BIAS IN FITTING THE SHARPE MODEL TO TIME SERIES DATA

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I. Introduction

The Sharpe model of capital asset pricing under conditions of risk has received wide theoretical acclaim and empirical support. This paper presents an econometric study of the model with the following objectives: (a) to show the effect of measuring the model's independent variables incorrectly; (b) to derive and use a new procedure for empirically testing the adequacy of the model as it is currently formulated.

The model is founded on the principle that every investor's utility is solely a function of the expected return and risk on his total portfolio of assets. Each asset in the portfolio is valued by its contribution to the return and risk of the whole. Its identity and other characteristics are irrelevant. In addition to this fundamental behavioral axiom, the following assumptions are utilized in deriving the model, that:

a. a riskless asset exists,
b. investors have homogeneous expectations,
c. time consists of only two periods,
d. the quantity of assets is fixed, and
e. risk is measured by the dispersion (variance) of returns.

Given these assumptions, Sharpe shows that capital asset prices will be defined in equilibrium by

$$(1) \qquad E(\tilde{R}_{j,t}|R_{F,t}) = R_{F,t}(1-\beta_j) + \beta_j E(\tilde{R}_{m,t}|R_{F,t}),$$

where E is the mathematical expectations operator, $\tilde{R}_{j,t}$ is the return on asset j in time t, $R_{F,t}$ is a riskless rate of interest, $\tilde{R}_{m,t}$ is the return

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on a "market" portfolio, and $\beta_j = \mathrm{Cov}(\tilde{R}_m, \tilde{R}_j | R_F)/\mathrm{Var}(\tilde{R}_m R_F)$ is a response (or risk) coefficient.[1] Since (1) is a static model, each of the three returns is defined for a given calendar period during which $R_{F,t}$ is nonstochastic.

Empirical studies[2] have operationalized the capital asset pricing model by replacing it with a "market model"[3] -- i.e., a regression equation[4]

$$(2) \qquad E(\tilde{R}_{j,t} | R_{F,t} ; R_{m,t}) = R_{j,t} - \epsilon_{j,t} = R_{F,t}(1-\beta_j) + \beta_j R_{m,t},$$

where the random disturbance term $\tilde{\epsilon}$ satisfies standard spherical assumptions.[5] R_m is usually measured by a stock market average and R_F is either

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[1]See William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance,* September 1964.

[2]For examples, see Marshall Blume, "The Assessment of Portfolio Performance, An Application to Portfolio Theory," unpublished Ph.D. dissertation (Chicago: Graduate School of Business, University of Chicago, March 1968); Eugene F. Fama, Lawrence Fisher, Michael C. Jensen, and Richard Roll, "The Adjustment of Stock Prices to New Information," *International Economic Review,* January 1969; or Michael C. Jensen, "The Performance of Mutual Funds in the Period 1945-1964," *Journal of Finance,* May 1968.

[3]A term used first in Eugene F. Fama, "Risk, Return, and Equilibrium: Some Clarifying Comments," *Journal of Finance,* March 1968.

[4]Note, however, that (1) is not a regression equation and it is not always appropriate to apply (2) to data. One time when it is appropriate is when $\tilde{R}_j$ and $\tilde{R}_m$ are distributed as bivariate normal. In that case, we know from distribution theory that

$$E(\tilde{R}_{j,t} | R_{F,t}; R_{m,t}) = E(\tilde{R}_{j,t} | R_{F,t}) + \beta_j [R_{m,t} - E(\tilde{R}_{m,t} | R_{F,t})],$$

where $\beta_j \equiv \mathrm{Cov}(\tilde{R}_m, \tilde{R}_j | R_F)/\mathrm{Var}(\tilde{R}_m | R_F)$ in accordance with Sharpe's theory. Substituting from (1) for $E(\tilde{R}_{j,t} | R_{F,t})$ and cancelling, provides the regression equation (2). The same development probably holds for any bivariate symmetric distribution of $\tilde{R}_j$ and $\tilde{R}_m$. However, when the joint distribution is asymmetric, (2) is not an appropriate operational counterpart to the theoretical model (1) and the application of (2) to date will give incorrect or misleading results.

