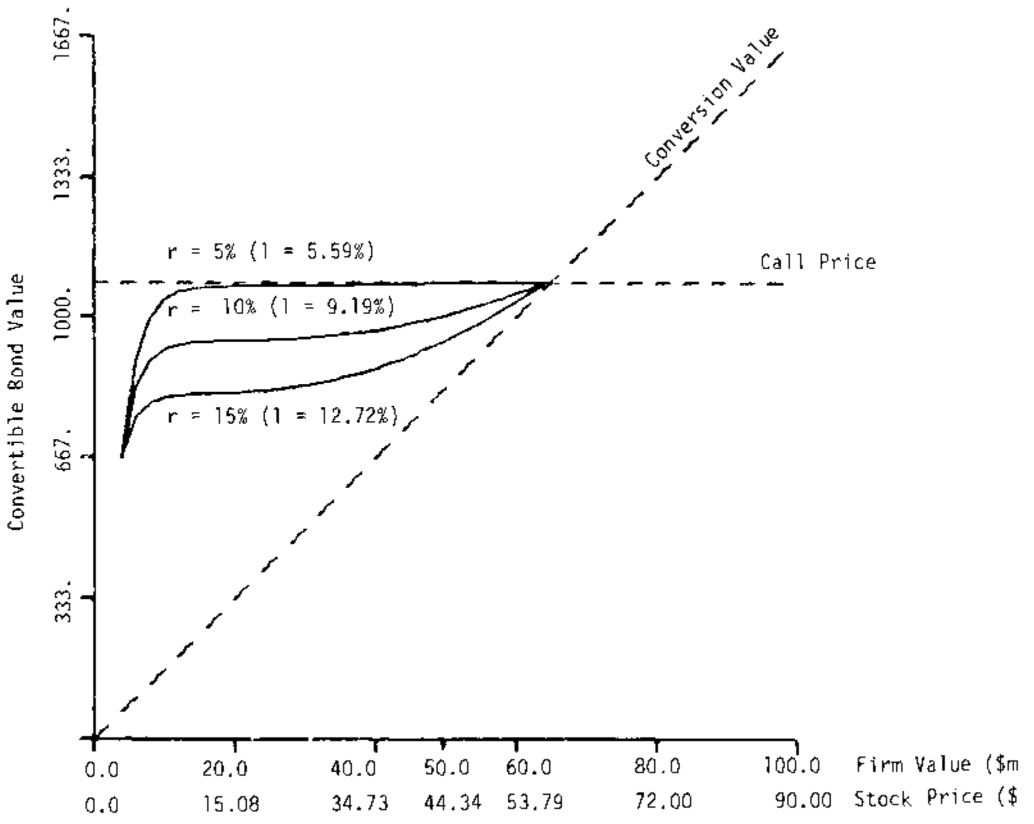
V = $50m

Figure 6



Bond Value at Time of Issue for Different Short-Term (r) and 10 year (l) Riskless Interest Rates.

Figure 7



Bond Value of Time if First Call for Different Short-Term (r) and 5 year (1) Riskless Interest Rates.

Figure 8

since for this firm value the coupon on the bond exceeds the dividend rate on the shares into which it is convertible. In Figure 6 the same relationship is shown at the time of first call; now the bond value is not only bounded from below by the conversion value, but is also bounded from above by the call price. Figures 7 and 8 show how the bond value is related to the firm value for different interest rates at time of issue and time of first call respectively. As one would expect, the bond value is a decreasing function of the interest rate.

V. Sensitivity Analysis

In addition to providing an estimate of the value of a particular convertible bond, the model may be used to assess the sensitivity of the bond value to changes in environmental, firm, and bond parameters. The results of some representative calculations are shown in Table 2: in this table various parameters are changed one at a time.

TABLE 2

COMPARATIVE STATICS OF BOND VALUE AT ISSUE (PARAMETER VALUES OF BASIC EXAMPLE IN PARENTHESES)

	Bond Value	% Change
Basic Example	997	-
Non-callable	1032	3.5
Non-callable, non-convertible	787	-21.1
Stock Price: $48.42 ($44.02)	1045	4.9
10-year Riskless Rate: 12.5% (11.4%)	973	2.3
σ_V: .22 (.20)	1007	1.0
d_1 = .055 (.050); d_2 =-.132 (-.12)	987	1.0
Coupon: 8.8% (8.0%)	1028	3.2
Conversion Price: $48.60 ($54.00)	1047	5.0
Bankruptcy Value: $333 ($666)	997	-0.0
First Call Date: 6 years (5 years)	1005	0.8
First Call Price: $1188 ($1080)	1005	0.8

First, it may be seen that removal of the corporation's right to call the bond for redemption would increase its value by 3.5 percent; on the other hand, removing the conversion privilege as well, which would make the bond a straight non-callable bond, would reduce the bond value by 21.1 percent. Thus, in this case the straight bond value at issue is about 79 percent of the equilibrium issue value.

