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Richard Roll

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AMBIGUITY WHEN PERFORMANCE IS MEASURED BY THE SECURITIES MARKET LINE

RICHARD ROLL*

I. INTRODUCTION

IMAGINE AN IDEALIZED ANALOG to the activities of professional money managers, a contest whose rules are as follows:

(a) Each contestant selects a portfolio from a specified set of individual assets.
(b) Returns are observed on the assets.
(c) After each period of return observation, the portfolios are re-balanced to the initial selections.
(d) After an interval consisting of several periods, "winners" and "losers" are declared for that interval.
(e) Contestants choose a new portfolio, or keep the old one, and the process (b) through (d) is repeated.
(f) After several intervals, consistent winners are declared to be superior portfolio managers and consistent losers are declared inferior. In the absence of any consistency, everyone is declared non-superior.

The sponsors of the contest face only a single problem of intellectual interest. They must develop criteria to partition contestants at step (d) into winners and losers. Of course, the criteria must be acceptable to participants and to disinterested observers. There should be a correspondence between "consistency in winning" and an intuitive notion of ability in portfolio selection.

We might think of many desirable qualities to be possessed by such criteria. For example, they should be robust to stochastic changes in the return sequence; true ability should be detectable over many intervals regardless of the sequence. If the criteria are employed by different sponsors, the same judgements about ability should be obtained. It should not be possible to reverse judgements by making changes in the computation of the criteria, if such changes are deemed insignificant by all observers. In other words, the criteria must provide decisions about ability that are *unambiguous* to rational judges.

A criterion that is widely employed in the financial community for assessing portfolio performance is the "securities market line," the (linear) relation between

*Graduate School of Management U.C.L.A.. Comments and suggestions by Alan Kraus, David Mayers, Stephen Ross and Eduardo Schwartz are gratefully acknowledged.

mean returns on assets or portfolios and the betas[1] of these assets or portfolios calculated against a market index. Judging from the academic literature, this criterion is even more widely-accepted by scholars as a tool for assessing the ex ante or ex post qualities of securities, portfolios and investment projects. There seems little doubt that it is currently the most widely-accepted criterion for inferences about the quality of risky assets.

It is quite simple to employ, particularly in a situation such as the contest mentioned above. There are only two steps to accomplish. First, an index must be chosen by the judges; second, the betas must be computed against this index for each asset (and portfolio). The second step can be accomplished in many ways, including purely subjective, but the most common method is to employ historical data over the same interval as the contest itself.

Given the computed betas and the returns on assets or portfolios, the criterion can be illustrated as in Figure 1. A line, $R = \gamma_0 + \gamma_1 \beta$ is fit[2] to the observations and assets are declared "winners" if they are above the line (such as A and B) or losers if they are below (C or D).

My purpose here is to expose the ambiguity in this criterion. It is not robust, is likely to yield different judgements when employed by different judges, and can completely reverse its judgements after seemingly innocuous changes in its compu-

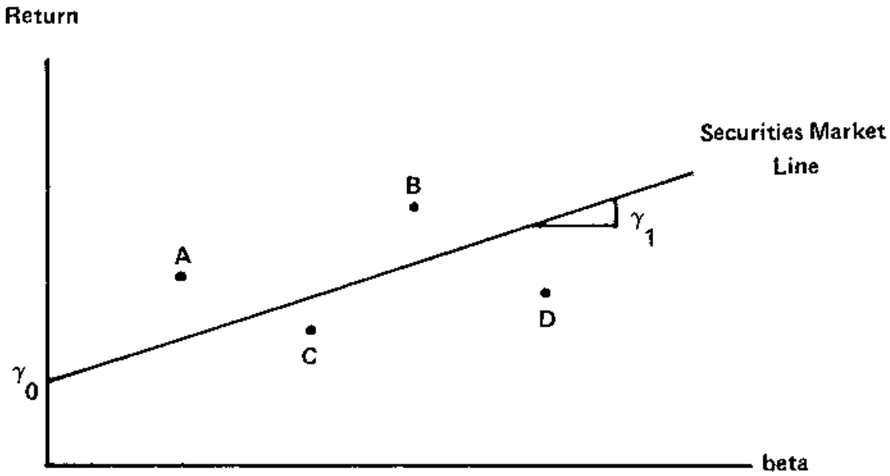

FIGURE 1. The Securities Market Line Employed as a Criterion for Assessing Asset Quality

1. The "beta" is the covariance of the asset and the market index returns divided by the variance of the market index' return.

