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CAPITAL BUDGETING OF RISKY PROJECTS WITH "IMPERFECT" MARKETS FOR PHYSICAL CAPITAL

MARCUS C. BOGUE AND RICHARD ROLL*

I. THE CAPITAL BUDGETING PROBLEM

SELECTING THE BEST from among competing investment proposals is the most important financial decision problem faced by the managers of enterprises. It confronts every high-ranking decision-maker regardless of the nature of his organization's activity. Corporate executive, public official and university administrator alike must determine, with limited information, how much to invest in a variety of risky projects.

Managers have been making these decisions since an overfed Australopithicus Robinson Crusoe decided to save a bone for a weapon rather than crack it for its marrow, but there still exists no fully-accepted objective procedure for optimal investment choice, even on a theoretical level where all problems of measurement are neglected. We believe, however, that the direction of research is clear and that a rapid rate of convergence toward an accepted capital budgeting procedure will soon become apparent. Our paper is intended as a single step (in the right direction). We have not solved the problem completely but we hope to have (a) clarified the remaining unsolved theoretical issues and (b) pointed the way toward practical interim techniques which approximate the exact procedures that will be found someday.

The Objective Function of Project Selection

A "best" technique for rating investment projects is heavily dependent on the decision maker's objective. For the corporate executive, perhaps the most straightforward objective is to maximize the current market value of common stock (an equivalent objective for managers of non-business enterprises is to maximize aggregate financial benefits minus financial costs). Although these rules are simple, many other rules have been prescribed on normative grounds and are supposedly in use by actual decision makers. Just in the corporate area, there is an extensive literature on conflicts between the objectives of managers and the objectives of stockholders.[1] In addition, some recent writers have questioned whether the objective of maximizing the value of common shares is in full agreement with the objective of maximizing stockholder welfare.[2]

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1. Notable examples of this literature include Berle and Means [1932], Penrose [1959], and Cyert and March [1963]. The number of distinct manager objectives proposed by various authors is at least as large as the number of authors.

2. Whitmore [1970] and Jensen and Long [1972] contain the earliest statements of this important problem. (Also, see Stiglitz [1972] and Fama [1972].) The argument used to demonstrate a possible conflict between share value maximization and stockholder welfare maximization involves the propensity of stockholders to diversify risk by holding a portfolio of stocks. It is argued that a

In this paper, however, we intend to utilize the objective of maximizing the market value of common stock. Our choice is motivated by three considerations: First, although we hope the paper will be useful to most managers, we have no desire to aid managers who do not want to increase stockholder welfare. Second, we believe the capital budgeting problem is wholly unsolvable for a general and unspecified objective function. Third, the possible conflict between maximizing stockholder welfare and maximizing common stock value has been partially mitigated by the work of Jensen and Long [1972, pp. 168-172] and Merton and Subrahmanyam [1973] who have shown that the two objectives are in complete accord under "perfect competition" in capital markets.[3] We therefore conclude, based on empirical evidence indicating the slight extent of capital market imperfections, that the market value maximization objective is a good approximation to what must be considered, on legal and empirical grounds, the appropriate objective of corporate managers. For non-business enterprises, when the ownership of assets resides with coalitions of uncertain composition, the procedures given here are of doubtful validity. Therefore, the discussion hereinafter will be limited to the language of corporation problems.

Characteristics of the Corporate Investment Process

Three distinct activities compose the investment process. Perhaps the most important activity is dreaming up project ideas; But academic literature has little advice to offer in this area and our paper is, unfortunately, no exception. The second problem is forecasting the cash flows for each potential investment. Since this task involves deriving subjective prior distributions, the literature of Bayesian statistics has a lot to offer and a comprehensive study of the available techniques[4] should be well-rewarded by higher profit.

The third activity, which is traditionally reserved for "finance," is ascertaining the value of a project, given its probability distribution. Here a facility is required for comparing future uncertain revenues and costs with current capital expenditures. Usually, this has been provided by a set of "discount" rates or "hurdle" rates with the following property: If the discounted present value of all expected cash flows from the project is positive, undertaking the project will increase the market price of the firm's current shares; while if the discounted value of expected cash flows is negative, undertaking the project will decrease the share price. Theory must tell us how these discount rates are measured, how far into the future they should be obtained, and what stockholder preferences they reflect. In essence, it must tell us how to assess the

particular choice of a risky investment project by one firm, although it enhances *that* firm's value, can cause a detrimental effect by increasing aggregate risk through the project's interaction with returns of other securities in investors' portfolios.