The next three lines of the table show the responsiveness of the bond price to environmental parameters beyond the control of the firm: of these, the stock price is the most critical, a 10 percent increase in the stock price raising the bond value by 4.9 percent. As one would expect, the bond value decreases as the long-term rate of interest increases. In this example the bond value rises with the risk of the firm, σ_V: it may be noted that the effect of an increase in firm risk is to raise both the probability of bankruptcy (which decreases the bond value) and the value of the conversion option (which raises the bond value). In this instance the effect of the latter outweighs the former. An increase in the firm's dividend payout also reduces the bond value since this both increases the probability of bankruptcy and reduces the expected rate of appreciation in the stock price, decreasing the value of the conversion privilege.

Turning to the characteristics of the bond itself, only the coupon rate and the conversion price have a major effect on the bond value, a 10 percent reduction in the conversion price raising the estimated bond value by 5.0 percent, and a 10 percent increase in the coupon rate raising the value by 3.2 percent. Changes in the security provisions have a negligible effect, reflecting the low probability that bankruptcy will occur, and small changes in the call price and date of first call have only modest effects on the bond value.

One use of the model is to assess the feasible tradeoffs between different bond characteristics. Thus, assuming that the relationships are approximately linear, Table 2 suggests that a 10 percent reduction in the coupon rate could be offset by a 6 percent reduction in the conversion price (10 percent x 3.2/5.0), or a 31 percent reduction in allowed future dividend payouts, and similar comparisons may be made between other parameters.

VI. Non-Stochastic Interest Rates

The advantage of the model developed in this paper over the earlier models of Brennan and Schwartz and Ingersoll is that it allows explicitly for uncertainty about future interest rates. On the other hand, this makes the model more complex and substantially increases the computational cost of solving the differential equation. It is therefore of interest to determine the error that would be caused by assuming a single known constant interest rate.

For this purpose, the 10-year riskless rates corresponding to short-term riskless rates of 0, 5, 10, 15 and 20 percent were calculated. The bond was then revalued assuming a constant interest rate equal to the 10-year rate. Comparison of the resulting bond value with that generated by the model allowing for uncertainty in interest rates yielded the valuation errors reported in Table 3 for different interest rates and firm values. A negative value in this table implies that the certain interest rate model value exceeds the uncertain interest rate model value.

TABLE 3

ERRORS IN BOND VALUATION (%) GENERATED BY
A MODEL WITH NON-STOCHASTIC INTEREST RATES

Short-term Riskless Rate:	r(%)	0	5	10	15	20
10-year Riskless Rate:	l(%)	3.12	5.96	8.74	11.46	14.17
V ($m)						
20		3.61	-1.61	-1.60	-1.26	0.73
50		3.82	-0.42	-0.37	-0.40	-0.63
80		2.15	0.07	0.00	-0.07	-0.15

The table suggests in a striking manner that for a reasonable range of interest rates the errors from the certain interest rate model are likely to be slight, and, therefore, for practical purposes it may be preferable to use this simpler model for valuing convertible bonds.

APPENDIX

THE PARTIAL DIFFERENTIAL EQUATION FOR CONVERTIBLE BONDS

The value of a convertible bond is subject to two distinct sources of uncertainty: the uncertainty in the value of the underlying firm value, V, and the uncertainty inherent in the rate of interest. It is necessary for us first to investigate the interest rate uncertainty.

Taking the limit of equation (10) as the time interval approaches zero, the process for the interest rate, r, is given by the stochastic differential equation

$$\text{(A1)} \qquad dr = \alpha(\mu_r - r)dt + r\sigma_r dZ_r .$$

The value of any default-free discount bond will be a function only of the interest rate and time. Let $G^i(r,t)$ $(i = 1,2)$ denote the market values of any two such bonds. Then, by Ito's Lemma, the instantaneous change in the value of such a bond is given by

$$\text{(A2)} \qquad \frac{dG^i}{G^i} = \mu_{G_i} dt + \sigma_{G_i} dZ_r$$

where

$$\mu_{G_i} = (G_t + G_r\alpha(\mu_r - r) + 1/2G_{rr}r^2\sigma_r^2)/G^i$$

$$\mu_{G_i} = G_r r\sigma_r/G^i .$$

Consider a zero net investment portfolio formed by investing an amount x_i $(i = 1,2)$ in bond i and borrowing an amount $(x_1 + x_2)$ at the instantaneous interest rate, r. The instantaneous return on this portfolio is

$$[x_1(\mu_{G_1} - r) + x_2(\mu_{G_2} - r)]dt+(x_1\sigma_{G_1} + x_2\sigma_{G_2})dZ_r .$$