2. The method of fit is not crucial for this discussion. It is often done by regression, sometimes with sophisticated econometric corrections of the betas. It could also be done by choosing γ_0 and γ_1 based on theories of market equilibrium. For example, the Sharpe [1964], Lintner [1965] theory requires that $\gamma_0 = R_F$, the riskless rate of interest, and that $\gamma_1 = E(R_m) - R_F$, the difference between the expected return on the market index and the riskless rate of interest. Thus, *estimates* of $E(R_m)$ and of R_F can be used to fix the line. Alternatively, the Black [1972] theory would require $\gamma_1 = E(R_m - R_z)$ and $\gamma_0 = E(R_z)$ where z is a portfolio with minimal variance uncorrelated with m.

tation. Reasons for these deficiencies will be explained in detail. By implication, the concept of the beta as an unambiguous measure of risk will be disputed.

A Numerical Example

To illustrate the ambiguity of the securities market line criterion, let us consider a specific numerical example—the idealized contest conducted with 15 hypothetical contestants and a hypothetical four asset universe.

Table 1 begins the example with portfolios selected by the fifteen contestants. Nothing is unusual about the selections. The first four contestants plunged into the individual assets and contestants 14 and 15 sold short some securities, but there seems little reason to exclude such possibilities. The idealization of the contest is evident only because trading costs and restrictions on short sales have not even been mentioned.

TABLE 1

PORTFOLIOS SELECTED BY FIFTEEN CONTESTANTS

Contestant	Asset			
	1	2	3	4
	Percentage Invested in Individual Asset			
1	100.	0.	0.	0.
2	0.	100.	0.	0.
3	0.	0.	100.	0.
4	0.	0.	0.	100.
5	50.	50.	0.	0.
6	33 1/3	33 1/3	33 1/3	0.
7	40.	30.	20.	10.
8	25.	25.	25.	25.
9	10.	20.	30.	40.
10	0.	33 1/3	33 1/3	33 1/3
11	0.	0.	50.	50.
12	59.6	27.6	7.69	5.08
13	40.7	31.9	14.0	13.4
14	−4.4	42.2	29.0	33.3
15	−49.6	52.4	44.1	53.1

Note: The last four portfolios' weights may not sum to 100% because of rounding. The exact weights were used in all calculations.

After the portfolios were selected, a sample period was observed with the results for individual assets reported in Table 2. Again, there is nothing abnormal nor pathological about these numbers. The mean returns were different on different assets and the covariance matrix was non-singular, certainly the most usual and desirable feature of real asset return samples. Indeed, all of the qualitative results to be reported hereafter could be obtained from any other numbers with these same general characteristics. For the purposes of illustration, there is not even any need to be concerned with the statistical properties of the sample. The ambiguities are not related in any way to the sampling error in the estimates. The same problems are present when the true population means, variances and covariances are known and used in computing the criterion.

TABLE 2

SAMPLE STATISTICS FOR INDIVIDUAL ASSETS WHICH CONSTITUTED THE UNIVERSE OF SECURITIES FOR THE PORTFOLIO SELECTION CONTEST

Asset	Sample Mean Return (%/period)	Asset 1	2	3	4
		Sample Variance/Covariance Matrix			
1	6.	10.			
2	7.	2.	20.		
3	8.	4.	4.	40	
4	9.	5.	1.	10.	60.

The observed means and variances of the 15 portfolios are easy to compute by applying the compositions of Table 1 to the observed individual asset returns, variances and covariances of Table 2.

Because of its wide acceptance, the securities market line criterion will be used by the three hypothetical sponsors of the contest in order to distinguish winners from losers. Let us suppose, however, that a dispute arises about the best index to use in calculating the "betas." Sponsor/judge no. 1 admits total ignorance about this question. He concludes that an index composed of equal weights in the individual assets would be the most sensible portrayal of this ignorance and the fairest to all contestants. Sponsor/judge no. 2, however, has studied asset pricing theory and argues that the appropriate index should have weights proportional to the aggregate market values of individual assets. He finds that the aggregate values of assets 2 and 3 are roughly four times larger than those of assets 1 and 4, so he suggests the index (10%, 40%, 40%, 10%). Sponsor/judge no. 3, also a theorist, thinks that a good index should be mean-variance efficient, in the sense of Markowitz (1959)[3], so he makes some calculations and obtains an index with the same mean return as the indices of the other judges but with a different composition. His index turns out to be (18.2%, 37.0%, 21.5%, 23.3%), the proportions being rounded to three significant digits.