[5]These assumptions are $E(\tilde{\epsilon}) = 0$; $\mathrm{Var}\ (\tilde{\epsilon})$ = constant over time; $\mathrm{Cov}\ (\tilde{\epsilon}_t, \tilde{\epsilon}_{t-j}) = 0$ for $j \neq 0$; $\tilde{\epsilon}$ is uncorrelated with the "independent" variables.

a short-term government bond[6] or is simply assumed constant over time.[7]

Of course, any time series fit of the Sharpe model violates the assumptions that assets are fixed in quantity and that there are only two periods. These assumptions are usually replaced by the contention that β_j is a constant over time.[8] In the following, we assume that β_j <u>is</u> a constant and that (2) is the appropriate operational form[9] of (1).

II. The Effect of Assuming a Constant Riskless Rate While Measuring the Market Return Without Error

If R_F is incorrectly assumed constant over time, the market model (2) becomes

$$R_{j,t} = \beta_j R_{m,t} + u_{j,t},$$

where $u_{j,t} = (1-\beta_j)R_{F,t} + \varepsilon_{j,t}$

and the fitted regression,

$$R_{j,t} = \hat{\alpha}_j + \hat{\beta}_j R_{m,t} + \hat{u}_{j,t} \tag{3}$$

contains estimates (denoted by circumflexes) that are inconsistent if $\tilde{R}_F$

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[6]A stock market average measures R_m with error because it only includes equity assets. A short-term government bond measures R_F with error since government bonds are not riskless, particularly when the bond's time until maturity does not match the investor's horizon period. Potential changes in the rate of inflation and in tax rates also influence the risk of holding government bonds.

[7]The latter was assumed in Blume, *op. cit.*, and in Fama et al., *op. cit.*

[8]And thus, β_j is not influenced by changes in R_F.

[9]Inter alia, this assumption means (a) no stochastic variable can have a probability distribution with non-finite moments, and (b) the "market horizon period" matches our period of observation.

and $\tilde{R}_m$ are correlated over time.[10] Specifically, the probability limits[11] of $\hat{\beta}_j$ and $\hat{\alpha}_j$ are

$$\text{(4)} \qquad \text{plim } \hat{\beta}_j = \beta_j + (1-\beta_j) \frac{\text{Cov}(\tilde{R}_m, \tilde{R}_F)}{\text{Var}(R_m)} ,$$

and

$$\text{(5)} \qquad \text{plim } \hat{\alpha}_j = (1-\beta_j) \left[E(\tilde{R}_F) - E(\tilde{R}_m) \frac{\text{Cov}(\tilde{R}_m, \tilde{R}_F)}{\text{Var}(\tilde{R}_m)} \right] .$$

There are several interesting features of the asymptotic biases[12] in $\hat{\alpha}_j$ and $\hat{\beta}_j$:

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[10] If $\tilde{R}_F$ and $\tilde{R}_m$ are correlated, $\text{Cov}(\tilde{R}_m, \tilde{u}_j) \neq 0$. Letting x denote the matrix of observations of the "independent" variables, it is well-known that the least-squares estimators $\hat{\beta}$ are related to their corresponding true coefficients by $\text{plim } \hat{\beta} = \beta + \text{plim}(N^{-1}x'x)^{-1} \text{ plim}(N^{-1}x'u)$. See Arthur S. Goldberger, *Econometric Theory* (New York: John Wiley and Sons, 1964), p. 270. It, therefore, follows for (3) that:

$$\text{plim } (N^{-1}x'x)^{-1} = \begin{bmatrix} E(\tilde{R}_m^2) & -E(\tilde{R}_m) \\ -E(\tilde{R}_m) & 1 \end{bmatrix} / \text{Var}(\tilde{R}_m), \quad \text{plim}(N^{-1}x'u) = \begin{bmatrix} (1-\beta_j)\ E(\tilde{R}_F) \\ (1-\beta_j)\ E(\tilde{R}_m\tilde{R}_F) \end{bmatrix} .$$

[11] If $\hat{x}_N$ is a sample statistic from a random sample of size N, the probability limit, plim $\hat{x}_N$, is defined by

$$\lim_{N\to\infty} \Pr[|\hat{x}_N - \text{plim } \hat{x}_N| < \xi] = 1,$$

where ξ is any arbitrarily small positive number.

[12] In (5), note that the expected value of $\hat{\alpha}_j$, if R_F is a constant over time, is $(1-\beta_j)R_F$. Therefore, it makes sense to define the asymptotic "bias" in $\hat{\alpha}_j$ as

$$(1-\beta_j)E(\tilde{R}_m) \frac{\text{Cov}(\tilde{R}_m, \tilde{R}_F)}{\text{Var}(\tilde{R}_m)} .$$

1. $\hat{\alpha}_j$ and $\hat{\beta}_j$ will be biased in opposite directions.