3. The two papers cited have defined "perfect" competition somewhat differently. Jensen and Long show that rules for stockholder wealth and welfare maximization become asymptotically equivalent as the number of firms grows indefinitely large (except in the special circumstance where the returns among different projects taken on by different firms are uncorrelated). Merton and Subrahmanyham define perfect competition as perfect positive correlation among the returns of shares of the same physical project taken on by different firms. This definition would seem to be quite in accord with the classic economic concept of perfect competition.

4. See, e.g., Winkler [1967] and Staël Von Holstein [1970].

value of the uncertain future cash flows arising from the proposed project. If this value exceeds the current cost, undertaking the project will increase the share price.

A reliable method for determining the value of uncertain future cash flows can be obtained by noticing how a market full of investors prices securities *when investors hold common subjective probability beliefs about returns.* The role of common beliefs or homogeneous expectations is of particular importance because (a) without them, there is no unique price which *all* investors would pay for a risky security; (b) the firm would be uncertain as to which particular stockholder's beliefs, and which stockholder's implicit project valuation, should be used in reaching a decision, and (c) from a practical viewpoint, there *are* actually homogenous expectations about investment projects in most cases because a unique set of probability assessments, that of management, is used in the decision process.

Given the assumption of homogenous expectations, there exists several theories of capital market equilibrium under risk and each one is a potential value assessment rule. The Sharpe [1964]-Lintner [1965] model is probably the most familiar and has been subjected to more empirical testing than any other.[5] Furthermore, it is known to be a reasonable approximation to reality whenever probability distributions of returns are compact,[6] that is, when they display only slight chance of returns deviating far from zero. Because of the model's familiarity and because it is an approximation at worse, we will use it exclusively hereafter.

Interrelated with the problem of selecting the best capital asset pricing model to measure investor risk preferences is the problem of how many future periods to consider. We do not yet fully understand how the application of two-period models such as the Sharpe-Lintner model in a multi-period situation may bias the final investment decision. However, some recent theoretical and empirical evidence[7] has suggested that two-period asset pricing models may not lead to grossly different decisions than would multi-period models, which as yet have not been adequately tested.

II. Capital Budgeting for Single-Period Projects

Consider a firm trying to decide whether to undertake a risky, single-period project. Such a project requires a known cash outlay at the beginning of the period and yields a single uncertain cash return at the end of the period. Even though this is the most simple of all risky investment problems, only within

5. Prominent alternatives include the growth-optimum model, Latane [1959], and Hakansson [1971], the Arrow [1964]-Debreu [1959, pp. 98-102] model and variations on the theme of the Sharpe-Lintner model, e.g., by Merton [1972], Vasicek [1971], Brennan [1971], and Black [1971].

6. Cf. Samuelson, [1970].

7. Merton [1972], using an apparatus where re-evaluations of investors' portfolios take place continually, has derived an equilibrium condition very similar to a continuous-time version of the Sharpe-Lintner equation (with the addition of a second premium to compensate investors for the risk of shifts in the investment opportunity set). Also, a close similarity between Sharpe-Lintner equilibrium and equilibrium under the growth optimum model, which is a multi-period model, has been shown by Roll [1973]. Fama and McBeth [1973], however, provide empirical evidence that not all investors in the U.S. market can be following growth-maximizing strategies.

the last few years has a theoretically acceptable solution been developed.[8] This section reviews that solution as a basis for building toward solutions to multi-period problems.

The single period problem can claim more interest than just the basis for more complicated problems, however, because in many cases the investment decision for a multi-period project can be made with a one-period forecast. Cash flows need not be forecast for more than one period ahead if perfect secondary markets exist for the project; that is, if markets are available without brokerage costs and with free transportation.[9] This has been long recognized in the finance literature[10] and, indeed is based on a very simple argument: Consider a machine that can be traded during any period, now or in the future, without incurring transaction costs (where the operational definition of zero transaction costs is: "The firm would receive from selling the machine exactly the same proceeds as it would have to expend for the machine's purchase."). To ascertain whether the machine should be purchased now, the firm only needs to compare its current cost with the value of forecasted cash flows during the first period and with forecasted end-of-period secondary market price. The decision next period as to whether the machine should be used in subsequent periods is completely unaffected by whether it is owned at the end of the present period. If the machine is purchased now, the "initial capital expenditure" in the investment decision problem that must be done at the beginning of next period in order to determine whether the machine should be retained is the opportunity cost of not selling it then. If the machine is not owned at the beginning of next period (i.e., not purchased now), the initial capital expenditure for next period's problem is its market price then. Given the assumption of zero transaction costs, these two capital expenditures are equal and thus every investment decision concerning capital equipment which enjoys a "perfect" market can be accomplished with one-period forecasts.

Even with imperfect secondary markets for physical capital, many investment decisions can be made with one period forecasts. Again considering the machine, the worst possible outcome for the firm is to receive its one-period cash flow and its net salvage value after the first period. But if the machine is acceptable on such a basis, it must be purchased; for the firm can only gain by the opportunity to retain it for additional periods.