The compositions of the indices, their observed mean returns and variances, and the securities market lines computed against them are reported in Table 3. Notice that the three indices have equal means, similar variances and closely adjacent securities market lines. (The securities market lines would have been identical if they had been fit using an asset pricing theory rather than by regression. See Note 2.)

3. I.e., a portfolio which has the smallest sample variance of return for a given level of sample mean return.

TABLE 3

INDICES AND SECURITIES MARKET LINES USED BY THE THREE JUDGES IN THE PORTFOLIO SELECTION CONTEST

Judge	1	2	3	4	Mean Return	Variance of Return	Slope	Intercept
	Composition of Index (Percentage of Index in Security)				Index		Securities Market Line[a]	
1	25	25	25	25	7.5	11.4	2.31 (.119)	5.22 (.127)
2	10	40	40	10	7.5	13.0	2.40 (.486)	5.46 (.444)
3[b]	18.2	37	21.5	23.3	7.5	11.0	2.57 (0.)	4.93 (0.)

[a]Standard errors are in parentheses. These lines were fitted cross-sectionally by ordinary least squares applied to mean returns and betas of the fifteen portfolios in the contest.

[b]Judge three actually specified more precise weights. The weights reported here have been rounded. (For example, the proportion of his index in asset 4 was actually 23.3214%.) The exact weights were used in all calculations.

Coefficients of Correlation Between Indices		
	2	3
1	.897	.982
2		.920

What about their assessments of "winners" and "losers"? Judge no. 1 ranks the 15 contestants from best (largest positive deviation from his securities market line) to worst as follows:

Rankings of Contestants by Judge no. 1

Winners (Best)→ (above the line)	2	15	14	5	10	/					
Losers (below the line)	6	13	9	8	7	12	4	11	3	1	←(Worst)

A graph of the contestants' positions relative to the judge's criterion is given by Figure 2(A).

The second judge has a different set of assessments, as shown in the following list and depicted in Figure 2(B):

Rankings of Contestants by Judge no. 2

Winners (Best)→	4	15	11	9	14	10	8	/
Losers	13	1	12	7	5	2	6	3 ←(Worst)

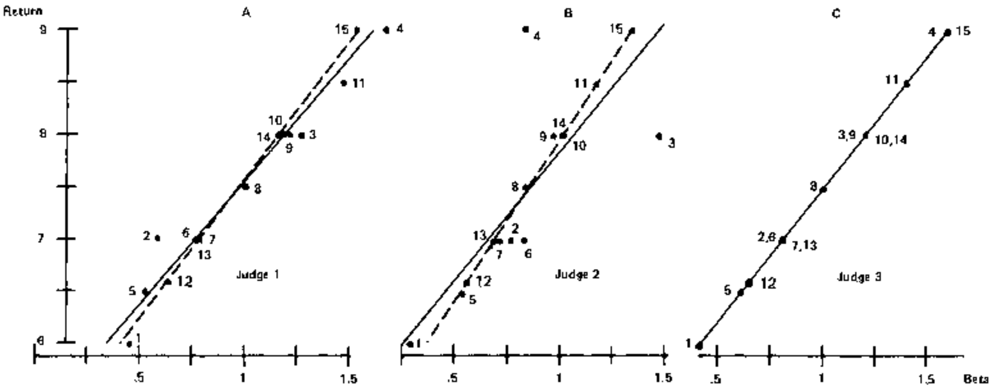

FIGURE 2. Securities Market Lines and Positions of Selected Portfolios as Perceived by the Three Judges of the Contest

The solid lines are ordinary least squares estimated securities market lines fitted through the fifteen contestants' portfolios. The dotted lines pass through portfolios 12–15, which were exactly mean/variance efficient ex post.

Although some contestants were similarly rated by both judges, (e.g., contestant 15 was ranked second by both and contestant 3 was ranked 14th by judge no. 1 and 15th by judge no. 2), other contestants were rated quite differently; (e.g., the number one winner according to judge no. 1 was a loser and ranked 13th out of 15 by judge no. 2.) The rank correlation between the decisions of these two judges is only .0036 and the lack of agreement is clearly evident in the figure.