2. if the correlation between the risk-free return and the market return is positive over time, $\hat{\beta}_j$ will be downward biased when $\beta_j > 1$ and upward biased when $\beta_j < 1$.

3. the risk-free rate is sometimes inferred from $\hat{\alpha}_j$ and $\hat{\beta}_j$ by asserting that $\hat{\alpha}_j = (1-\hat{\beta}_j)R_F$ and estimating R_F by

$$\hat{R}_F = \frac{\hat{\alpha}_j}{1-\hat{\beta}_j} .$$

From (4) and (5) however,

$$\text{(6)} \qquad \operatorname{plim} \hat{R}_F = \operatorname{plim} \frac{\hat{\alpha}_j}{1-\hat{\beta}_j} = E(\tilde{R}_F) \frac{1 - \dfrac{E(\tilde{R}_m)}{E(\tilde{R}_F)} \dfrac{\operatorname{Cov}(\tilde{R}_m, \tilde{R}_F)}{\operatorname{Var}(\tilde{R}_m)}}{1 + \dfrac{\operatorname{Cov}(\tilde{R}_m, \tilde{R}_F)}{\operatorname{Var}(\tilde{R}_m)}}$$

which is less than $E(\tilde{R}_F)$ if $\operatorname{Cov}(\tilde{R}_m, \tilde{R}_F) > 0$, and greater than $E(\tilde{R}_F)$ if $\operatorname{Cov}(\tilde{R}_m, \tilde{R}_F) < 0$, and if the market exhibits risk aversion -- i.e., if $E(\tilde{R}_m) > E(\tilde{R}_F)$.

III. Measuring the Market Return With Error While the Riskless Return is Incorrectly Assumed Constant

Let the market return, $r_{m,t}$, be measured with error and given by

$$\text{(7)} \qquad r_{m,t} = R_{F,t}(1-\beta_m) + \beta_m R_{m,t} + \varepsilon_{m,t},$$

which indicates that the observed market return is also related to the true market return through the Sharpe model. In this case, the fitted Sharpe model for security j is

$$\text{(8)} \qquad R_{j,t} = \alpha_j + \beta_j r_{m,t} + \mu_{j,t}.$$

Again, $\tilde{r}_m$ and $\tilde{\mu}_j$ are not independent, and the probability limits of the estimators for (8) are given by

$$(9)\quad \text{plim}\ \hat{\alpha}_j = \left(1-\frac{\beta_j}{\beta_m}\right)\left[E(\tilde{R}_F) - E(\tilde{r}_m)\frac{\text{Cov}(\tilde{r}_m,\ \tilde{R}_F)}{\text{Var}(\tilde{r}_m)}\right] + \frac{\beta_j}{\beta_m} E(\tilde{r}_m)(1-\rho_m^2),\ \text{and}$$

$$(10)\qquad \text{plim}\ \hat{\beta}_j = \frac{\beta_j}{\beta_m} + \left(1-\frac{\beta_j}{\beta_m}\right)\frac{\text{Cov}(\tilde{r}_m,\ \tilde{R}_F)}{\text{Var}(\tilde{r}_m)} - \frac{\beta_j}{\beta_m}(1-\rho_m^2),$$

$$(11)\qquad \text{where}\quad \rho_m^2 = 1 - \frac{\text{Var}(\tilde{\varepsilon}_m)}{\text{Var}(\tilde{r}_m)}\ .$$

Equations (9) and (10) are similar to results (4) and (5) of the last section where the "market" return, R_m, was assumed to be measured without error. The main differences are the last terms in (9) and (10). These are due to "attenuation bias," a familiar result with errors-in-variables models.