While this provides sufficient conditions of acceptance for many projects, those conditions are not necessary. Many worthwhile projects may be rejected because their salvage value at the end of the period poorly reflects their future earnings power either because they are costly to remove and transport or because

8. The two-period project selection criteria in the context of the Sharpe-Lintner model was evidently given first by Tuttle and Litzenberger [1968] and by Mossin [1969]. Since then, the criteria has appeared in several papers and the literature is growing (see, e.g., Hamada [1971], Rubinstein [1973], Weston [1973]), and is becoming specialized to certain contexts; for example, to regulatory proceedings; (see Myers [1972,b] and Breen and Lerner [1972] as improved by Myers [1972,a]).

9. With this definition, "perfect capital market" requires much more than just "perfect competition" and the two phrases should not be regarded as equivalent. See Stigler [1968, pp. 113-121] for a complete discussion.

10. See Fama and Miller [1972, pp. 122-125] for a recent discussion of this point.

it is difficult or takes too long to find a buyer. In these cases, further periods must be considered in arriving at a decision.

For the several reasons just given, we believe that single-period solution techniques for evaluating risky investment projects constitute a very important part of the set of all techniques. Fortunately, logically complete single-period techniques are now available. They are all based on theories of equilibrium in the capital markets; that is, on theories which describe how stock market prices are related to investor attitudes toward risk and to investor abilities to diversify. Each theory implies a valuation equation, which relates a stock price or a project *value*, to associated expected cash flows and to risk.

A Single Period Valuation Model

The particular valuation theory to be employed here was first developed by Sharpe [1964] and Lintner [1965]. As mentioned before, there are several competing theories and empirical testing to determine the "best" theory has not yet been completed. This really doesn't matter for our purposes, however, because the procedures presented below can be modified to accommodate whichever valuation theory is ultimately judged most correct. Specific decision rules would be affected by using another model but the basic ideas would remain unaltered.

Sharpe-Lintner theory implies that the equilibrium value of a firm in a single-period world of risk averse, expected utility maximizers is given by[11]

$$V^{(0)} = \frac{E[\tilde{V}^{(1)}] - \lambda^{(0)} \operatorname{cov}(\tilde{V}^{(1)}, V_m^{(1)})^{12}}{1 + r_f^{(0)}} \tag{2.1}$$

provided that the distribution of $\tilde{V}$ and $\tilde{V}_m$ is multivariate Gaussian with parameters known to everyone: The symbols are defined as follows:

$\tilde{V}^{(1)}$ = the uncertain end-of-period value of the firm (including any dividends paid over the period)

$\tilde{V}_m^{(1)}$ = the uncertain end-of-period value of all firms

$r_f^{(0)}$ = the risk-free rate of interest over the period

$$\lambda^{(0)} = \frac{E[\tilde{V}_m^{(1)}] - (1 + r_f^{(0)})\tilde{V}_m^{(0)}}{\sigma^2(V_m^{(1)})} = \text{the market price per unit of risk.}$$

Evaluation of a Single-Period Project

Consider a proposed single period project that will bring an incremental end-of-period net cash inflow of $\tilde{C}^{(1)}$ and will require a current net cash outlay

11. Herein, superscripts in parentheses are used exclusively to denote time, $\sim$ denotes random variable, E is mathematical expectation, "cov" is covariance, σ^2 is variance.

12. A more familiar expression in terms of returns is

$$\tilde{r}^{(0)} - r_f^{(0)} = \beta(E[\tilde{r}_m^{(0)}] - r_f^{(0)})$$

where $\tilde{r}_m^{(0)}$ = the return on the market portfolio over the period.

$r_f^{(0)}$ = the return on a non-risky asset

$\tilde{r}^{(0)}$ = the return on the risky asset (or project)

$$\beta = \frac{\operatorname{cov}(\tilde{r}^{(0)}, \tilde{r}_m^{(0)})}{\sigma^2(\tilde{r}_m^{(0)})} = \text{the systematic risk of the firm or asset.}$$

The two expressions are equivalent, but we used the form in 2.1 because it is much easier to work with in the multi-period case as will be seen in the next section.

of $C^{(0)}$. Including the project, the end-of-period value of the firm will be $\tilde{V}^{(1)} + \tilde{C}^{(1)}$. Using the valuation model

$$V^{(0)} + \Delta V^{(0)} = \frac{E[\tilde{V}^{(1)} + \tilde{C}^{(1)}] - \lambda^{(0)} \operatorname{cov}(\tilde{V}^{(1)} + \tilde{C}^{(1)}, V_m^{(1)})}{1 + r_f^{(0)}} \tag{2.2}$$

So

$$\Delta V^{(0)} = \frac{E[\tilde{C}^{(1)}] - \lambda^{(0)} \operatorname{cov}(\tilde{C}^{(1)}, \tilde{V}_m^{(1)})}{1 + r_f^{(0)}}, \tag{2.3}$$

which gives a value to the additional investment which must be compared with its cost.[13] If $\Delta V^{(0)} > C^{(0)}$ the project is clearly acceptable and should be undertaken, whereas if $\Delta V^{(0)} < C^{(0)}$ the project is unacceptable and should not be undertaken.

