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COMMENTS ON QUALITATIVE RESULTS FOR INVESTMENT PROPORTIONS

Richard ROLL*

University of California, Los Angeles, CA 90024, U.S.A.

Stephen A. ROSS*

Yale University, New Haven, CT 06520, U.S.A.

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Some mean/variance efficient portfolios will have positive investments in all individual assets when covariances satisfy the conditions given here. These conditions are more useful empirically than qualitative results that depend on the inverse covariance matrix. The prospect appears dim for general and useful qualitative results.

The theorem corrected by Rudd (this issue) provides an occasion to comment on the significance and potential scope of qualitative results for investment proportions.[1] Qualitative results would be helpful in testing asset pricing models; but they seem difficult to obtain in a usable form. This is well-illustrated by the corrected version of Roll's Theorem 3, Rudd's Theorem 3*.

The theorem now states that a totally positive vector of investment proportions for the global minimum variance portfolio implies a dominant diagonal covariance matrix in the weak sense (every individual variance greater than all its associated covariances), if two conditions obtain: (a) the covariance matrix is totally positive, and (b) its inverse has a positive diagonal and non-positive off-diagonal. The second requirement renders the theorem virtually worthless as a source of tests. Useful qualitative results would not depend on direct knowledge of the inverse covariance matrix. If the inverse can be computed, qualitative results are unnecessary because empirical tests can be devised from the quantitative elements of the inverse.

The covariance matrix required for testing asset pricing theory is the full matrix of *individual* assets, not a smaller matrix obtained from a subset of assets

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[1]The incorrect theorem was not used in the text of Roll (1977) so none of that paper's conclusions were affected.

nor from portfolios of assets. But inverting a matrix of the required size exceeds current technological limitations on accuracy and cost. Thus, inferring something about investment proportions directly from the structure of individual covariances might prove extremely useful. Furthermore, statistical tests of significance based on the sample covariance matrix would be simpler than tests based on the sample inverse.

Among the most useful results would be sign patterns for investment proportions of mean-variance efficient portfolios. These vectors depend only on the inverse covariance matrix, V^{-1}, and the mean return vector, R, of individual assets.[2] What conditions imposed on V and R would guarantee that at least some efficient vectors are totally positive? More importantly, what conditions would guarantee that *all* efficient portfolios have at least one non-positive investment? An inference about the true (but unknown) market portfolio, which is totally positive, would be possible if these questions could be answered.

Answers are not immediately obvious, so it seems best to begin with simple cases. Perhaps the simplest is the global minimum variance portfolio, for its investment weights are proportional to $V^{-1}\iota$, where ι is the unit vector.[3] This is the only efficient portfolio whose weights do not depend also on R. The following is a result which satisfies the criterion of usefulness:

Theorem. *Given a square matrix, V, of order n, if there exists a positive vector a, with $a'\iota = 1$, such that for all j,*

$$v_{jj} > \sum_k a_k v_{kj},$$

and for all $i \neq j$,

$$v_{ij} \leq \sum_k a_k v_{kj} \quad (<),$$

then there exists a semipositive (positive) vector x and a scalar $\lambda \geq 0$ such that

$$Vx = \lambda\iota.$$

Proof. Defining A as a diagonal matrix with the vector a on the diagonal, let

$$T = A(I-\iota a')V.$$

[2]All efficient vectors x satisfy

$$x = V^{-1}(R\iota)G^{-1}\begin{pmatrix} r_p \\ 1 \end{pmatrix}$$

where ι is the unit vector, r_p is the return on the efficient portfolio whose investment proportions are in x, and $G \equiv (R\iota)'V^{-1}(R\iota)$. See Roll (1977, p. 160).

[3]I.e., the jth asset's weight in the global minimum variance portfolio is proportional to the sum of the jth row of the inverse covariance matrix.

It is easily verified that $Vx = \lambda\iota$ for some $x \geq 0$ ($>$) if and only if $Tx = 0$. Furthermore, $t_{ii} > 0$, $t_{ij} \leq 0\,(<)$ for $i \neq j$, and $\iota'T = 0$.