IV. Measuring the Market and Riskless Rates With Error

Now consider the case when an attempt is made to include a variable risk-free rate. Specifically, let the measured market rate contain errors per (7) and let the measured "riskless" rate be given by a similar equation,

$$(12)\qquad r_{F,t} = (1-\beta_F)R_{F,t} + \beta_F R_{m,t} + \varepsilon_{F,t},$$

where $R_{F,t}$ and $R_{m,t}$ are the true riskless and market rates as before.[13]

Equations (2), (7), and (12) contain eight variables

$$(R_{j,t};\ r_{F,t};\ r_{m,t};\ R_{F,t};\ R_{m,t};\ \varepsilon_{j,t};\ \varepsilon_{F,t};\ \text{and}\ \varepsilon_{m,t}),$$

and the two unobservable returns ($R_{j,t}$ and $R_{m,t}$) can be eliminated to obtain a regression equation involving observable returns,

$$(13)\qquad R_{j,t} = \gamma_o + \gamma_1 r_{F,t} + \gamma_2 r_{m,t} + \mu_{j,t},$$

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[13]It seems reasonable to assume that the measured "riskless" and market rates conform to models (12) and (7). If the Sharpe market model is valid, every capital asset, including the assets used for these observed rates, will be related to the true but unobserved rates by expressions such as (7) or (12).

where

(14)
$$\begin{cases} \gamma_o = 0, \\ \gamma_1 = \dfrac{\beta_m - \beta_j}{\beta_m - \beta_F}, \\ \gamma_2 = \dfrac{\beta_j - \beta_F}{\beta_m - \beta_F} = 1 - \gamma_1, \text{ and} \end{cases}$$

$$\mu_{j,t} = \varepsilon_{j,t} - \gamma_1 \varepsilon_{F,t} - \gamma_2 \varepsilon_{m,t}.$$

The asymptotic properties of the least-squares estimators for (13) are given by

(15)
$$\text{plim } \hat{\gamma}_o = \gamma_o + \frac{\gamma_1(1-\rho_F^2)a_{F|m} + \gamma_2(1-\rho_m^2)a_{m|F}}{1 - \rho_{Fm}^2},$$

(16)
$$\text{plim } \hat{\gamma}_1 = \gamma_1 + \frac{\gamma_2(1-\rho_m^2)b_{m|F} - \gamma_1(1-\rho_F^2)}{1 - \rho_{Fm}^2}$$

(17)
$$\text{plim } \hat{\gamma}_2 = \gamma_2 + \frac{\gamma_1(1-\rho_F^2)b_{F|m} - \gamma_2(1-\rho_m^2)}{1 - \rho_{Fm}^2},$$

where

(18)
$$\left\{ \begin{array}{l} \rho_j^2 = 1 - \dfrac{\text{Var}(\tilde{\varepsilon}_j)}{\text{Var}(\tilde{r}_j)} \; ; \; b_{x|y} = \dfrac{\text{Cov}(\tilde{r}_x, \tilde{r}_y)}{\text{Var}(\tilde{r}_y)} \\ a_{x|y} = E(\tilde{r}_x) - E(\tilde{r}_y) b_{x|y}; \; \rho_{Fm} = \dfrac{\text{Cov}(\tilde{r}_F, \tilde{r}_m)}{[\text{Var}(\tilde{r}_F)\ \text{Var}(\tilde{r}_m)]^{1/2}} \end{array} \right\}.$$

Large sample estimates of γ_1 and γ_2 can be obtained from the five equations (15), (16), (17); $\gamma_o \equiv 0$, and $\gamma_1 \equiv 1 - \gamma_2$. This completes a system of five equations in the five unknowns γ_o, γ_1, γ_2, ρ_F^2, and ρ_m^2. The other terms, (ρ_{Fm}^2, $a_{F|m}$, $a_{m|F}$, $b_{F|m}$, $b_{m|F}$, plim $\hat{\gamma}_0$, plim $\hat{\gamma}_1$, plim $\hat{\gamma}_2$), are probability limits of observable variables, ($\tilde{R}_j$, $\tilde{r}_F$, and $\tilde{r}_m$). In large samples, the three plims are estimated by calculated least-squares coefficients. The other terms are estimated by using large sample moments where population moments appear in (18).

A solution to the system includes

$$1 - \gamma_1 = \gamma_2 = \frac{\text{plim}(\bar{R}_j - \bar{r}_F)}{\text{plim}(\bar{r}_m - \bar{r}_F)}\,, \tag{19}$$

where plim $(\bar{r}_x)$ is the large sample mean,[14] and also provides

$$1 - \rho_F^2 = \frac{(\gamma_1 - \text{plim}\ \hat{\gamma}_1) + b_{m|F}(\gamma_2 - \text{plim}\ \hat{\gamma}_2)}{\gamma_1}\,, \tag{20}$$

and

$$1 - \rho_m^2 = \frac{(\gamma_2 - \text{plim}\ \hat{\gamma}_2) + b_{F|m}(\gamma_1 - \text{plim}\ \hat{\gamma}_1)}{\gamma_2}\,. \tag{21}$$