Thus, the one-period case has a very simple solution, ignoring measurement problems: Just forecast the expected cash flow from the project and its dependence on the market; then, using the market price of risk and risk-free rate, convert these forecasts to current values which can be compared with current costs.

It is instructive to examine this project valuation in some detail because it forms the basis of generalization to the multi-period case. For an uncertain cash flow $\tilde{C}^{(1)}$, one should construct a certainty equivalent value, $E[\tilde{C}^{(1)}] - \lambda^{(1)} \operatorname{cov}(\tilde{C}^{(1)}, V_m^{(1)})$ from his forecasts. This end-of-period certainty equivalent value should be discounted at the risk-free rate to determine its current value.

It is important to realize that the certainty equivalent is not a function of the preferences of some manager or group of managers, but is the market's assessment. Neither the managers' utility function nor the utility function of any individual shareholder is considered. By deferring to a general equilibrium valuation model, the individual's utility is removed from consideration and replaced by his current wealth.

III. Essential Features of the Multi-Period Problem

Assuming perfect secondary markets, we have observed that the multi-period problem collapses to a one-period problem. For most projects, however, we are not likely to find perfect secondary markets and we must turn to a multi-period evaluation of the project.[14]

13. In writing 2.2 and 2.3, we have assumed that the impact of the additional project on the market is negligible. This assumption leaves

$$r_f^{(0)}, \lambda^{(0)} \text{ and } V_m^{(1)}$$

unchanged with the addition of the project. This is a very important assumption in obtaining simple results and it is to be regarded as in the spirit of an assumption of perfect competition.

14. We have found several other treatments of the multi-period capital budgeting problem using a capital asset pricing model. The best one is provided by Brennan [1973]. In some respects, Brennan's model is more sophisticated than ours because he uses a framework very similar to Merton's [1972] more general continuous-time capital asset pricing model. To obtain useful solutions, however, Brennan adopted a rather restrictive assumption about the form of expectations of future cash flows. In that respect, our model is more general because we do not make any assumption about the subjective probability distribution of cash flows except that their moments exist.

Another article is by Stapleton [1971]. He imposed a mean-variance criteria on the stochastic

Suppose the firm is faced with a project lasting over n periods. Let us designate the net uncertain cash flows due to this project by $\tilde{C}^{(t)}$ and the increments to the value of the firm over the project's life by $\Delta\tilde{V}^{(t)}$ $(t = o, 1, \ldots n)$. If we can determine $\Delta V^{(0)}$ the problem is solved. For if $\Delta V^{(0)}$ is greater than the initial cash outlay, the project should be accepted, otherwise not.

Consider the last period of the project. For that period trivially,

$$\Delta\tilde{V}^{(n)} = \tilde{C}^{(n)}.$$

In the next to the last period we have a one period valuation problem. But if we assume that the market for corporate equities will always be in equilibrium, we can apply our one-period valuation model to find the value of the final cash flow at the end of the next to the last period. This will result in the following discounted certainty equivalent of $C^{(n)}$ for period $n - 1$:

$$\frac{E[\tilde{C}^{(n)}|\epsilon^{(n-1)}] - \tilde{\lambda}^{(n-1)} \operatorname{cov}(\tilde{C}^{(n)}, \tilde{V}_m{}^{(n)}|\epsilon^{(n)})}{1 + r_f{}^{(n-1)}}, \tag{3.1}$$

which contains nothing but the stochastic elements,

$$\begin{aligned} \tilde{\epsilon}^{(n-1)} &= \text{state of the world at time } (n-1) \\ \tilde{\lambda}^{(n-1)} &= \text{market price of risk at } (n-1) \\ \tilde{r}_f{}^{(n-1)} &= \text{risk-free rate at } (n-1) \\ \tilde{E}[\tilde{C}^{(n)}|\tilde{\epsilon}^{(n-1)}] &= \text{conditional expectation at } n-1.^{15} \end{aligned}$$

This provides an expression for the incremental value at $n - 1$.