If x_1 and x_2 are chosen so that $(x_1\sigma_{G_1} + x_2\sigma_{G_2}) = 0$, then the return on the portfolio is non-stochastic. To avoid the possibility of arbitrage profits the return must be zero. This will be so if and only if

$$\text{(A3)} \qquad \frac{\mu_{G_1} - r}{\sigma_{G_1}} = \frac{\mu_{G_2} - r}{\sigma_{G_2}} = \lambda(r,t)$$

where $\lambda(r,t)$ is the same for all bonds. (A3) says that the reward to variability ratio for all default-free bonds is the same. With this result we are now prepared to derive the partial differential equation for convertibles (or for the senior firm debt) by a similar argument.

Taking the limit of equation (11) as the time interval approaches zero, the instantaneous change in the value of the firm is given by

$$\text{(A4)} \qquad \frac{dV}{V} = \left[\mu_V - \frac{Q(V,t)}{V} \quad dt + \sigma_V dZ_V\right].$$

Dropping the superscript i from equation (A2) the instantaneous rate of return on any default-free bond is given by

$$\text{(A5)} \qquad \frac{dG}{G} = \mu_G dt + \sigma_G dZ_r \ .$$

Finally, given that the only two sources of uncertainty are the value of the underlying firm, V, and the interest rate, r, it follows that the value of a convertible bond may be written as a function of these two variables and time: $C(V,r,t)$. Then using Ito's Lemma, the instantaneous rate of capital gain on the convertible is given by

$$\text{(A6)} \qquad \frac{dC}{C} = \mu_C dt + \frac{C_V V \sigma_V}{C} dZ_V + \frac{C_r r \sigma_r}{C} dZ_r$$

where

$$\mu_C = [C_t + C_r \alpha(\mu_r - r) + C_V(\mu_V - Q(V,t))$$

$$+ 1/2 C_{rr} r^2 \sigma_r^2 + C_{Vr} V r \rho \sigma_V \sigma_r + 1/2 C_{VV} V^2 \sigma_V^2]/C$$

and ρ is the instantaneous correlation between dZ_V and dZ_r.

Consider forming a zero net investment portfolio by investing amounts x_C, x_G, x_V in the convertible, the default-free bond and the firm respectively, and borrowing $(x_C + x_G + x_V)$ at the instantaneous interest rate r. The instantaneous return on this portfolio is then, using (A4), (A5) and (A6) and taking account of the rate of coupon payment on the convertible cF,

$$\text{(A7)} \qquad [x_C(\mu_C + cF/C - r) + x_G(\mu_G - r) + x_V(\mu_V - r)]dt$$

$$+ [x_C C_V V \sigma_V/C + x_V \sigma_V] dZ_V$$

$$+ [x_C C_r R \sigma_r/C + x_G \ \sigma_G] dZ_r.$$

Let

$$\text{(A8)}\quad \begin{aligned} x_C C_V V\sigma_r/C + x_V\sigma_V &= 0 \\ x_C C_r r\sigma_r/C + x_G\sigma_G &= 0. \end{aligned}$$

Then the instantaneous rate of return on the portfolio is certain, and to avoid the possibility of arbitrage profit it must be equal to zero so that

$$\text{(A9)}\quad x_C(\mu_C + cF/C - r) + x_G(\mu_G - r) + x_V(\mu_V - r) = 0.$$

Eliminating x_C, x_V and x_G between (A8) and (A9) and using the definition μ_C, we obtain

$$\text{(A10)}\quad \begin{aligned} &1/2C_{VV}V^2\sigma_V^2 + C_{Vr}Vr\rho\sigma_V\sigma_r + 1/2C_{rr}r^2\sigma_r^2 \\ &+ C_r\alpha(\mu_r - r) + C_V(rV - Q(V,t)) + cF - rC + C_t \\ &- \frac{C_r r\sigma_r}{\sigma_G}(\mu_G - r) = 0. \end{aligned}$$

Finally we may use equation (A3) to write the last term as $-C_r\lambda_r\sigma_r$ which is then equivalent to equation (13) of the text.

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