Equations (20) and (21) enable us to empirically measure the correlations between the proxy riskless and market rates, r_F and r_m, and their unobservable counterparts, R_F and R_m. Even alone, this would be unusual for an errors-in-variables situation, but (20) and (21) provide an additional opportunity; a test of the validity of the Sharpe market model itself. When the model is valid, large sample estimates of ρ_F^2 and ρ_m^2, calculated by (20) and (21), will lie in the range zero to one (by the definition of the squared correlation coefficient). If the model is not valid, calculations of (20) and (21) may result in an impossible value for ρ_F^2 or ρ_m^2. A value outside the zero-one range will be condemning evidence for the model.

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[14] The simple result of (19) makes intuitive sense when one considers the original form of the Sharpe model, $E(R_j) = R_F + \beta_j E(\tilde{R}_m - \tilde{R}_F)$ or

$$\beta_j = \frac{E(\tilde{R}_j - R_F)}{E(\tilde{R}_m - R_F)}.$$

The result can also be obtained directly from (13) using $\gamma_1 = 1 - \gamma_2$, $\gamma_o = 0$.

V. An Application of the Sharpe Model to U. S. Treasury Bills

As an illustration, we now present a fit of the Sharpe market model to a sample of rates of return on a particular class of capital assets, United States Treasury bills. Referring to (13),

$$R_{j,t} = \gamma_o + \gamma_1 r_{F,t} + \gamma_2 r_{m,t} + \mu_{j,t}, \tag{13}$$

where the observed variables are defined as follows:

$r_{F,t}$ is the spot rate at the beginning of week t on a Treasury bill with one week to maturity.[15]

$r_{m,t}$ is the natural logarithm of the price relative of the Dow-Jones Composite Average from the beginning of week t to the beginning of week t+1.

$R_{j,t} \equiv \log_e(p_{j,t+1}/p_{j+1,t})$, where $p_{j,t+1}$ is the price of a Treasury bill with j weeks to maturity at the beginning of week t+1.

The sample, consisting of 793 consecutive weeks beginning on October 4, 1949, was split into four consecutive sub-samples of 198 weeks each.[16] Estimated coefficients from model (13) are presented in Table 1 for the entire period and the four sub-periods. For completeness, standard errors are reported, but they should not be used because of known biases.[17] Estimates of ρ^2_{Fm}, $b_m|_F$, etc., using the sample counterparts to (18), are given in Table 2.

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[15]The rates are in units of percent per annum. A simple arithmetic mean of bid and asked rates was used. The beginning of the week is defined as the closing bell on Tuesday. If Tuesday is a holiday, Monday closing quotations are used.

[16]One week was lost due to taking relative changes.

[17]The regressions were also checked to determine if auto-correlated disturbance terms were present. The Durbin-Watson statistic ranged from 1.5 to 1.8, indicating slight positive auto-correlation. Normal probability plots of the residuals indicated their distribution was symmetric but had fat tails, a result reported by other studies of financial asset prices, e.g. in Benoit Mandelbrot, "The Variation of Certain Speculative Prices," *Journal of Business*, October 1963.