$$\Delta V^{(n-1)} = C^{(n-1)} + \frac{\tilde{E}[\tilde{C}^{(n)}|\tilde{\epsilon}^{(n-1)}] - \tilde{\lambda}^{(n-1)} \widetilde{\operatorname{cov}}(\tilde{C}^{(n)}, \tilde{V}_m{}^{(n)}|\tilde{\epsilon}^{(n)})}{1 + \tilde{r}_f{}^{(n-1)}}. \tag{3.2}$$

It should be clear that we can again apply our single period valuation theory to determine $\Delta V^{(n-2)}$. In general, we will have

$$\Delta\tilde{V}^{(k)} = \tilde{C}^{(k)} + \frac{\tilde{E}[\Delta\tilde{V}^{(k+1)}|\tilde{\epsilon}^{(k)}] - \tilde{\lambda}^{(k)} \widetilde{\operatorname{cov}}(\Delta\tilde{V}^{(k+1)}, \tilde{V}_m{}^{(k+1)}|\tilde{\epsilon}^{(k)})}{1 + \tilde{r}_f{}^{(k)}}. \tag{3.3}$$

For an n-period project, therefore, we have an n-period infinite state dynamic programming problem to solve. Each step in the solution is a single application of the one-period valuation model, with the parameters depending on the state of the world at the beginning of that period.

It is interesting to contrast this solution with that obtained when a perfect secondary market exists for the project. At each point in time, the perfect secondary market value fully reflects the future earning power of the project. That value, of course, is equal to $\Delta\tilde{V}^{(k)}$. So that the perfect secondary market does nothing but provide us with the solution at each stage in time to the

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present value of a dividend stream discounted at a fixed interest rate. This approach ignores uncertainty in the rate of interest and suffers from several other problems which were carefully analyzed by Keeley and Westerfield [1972]. Stapleton also assumed that new project returns were perfectly correlated with returns from the firm's existing assets. Bierman and Hass [1973] also ignore interest rate uncertainty and the Keeley-Westerfield problems.

15. N.B. Expression (3.1) is not observable in the first period. It is the conditional certainty equivalent of $c^{(n)}$ which will obtain in period $n - 1$ *if* ε^{n-1} is realized; of course, in the first period, ε^{n-1} is itself a random variable.

dynamic programming problem *for the paths actually taken.* The market solves one path forward, while we must solve all paths back to get $\Delta\widetilde{V}^{(k)}$.

IV. Capital Budgeting for Two-Period and Three-Period Risky Projects

Although the general structure of multi-period project evaluation is clear, the solution to the problem in terms of currently known values and estimates is far from clear. We present here the solution for a simple special case: a project with a single cash inflow two periods in the future.

Two-period Problem

From the preceding section

$$\Delta\widetilde{V}^{(1)}(1+\widetilde{r}_f^{(1)}) = \widetilde{E}[\widetilde{C}^{(2)}|\widetilde{\epsilon}^{(1)}] - \widetilde{\lambda}^{(1)}\,\mathrm{cov}(\widetilde{C}^{(2)},\widetilde{V}_m^{(2)}|\widetilde{\epsilon}^{(1)}) \tag{4.1}$$

$$\Delta V^{(0)}(1+r_f^{(0)}) = E[\Delta\widetilde{V}^{(1)}] - \lambda^{(0)}\,\mathrm{cov}(\Delta\widetilde{V}^{(1)},\widetilde{V}_m^{(1)}) \tag{4.2}$$

Taking expectations of 4.1 and re-arranging, we obtain

$$E[\widetilde{\Delta V}^{(1)}]E[1+\widetilde{r}_f^{(1)}] + \mathrm{cov}(\widetilde{\Delta V}^{(1)},\widetilde{r}_f^{(1)}) = E[\widetilde{C}^{(2)}] - E[\widetilde{\lambda}^{(1)}\,\widetilde{\mathrm{cov}}(\widetilde{C}^{(2)},\widetilde{V}_m^{(2)}|\widetilde{\epsilon}^{(1)})]$$

Substituting for $E[\Delta\widetilde{V}^{(1)}]$ from (4.2) and rearranging

$$\begin{aligned}\Delta V^{(0)}(1+r_f^{(0)})E[1+\widetilde{r}_f^{(1)}] = {} & E[\widetilde{C}^{(2)}] - E[\widetilde{\lambda}^{(1)}\,\widetilde{\mathrm{cov}}(\widetilde{C}^{(2)},\widetilde{V}_m^{(2)}|\widetilde{\epsilon}^{(1)})] \\ & - \lambda^{(0)}\,\mathrm{cov}(\Delta\widetilde{V}^{(1)},\widetilde{V}_m^{(1)})E(1+\widetilde{r}_f^{(1)}) - \mathrm{cov}(\widetilde{\Delta V}^{(1)},\widetilde{r}_f^{(1)})\end{aligned} \tag{4.3}$$