By the theorem of separating hyperplanes [see Gale (1960, pp. 48–49)], $Tx = 0$ has a solution with $x \geq 0$ ($>$), if and only if there does not exist y such that $y'T > 0$ ($\geq$). Suppose to the contrary that such a y exists and without loss of generality let $y_1 = \min_i y_i$. Since $t_{11}y_1 + \ldots t_{n1}y_n > 0$, ($\geq$) and $t_{11} > 0$, we must have $y_1 > [(-t_{21}/t_{11})y_2 + \ldots + (-t_{n1}/t_{11})y_n]$ ($\geq$). But, $(-t_{i1}/t_{11}) \geq 0$ ($>$) for $i > 1$ and from $\iota'T = 0$ we have that $(-t_{21}/t_{11}) + \ldots + (-t_{n1}/t_{11}) = 1$. Hence y_1 must exceed a (strict) convex combination of $y_2, \ldots, y_n$ and cannot be the minimum element of y. This is a contradiction and implies that no y satisfies $y'T > 0$. Hence, there exists an x satisfying $Tx = 0$, or equivalently, $Vx = \lambda\iota$. Q.E.D.

This theorem can be used to obtain a number of useful conditions which guarantee that every asset is represented non-negatively (positively) in the minimum variance portfolio. For example, setting $a_i = 1/n$ we obtain the condition that every diagonal element must exceed the arithmetic mean of all elements in its column and the off-diagonals of the column must not exceed (must be strictly less than) the same mean. (This restricted version of the theorem is minimal within the set of conditions that treat off-diagonal elements identically and apply equally to all columns). Similarly, if all assets are uncorrelated or negatively correlated then every asset is represented non-negatively in the minimum variance portfolio.

The conditions of the theorem, however, are not necessary for the existence of a non-negative (positive) minimum variance portfolio. One implication of the conditions is that (after permutation) every diagonal element must exceed each off-diagonal in its column. The following covariance matrix

$$V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 10 & 11 \\ -1 & 11 & 30 \end{bmatrix}, \qquad (1)$$

has a column with an off-diagonal element that exceeds the diagonal element, but for $x' = (178, 7, 9)$; $Vx = (169)\iota$.

Further counter-examples may save someone a bit of trouble; a natural conjecture would have been whether total positivity was guaranteed when the theorem's conditions are weakened to require that each variance be greater than all of its associated covariances. This conjecture is false and a counter-example is provided by the matrix

$$V = \begin{bmatrix} 10 & 9 & 2 \\ 9 & 200 & 20 \\ 2 & 20 & 40 \end{bmatrix}, \qquad (2)$$

whose associated minimum variance portfolio has a negative investment in the second asset.

Unfortunately, then, this theorem is merely a beginning to the qualitative proportions problem.[4] There may be many positive efficient portfolio vectors even when the global minimum variance portfolio contains negative investments in some assets; and there may be situations where no totally positive efficient vector exists. The role of the mean return vector, R, in these problems can be illustrated by example. Using the 3×3 matrix V given by (2) above, it is straightforward to show that $R' = (0\ 1\ 2)$ implies a range of strictly positive efficient portfolios lying between 1.5 and 1.995 percent. (The global minimum variance portfolio would have a return of 0.346 percent.) However, changing the mean return vector to $R' = (0\ 1\ 4)$, while using the same covariance matrix, creates an efficient frontier on which *every* portfolio has a negative investment in at least one asset. Thus, interactions of the elements of V and R pose difficulties and the scope of qualitative results for investment proportions still appears to be rather narrow.

[4]The possibility of a complete solution seems remote. To convince oneself the reader can examine the 3×3 case by projecting the columns onto the simplicial plane. Any three points on the plane whose interior triangle contains the center will yield a positive minimum variance portfolio. Consequently any points drawn from alternating regions defined by any three lines through the center will yield a positive minimum variance portfolio.

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