Table 1

Sharpe Market Model Applied to U.S. Treasury Bill Rates*

	Total Sample (792 weeks)			Sub-Periods (198 Weeks each) 1 Weeks 1-198			2 Weeks 199-396			3 Weeks 397-594			4 Weeks 595-792		
Maturity, j (Weeks)	$\hat{\gamma}_o$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_o$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_o$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_o$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_o$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
1	.0353	.9943	$-.0^4400$	-.104	1.108	$.0^42009$	-.0201	1.020	$.0^3347$	-.0136	1.017	$-.0^31022$	.193	.9336	$-.0^3257$
	.9448	$.0^2831$	$.0^4885$	.8074	.0372	$.0^3143$	.9339	.0173	$.0^3167$	.9096	.0214	$.0^3209$	.8876	.0221	$.0^3171$
3	.0827	1.046	$-.0^4480$	-.245	1.265	$-.0^3160$	.0271	1.067	$.0^3682$	.0175	1.116	$.0^4889$	.622	.8299	$-.0^3435$
	.7876	.0193	$.0^3205$	.5097	.0879	$.0^3338$	.7693	.0408	$.0^3393$	.6908	.0531	$.0^3518$	.6797	.0398	$.0^3378$
5	.130	1.102	$.0^4688$	-.218	1.302	$.0^4378$	.0649	1.110	$.0^3865$	.0515	1.243	$.0^3348$	1.09	.7036	$-.0^3308$
	.6506	.0288	$.0^3306$	.3714	.120	$.0^3461$	.6401	.0587	$.0^3566$	.5386	.0823	$.0^3802$	.4874	.0509	$.0^3393$
7	.130	1.139	$-.0^4159$	-.302	1.376	$.0^4209$	.0240	1.177	$.0^2117$	.109	1.285	$.0^4178$	1.19	.6880	$-.0^4411$
	.5470	.0370	$.0^3394$	.3060	.147	$.0^3565$	.5526	.0749	$.0^3722$	.4090	.111	$.0^2109$	.4416	.0547	$.0^3422$
9	.0744	1.191	$-.0^3120$	-.422	1.452	$.0^3419$	$.0^4945$	1.191	$.0^2132$	.127	1.358	$-.0^3580$	1.12	.7208	$-.0^3552$
	.4679	.0454	$.0^3483$	.2766	.166	$.0^3640$	.4667	.0901	$.0^3869$	.3324	.140	$.0^2137$	.4050	.0618	$.0^3478$
11	.0910	1.174	$-.0^3345$	-.473	1.495	$.0^3423$	.0115	1.193	$.0^2174$	.0703	1.356	$-.0^2142$	1.03	.7407	$-.0^3852$
	.3899	.0526	$.0^3560$	.2248	.197	$.0^3757$	.3932	.105	$.0^2101$	.2671	.167	$.0^2163$	.3965	.0649	$.0^3501$

*Coefficients are given in the first row of each panel. Standard errors are given below $\hat{\gamma}_1$ and $\hat{\gamma}_2$. Multiple R^2 adjusted for degrees of freedom is given below $\hat{\gamma}_0$. Week number 1 starts October 4, 1949.

Table 2

Estimates of Bias Components* in the

Mis-Specified Sharpe Model

Components	Total Period	Sub-Periods			
		1	2	3	4
$a_{F\|m}$	2.00	1.37	1.64	2.26	2.72
$a_{m\|F}$	30.2	33.5	44.9	52.1	-10.5
$b_{F\|m}$	$-.0^{2}115$	$-.0^{3}228$	$-.0^{2}184$	$-.0^{2}194$	$-.0^{3}419$
$b_{m\|F}$	-10.1	-15.4	-19.8	-20.4	7.02
$\bar{r}_F$	1.99	1.37	1.61	2.25	2.72
$\bar{r}_m$	10.1	12.3	13.1	6.30	8.58
ρ^2_{Fm}	.0116	$.0^{2}352$	.0363	.0394	$.0^{2}295$

*See p. 276 for definitions of components.

We are interested in using these results to answer the following questions:

a) is the risk coefficient $\gamma_2 = 1 - \gamma_1$ constant over time?

b) does the Sharpe model fit these data?

First, the least-squares coefficients (γ's) are certainly not stationary over the four sub-periods (each sub-period contains almost four years of weekly observations). There are several potential explanations for this: (a) the Sharpe model is not valid at all, (b) the true coefficients fluctuate over time, and (c) the process which generates the true riskless and market returns, R_F and R_m, is not stationary and the least-squares coefficients fluctuate because they are related to this non-stationarity through the errors-in-variables model as shown by (15) - (17).

Table 2 shows that the mean values of r_F and r_m and their interrelationships are not temporally constant. Will accounting for these fluctuations lower the variability of γ_1? Table 3 provides a partial answer. Here γ_1 has been calculated by

$$\gamma_1 = \frac{\bar{r}_m - \bar{R}_j}{\bar{r}_m - \bar{r}_F}, \tag{22}$$

which corresponds to the asymptotic estimator of (19). Essentially, we hypothesize that the process generating $\tilde{r}_F$ and $\tilde{r}_m$ is relatively stationary during each sub-period, and that a superior estimate of γ_1 can be obtained from (22). In fact, Table 3 shows that the γ_1's calculated from (22) are considerably more uniform over time than the least-squares estimates. They are not constant, however, and are considerably lower in sub-period 3 than in the other sub-periods, Table 3 only suggests that <u>part</u> of the variation in the least-squares coefficients is caused by long-term changes in the process generating $\tilde{R}_F$ and $\tilde{R}_m$.