This result says that the value of the single uncertain cash flow two periods in the future is given by the current expectation of that flow less three risk premiums discounted at the product of one plus the current risk-free rate and one plus the second period's *expected* risk-free rate. The first premium, $E[\widetilde{\lambda}^{(1)}\widetilde{\mathrm{cov}}(\widetilde{C}^{(2)},\ \widetilde{V}_m^{(2)}\widetilde{\epsilon}^{(1)})]$ accounts for covariation risk within the second period. (This is analogous to the β-type risk premium in the single period model). The second term, $\lambda^{(0)}\mathrm{cov}(\widetilde{\Delta V}^{(1)},\ \widetilde{V}_m^{(1)})\ E[1+\widetilde{r}_f^{(1)}]$, accounts for the covariation risk of the intermediate value of the project. This term, which plays the roll of a reinvestment opportunity cost, accounts for the possibility that title to the project could be transferred before the final cash flow is realized. The last term, $\mathrm{cov}(\widetilde{\Delta V}^{(1)},\ \widetilde{r}_f^{(1)})$, assesses the risk of interest rate fluctuation. The basic notion of the valuation model is reduction of risk through diversification. Yet the model also introduces a term for interest rate risk, which measures the possibility that interest rate changes could bring about changes in the project's value at intermediate periods even if the probability assessments of the project's cash flows remain constant.[16]

16. For comparison, we also present the solution for a project with a single cash inflow *three* periods in the future (denoted $C^{(3)}$). The initial value of this project is given by $\Delta V^{(0)}$ in the equation below.

$$\begin{aligned}\Delta V^{(0)}(1+r_f^{(0)})E[1+\widetilde{r}_f^{(1)}]E[1+\widetilde{r}_f^{(2)}] = {} & E[\widetilde{C}^{(3)}] \\ & - \{E[\widetilde{\lambda}^{(2)}\widetilde{\mathrm{cov}}(C^{(3)},V_m^{(3)}|\widetilde{\epsilon}^{(2)})] + \mathrm{cov}(\widetilde{\Delta V}^{(2)},\widetilde{r}_f^{(2)})\} \\ & - \{E[\widetilde{\lambda}^{(1)}\widetilde{\mathrm{cov}}(\Delta V^{(2)},V_m^{(2)}|\widetilde{\epsilon}^{(1)}] + \mathrm{cov}(\widetilde{\Delta V}^{(1)},\widetilde{r}_f^{(1)})\}E[1+\widetilde{r}_f^{(2)}]\} \\ & - \{\lambda^{(0)}\mathrm{cov}(\widetilde{\Delta V},^{(1)}\widetilde{V}_m^{(1)})\}E[1+\widetilde{r}_f^{(1)}]E[1+\widetilde{r}_f^{(2)}]\end{aligned}$$

Since there are now two possible reinvestment dates, there is an extra set of risk premia. Their interpretation is perfectly like the interpretation of premia for the two-period project in the text

V. Some Implications for the Term Structure of Interest Rates

We have been speaking of equilibria in the capital markets, as described by a system of equations, one for every pair of adjacent future periods. This model actually combines two divergent areas of the field of finance—the first area is portfolio theory, or the theory of diversification of risky investments *within* a given period. The second area is the term structure of interest rates, which in its purest sense is the theory of the relation between long- and short-maturity bond prices when the only source of randomness is intertemporal variation in the short-term rate of interest. To put it another way, term structure analysis assumes that a particular riskless security is available over every future time interval; namely, the pure discount[17] bond of that maturity. One of these bonds, and only one, exists for each maturity. As each period arrives, the shortest-term bond pays a *fixed* cash amount and then passes out of existence. This framework is to be contrasted with that of portfolio theory, in which securities pay random and perhaps correlated cash amounts in each period. One must notice, however, that the two ideas are not completely divergent because the longer-term bonds in term structure theory can be traded each period at uncertain prices. Even if the only available securities were pure discount bonds, an investor might find it advantageous to diversify by holding a portfolio of various maturities.

Our intention is to show that the model enunciated earlier in the paper captures the essential features of the pure term structure problem. It is sufficient to use the two-period case, which was worked out in the preceding section, as an example.

Imagine that the "project" is a discount bond. It pays a fixed dollar amount after two periods and nothing before. Using the notation of the valuation equation for a two-period project, eq. (4.3), the final payment, $C^{(2)}$, is nonstochastic even in period zero.