As a further check on the model, we can solve (20) and (21) for ρ_F^2 and ρ_m^2. Recall that

$\rho_F^2 = 1 - \dfrac{\mathrm{Var}(\tilde{\varepsilon}_F)}{\mathrm{Var}(\tilde{r}_F)}$ is defined by (18) as the proportion of variation in the <u>measured</u> riskless rate "explained" by its correspondence to the true riskless and market rates through the Sharpe model of

Table 3

Large Sample Estimates of Sharpe

Market Model Parameter γ_1*

Maturity, j (Weeks)	Total Period	Sub-Period 1	2	3	4
1	.9971	.9960	.9985	.9942	.9984
3	.9785	.9893	.9875	.9311	.9736
5	.9587	.9820	.9779	.8518	.9513
7	.9497	.9805	.9716	.8150	.9426
9	.9440	.9814	.9716	.7710	.9395
11	.9464	.9807	.9698	.7872	.9464

*Since $\gamma_2 \equiv 1 - \gamma_1$, only γ_1 from (22) is listed.

(12). ρ_m^2 is similarly defined for the measured market rate. (See (7).) To estimate ρ_F^2 and ρ_m^2 using the entire sample period, long-term sample means of R_j, r_F, and r_m are used to estimate γ_1, per (22). Then, this result is combined in (20) and (21) with $b_{F|m} = -.0^2 115$ and $b_{m|F} = -10.1$ from Table 2. These results, reported in Table 4, make absolutely no sense because the implicit ρ_F^2 is greater than one and the implicit ρ_m^2 is negative! The identical pattern also occurs in every sub-period when we use the corresponding $b_{F|m}$, $\bar{r}_F$, etc. for each sub-period.[18]

A key to the source of these results is the original simultaneous solution to (15) - (17) which required that the intercept term, γ_0, be zero. γ_0 will not be zero if traders have some motive for holding Treasury bills other than portfolio considerations. In fact, γ_0 is the extra return traders demand ($\gamma_0 > 0$) or forego ($\gamma_0 < 0$) due to other motives for holding Treasury bills.

A rough estimate of γ_0 can be made by assuming that $\bar{r}_F$ and $\bar{r}_m$ measure $E(\tilde{r}_F)$ and $E(\tilde{r}_m)$ in a pair of sub-periods. Using $\gamma_1 \equiv 1 - \gamma_2$, we write equation (13) as

$$\bar{R}_{ja} - \bar{r}_{ma} = \gamma_o + \gamma_1(\bar{r}_{Fa} - \bar{r}_{ma}) \tag{23}$$

where $\bar{R}_{ja}$ is the mean return on security j, and $\bar{r}_{Fa}$ and $\bar{r}_{ma}$ are means of the measured riskless and market returns during sub-period a. Estimates for a pair of sub-periods can be used in

$$\frac{\bar{R}_{ja} - \bar{r}_{ma} - \gamma_o}{\bar{r}_{Fa} - \bar{r}_{ma}} = \frac{\bar{R}_{jb} - \bar{r}_{mb} - \gamma_o}{\bar{r}_{Fb} - \bar{r}_{mb}}, \tag{24}$$

which can be solved for γ_o, assuming that γ_o is constant and equal in both sub-periods. Table 5 presents the results of this calculation[19] for a representative maturity of five weeks.[20] It indicates that γ_o, the

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[18] There was an exception. For sub-period 4, $\rho_F^2 \doteq 3$.

[19] γ_1 is calculated from (23) after γ_0 is obtained from (24).

[20] Results for other maturities are very similar.

Table 4

Implied Multiple R^2's For Measured Riskless and Market Returns (Total Period, 792 Observations)

Maturity, j	ρ_F^2	ρ_m^2
1	1.027	-.0126
3	1.292	$-.0^2584$
5	1.587	$-.0^2398$
7	1.736	$-.0^2464$
9	1.865	$-.0^2720$
11	1.819	-.0113

Table 5

Estimated Parameters Obtained by Allowing γ_o to be Non-Zero (Maturity-Five Weeks)

Sub-Periods		Estimated Parameters	
a	b	γ_o	γ_1
1	2	-1.026	.8883
1	3	.8387	1.059
1	4	.3873	1.017
2	3	.7913	1.047
2	4	.3186	1.006
3	4	1.309	1.175

intercept in (13), was probably negative in the first sub-period,[21] positive in the last three and highest in sub-period three. We also note that allowing γ_o to be non-zero results in more sensible implicit values for the multiple squared correlation coefficients, ρ_F^2 and ρ_m^2, from the Sharpe model, (7) and (12), that determine our riskless and market rates. The estimates of ρ_F^2 and ρ_m^2 in Table 6 are obtained by replacing the riskless rate coefficient γ_1 in (20) and (21) by the arithmetic mean of the six γ_1 values estimated for sub-periods and reported in Table 5.