Thus, we have

$$E[\tilde{C}^{(2)}] = C^{(2)}$$

and

$$\mathrm{Cov}(\tilde{C}^{(2)}, \tilde{V}_m^{(2)}) = 0.$$

Without loss of generality, let us assume that the final payment is $\$1 = C^{(2)}$. Then (4.3) becomes

$$\Delta V^{(0)}(1 + r_f^{(0)})E(1 + \tilde{r}_f^{(1)}) = 1 - \lambda^{(0)}\,\mathrm{Cov}(\Delta\tilde{V}^{(1)}, \tilde{V}_m^{(1)})E(1 + \tilde{r}_f^{(1)}) - \mathrm{Cov}(\Delta\tilde{V}^{(1)}, \tilde{r}_f^{(1)}) \quad (5.1)$$

Since $\Delta V^{(t)}$ is the price of the discount bond in period t and since the reciprocal relation between price and yield-to-maturity is

$$\Delta V^{(t)} = \frac{1}{(1 + R_t)^t}$$

but there is one extra premium for reinvestment opportunity cost and one extra premium for interest rate fluctuation risk.

17. A pure discount bond is a bond that pays a fixed amount after n period and nothing before or afterward. Coupon bonds, which promise a stream of cash payments at several dates, should be considered hybrid securities—combinations of several pure discount bonds.

where R_t is the yield-to-maturity for a bond with t periods to maturity, we note that

$$\Delta V^{(1)} = \frac{1}{1 + R_1} = \frac{1}{1 + r_f^{(1)}}.$$

Thus,

$$\mathrm{Cov}\left[\frac{1}{1 + \tilde{r}_f^{(1)}}, \tilde{r}_f^{(1)}\right] = 1 - E(1 + \tilde{r}_f^{(1)})E(\tilde{\Delta V}^{(1)})$$

and (5.1) simplifies to

$$\frac{1 + r_f^{(0)}}{(1 + R_2)^2} = E(\tilde{\Delta V}^{(1)}) - \lambda^{(0)}\,\mathrm{Cov}(\tilde{\Delta V}^{(1)}, \tilde{V}_m^{(1)}) \qquad (5.2)$$

The quantity on the left side of (5.2) is the reciprocal of unity plus the "forward rate" of interest to begin after one period while the first term on the right is the expected reciprocal of one plus the "spot" rate after one period. Thus, (5.2) is a solution to the central question of term structure analysis: "What is the relation between forward rates and expected future spot rates?"

Denoting the forward rate for t-period hence as $F_t^{(0)}$, (5.2) can be rewritten as

$$\frac{1}{1 + F_1^{(0)}} = E\left[\frac{1}{1 + \tilde{R}_1^{(1)}}\right] - \lambda^{(0)}\,\mathrm{Cov}\left[\tilde{V}_m^{(1)}, \frac{1}{1 + \tilde{R}_1^{(1)}}\right] \qquad (5.3)$$

which can be considered the portfolio theory solution to the term-structure problem. The one-period-hence forward rate must be related to the expected reciprocal of the corresponding one-period spot rate less a risk premium which measures the contribution to portfolio risk made by a two-period discount bond. The greater the covariance between the price of a two-period bond and the market portfolio, the larger the forward rate (and the lower the *initial* price of the bond).[18]

It may be of interest to note that one-period-hence forward rates will be downward-biased estimates of future spot rates even if there is no portfolio risk associated with holding bonds, i.e., if the covariance term in (5.3) is zero. This follows directly from equation (5.3) by noting that

$$E\left(\frac{1}{1 + R_1^{(1)}}\right) > \frac{1}{1 + E(R_1^{(1)})} \quad \text{for} \quad R_1^{(1)} > -1.$$

Unfortunately, forward rates of longer-term than one period bear more complex relations to their corresponding expected future spot rates. The reason for this is the possibility that a forward contract made for two, three, four or more periods hence can be traded or bought back at any intervening period. Thus, the risk associated with these potential intervening trades must be taken into consideration in advance.[19]

18. This same result has been obtained in an earlier paper (Roll [1971]). That paper emphasized an assumption which has not been mentioned yet here even though it has been made implicitly, viz., the investors' horizon is the shortest period of observation.

19. As an example, consider the case of the forward rate for a loan to begin two periods

VII. Summary and Remark on Practical Implementation

The single most difficult problem facing the manager who wishes to apply our model in a decision-making capacity is the assessment of the necessary probability distributions of cash flows. The crucial question here is, "How much should the manager invest in sharpening his assessments?" He might simply sit back at his desk and spin out assessments rapidly at a low cost; or he might allocate a sizeable fraction of staff resources to a given project in order to obtain more accurate estimates of its cash flow expectations and covariances. In any case, the manager needs to know what benefits more accurate assessments are likely to bring. Here, once again, the theory of finance has something to offer.