Table 6
Implied Values of ρ_F^2 and ρ_m^2 Allowing
γ_o to be Non-Zero (Maturity-Five Weeks)

Sub-period	ρ_F^2	ρ_m^2
1	.784	$.0^2193$
2	.464	$.0^2448$
3	.574	$.0^2128$
4	.899	$.0^2431$

These estimated correlation coefficients are within the zero-one range. Other maturities give similar results.[22] If we assume that the ρ^2's in Table 6 are not orders of magnitude in error, our measured riskless rate, the one-week bill rate, is correlated fairly well with the true riskless rate, but our measured <u>market</u> rate, the Dow-Jones composite average log relative, bears practically no relation to the correct "market portfolio" rate for Treasury bills. In fact, non-zero values for γ_o, and its temporal instability suggest that traders in Treasury bills are not entirely motivated by portfolio considerations.

[21]This sub-period includes about 75 observations before the Federal Reserve-Treasury accord of March 1951. Before the accord, the Federal Reserve was using open market operations to keep bill rates at artificially low levels.

[22]Only two of the 24 implied ρ^2's are outside the range 0-1. These are ρ_F^2 = 1.077 for sub-period one, maturity 3 weeks, and ρ_F^2 = 1.021 for sub-period four, maturity 9 weeks.

VI. Summary, Conclusions

Econometric difficulties may be encountered when attempting to fit the static equilibrium Sharpe model to time series. A major difficulty is due to measurement error in the "riskless" and "market" returns, the explanatory variables of the model. This paper has derived asymptotic biases of least-squares estimators of Sharpe model parameters for three different measurement error specifications:

(a) riskless rate incorrectly assumed constant,

(b) riskless rate incorrectly assumed constant and market rate measured with error, and

(c) both riskless and market rates measured with error.

In specifications (b) and (c), the measured rates were assumed related to true riskless and market rates through the Sharpe model.

In large samples, it is possible to estimate multiple correlations between the measured riskless and market rates and their true but unobservable counterparts. This feature can be utilized systematically as a check on the model's validity. Large sample empirical estimates of these squared correlation coefficients should lie between zero and one.[23]

In a sample of U. S. Treasury bills, these estimated correlation coefficients were outside the range zero to one, implying that the current formulation of the model is invalid. As a partial explanation of the model's failure, the data suggested that (a) the model's parameters were not temporally constant, and (b) some non-portfolio motives for holding Treasury bills are not accounted for by the model.

Despite these negative results, this paper does not diminish the great contribution that the Sharpe model has made in clarifying the theory of capital asset pricing. It has only confirmed a widely-held opinion that relaxation of the model's assumptions is necessary for a better adherence to empirical facts. Generalizing the simple model to allow the production of financial and physical assets, the existence of more than two periods, and

[23]These correlation coefficients are estimated by formulae that are not based on direct sample product moments. They can lie outside the range 0-1 if the model is invalid.

differences in investor opinions is likely to provide a better understanding of the capital asset pricing process and a greater ability to explain observed market prices.

REFERENCES

1. Blume, Marshall, "The Assessment of Portfolio Performance, An Application to Portfolio Theory," (unpublished Ph.D. dissertation, Graduate School of Business, University of Chicago), March, 1968.

2. Fama, Eugene F., "Risk, Return, and Equilibrium: Some Clarifying Comments," *Journal of Finance*, March, 1968.

3. ____________, Lawrence Fisher, Michael C. Jensen, and Richard Roll, "The Adjustment of Stock Prices to New Information," *International Economic Review* (January, 1969).

4. Goldberger, Arthur S., *Econometric Theory* (New York: John Wiley and Sons, 1964).

5. Jensen, Michael C., "The Performance of Mutual Funds in the Period 1945-1964," *Journal of Finance*, May, 1968.

6. Mandelbrot, Benoit, "The Variation of Certain Speculative Prices," *Journal of Business*, October, 1963.

7. Sharpe, William F., "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance*, September, 1964.