If errors in probability assessments are not systematically biased, they are not likely to be of overwhelming importance to stockholders; for stockholders can diversify away many of these errors. For example, consider a large number of one-period projects being analyzed by a large number of different firms whose shares are held by investors in well-diversified portfolios. We may suppose that an objective but unobservable probability distribution of cash flow is given by nature to each project; and we can presume that each manager's assessment differs by some additive amount from the truth. For example, managers j's expectation for project i might be

$$E_j(\tilde{\Delta}V_i^{(1)}) = E(\tilde{\Delta}V_i^{(1)}) + \epsilon_{ij}$$

where ϵ_{ij} is the "error" in subjective expectation of the future cash flow, and $E(\Delta V_i^{(1)})$ is the *true* first moment of nature's distribution of the cash flow.

If there are n of these projects in all, their aggregate true value to the stockholder is, according to decision rule (2.1), equal to

hence. Using the formulae at the end of Section IV (and in note 16) it can be shown that a condition analogous equation 5.3 must hold between the *expected* forward rate at the beginning of the second period and the expected spot rate two periods hence,

$$E\left[\frac{1}{1+\tilde{F}_1^{(1)}}\right] = E\left[\frac{1}{1+\tilde{R}_1^{(2)}}\right] - E\left\{\tilde{\lambda}^{(1)}\operatorname{cov}\left[\tilde{V}_m^{(2)}, \frac{1}{1+\tilde{R}_1^{(2)}}\right]\right\}.$$

In addition, there must be an expression between the forward rate in period one and the expected forward rate in period two; specifically

$$\frac{1}{1+F_2^{(0)}} = \frac{E(\Delta\tilde{V}^{(1)}) - \lambda^{(0)}\operatorname{Cov}(\tilde{\Delta}V^{(1)},\tilde{V}_m^{(1)})}{E\left[\frac{1}{1+\tilde{R}_1^{(1)}}\right] - \lambda^{(0)}\operatorname{Cov}\left(\frac{1}{1+\tilde{R}_1^{(1)}},\tilde{V}_m^{(1)}\right)}.$$

The term on the right side of this equation is related to a risk adjusted expected forward rate reciprocal. To see this, note that $\Delta V^{(1)}/[\frac{1}{1+R_1^{(1)}}]$would be exactly equal to $\frac{1}{1+F_1^{(1)}}$ in the absence of stochastic elements. Recall that $F_1^{(1)}$ is the realization after one period of $_{F2}(0)$; i.e., the forward rate sequence is $\{F_2^{(0)},F_1^{(1)},R_1^{(2)}\}$.

Under the pure expectations hypothesis of the term structure, forward rate sequences must be pure martingales. (See e.g., Roll [1970, pp. 35-36], Sargent [1972], and the comment by Shiller [1973]). In the current framework, it is easy to observe that this can occur if portfolio risk coefficients for bonds are zero. If the risk coefficients are non-zero, and even if they are stationery, forward rates need not follow martingale sequences. As each trading date goes by, the necessity to hedge against interest rate uncertainty on that date vanishes and the forward rate may change in a predictable way. This is true even if all expectations happen to be realized exactly.

$$\sum_{i=1}^{n} \Delta V^{(0)} = \frac{1}{1+r_f^{(1)}} \sum_{i=1}^{n} [E(\Delta V_i^{(1)}) - \text{Cov}(\Delta V_i^{(1)}, V_m^{(1)})]$$

and their aggregated *subjective* value is

$$\sum_{i=1}^{n} \Delta V^{(0)} + \sum_{i=1}^{n} \epsilon_{ii}{}^{20}$$

Since the investor normally would own only a small fraction of each firm, he would acquire only a small fraction of each new project; and if managers were unbiased and unrelated to each other in their assessments, the mean square error in project value assessment would decline toward zero as the number of projects became indefinitely large.[21]

In summary, the framework here is available to help a manager make multi-period investment choices. To the extent that the Sharpe-Lintner model is valid, the equations given in Section IV *must* be used if stockholder wealth is to be maximized. Furthermore, the use of a different technique will cause more serious errors than mis-assessments of the probability distributions of cash flows. This is because unbiased mis-assessments can be diversified away in the personal portfolios of stockholders while the use of an improper capital budgeting technique (such as risk-adjusted discount rates chosen without the benefit of portfolio consideration) will result in *aggregate* errors which stockholders will not be able to eliminate.

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20. For simplicity, we have assumed that no errors exists in the covariance assessments. One can observe that additive errors in the one-period covariance term (or additive errors in any of the terms of a multi-period project), would have exactly the same form as the error terms of the text's example.

21. To put it more precisely, suppose that an investor owned 1/n of each firm, where n is the same size as the total number of projects being considered. The total error in the investors portfolio would then be

$$\frac{1}{n} \sum_{1} e_{ii}$$

which has a distribution that becomes a spike at zero as n increase if managers make unbiased and independent judgments.

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