As for judge no. 3, after calculating his securities market line and plotting the selected portfolios, he observes Figure 2(C). Every single contestant is exactly on the line. Of course, this judge is unable to construct a ranking and can draw no inference about the relative abilities of contestants.

This example was not constructed to generate bizarre results. The same results can be obtained from every sample of asset returns. They were not caused by any of the example's parameters, by the numbers of assets and contestants, nor by the pattern of returns.

The results in the example and the ambiguity in the securities market line criterion can be attributed to the following fact: corresponding to every index, there is a beta for every individual asset (and thus for every portfolio); but these betas can be different for different indices and will be different for most. To consider the beta as an attribute of the individual asset alone is a significant mistake. For every asset, an index can be found to produce a beta of any desired magnitude, however large or small. Thus, for every asset (or portfolio) judicious choice of the index can produce any desired measured "performance," (positive or negative), against the securities market line.

II. MATHEMATICS OF THE SECURITIES MARKET LINE CRITERION

In this section, the mathematical causes of the preceding numerical results will be made more precise. Since there are general principles involved that would bring similar results to every ex post sample and ex ante application of the criterion, it seems worthwhile to have a unified list of the principles in one place.

Let us presuppose the existence of a mean return vector, R, and a non-singular covariance matrix, V, of N *individual* assets. There is no need to require that these be of any particular dimension nor that they be ex ante (nor ex post sample statistics). The following statements are true for any R and V, however obtained. These statements constitute the mathematics that explains the behavior of the criterion:

S1: Let X_p be the $N \times 1$ column vector of investment proportions (or "weights") that defines a portfolio p. Let X_I define the index I used by a judge. Then the "beta" for portfolio p with respect to the index is given by

$$\beta_{p,I} = X_p' V X_I / \sigma_I^2 \tag{1}$$

where $\sigma_I^2 = X_I' V X_I$.
Proof: obvious.

S2: If the portfolio q is mean/variance efficient (in the Markowitz sense) with respect to R and V, then its weights are given by

$$X_q = V^{-1}(R\,\iota)A^{-1}(r_q\,1) \tag{2}$$

where ι is the $N \times 1$ unit vector, A is the efficient set information matrix[4] and $r_q \equiv X_q' R$ is the return of portfolio q.
Proof: Merton [1972] or Roll [1977].

S3: If the selected index is mean/variance efficient, then the betas of all assets are related to their mean returns by the same linear function.
Proof: Set $q = I$ in (2) and use the results in (1). Then simplify to

$$\beta_{p,I} = (r_p\,1)A^{-1}(r_I\,1)/\sigma_I^2 \tag{3}$$

A, r_I, and σ_I^2 are cross-sectional constants.[5] *Q.E.D.*

S4: If, for some selected index, the betas of all assets are related to their mean returns by the same linear function, then that index is mean/variance efficient.
Proof: Roll [1977, p. 165.]

S5: The betas of all mean/variance efficient portfolios are related to their mean returns by the same linear function, even if the index used to compute the betas is not itself mean/variance efficient.
Proof: Identical to S3.

S6: For every ranking of performance obtained with a mean/variance non-efficient index, there exists another non-efficient index which reverses the ranking.

4. i.e.,

$$A = \begin{bmatrix} R'V^{-1}R & \iota'V^{-1}R \\ \iota'V^{-1}R & \iota'V^{-1}\iota \end{bmatrix}$$

5. It can be shown easily that $r_p = \gamma_0 + \gamma_1 \beta_{p,I}$ where γ_0 is the return on a portfolio orthogonal to I and $\gamma_1 = r_I - \gamma_0$.

Proof: Let Z be the $(K \times N)$ matrix of selected portfolios (each row containing the weights for N individual assets in the portfolio selected by one of the K contestants). Let $_A$ be a subscript which denotes the first ranking, (based on index "$_A$"). The scatter of selected portfolios in the return/beta space can be denoted

$$ZR = \alpha_A + \gamma_{0A}\iota_k + \gamma_{1A}(ZVX_A/\sigma_A^2) \tag{4}$$

where γ_{0A} and γ_{1A} are, respectively, the (estimated) intercept and slope of the securities market line and α_A is the $(K \times 1)$ vector of deviations. The jth element of α_A is the "performance measure" for contestant j. (ι_k is the K-element unit vector).

A reversal of rankings can be achieved by finding a second index, denoted B, such that $\alpha_B = -d\alpha_A$ where d is a positive scalar constant. Choose $d = \sigma_A^2/\sigma_B^2$. Then (4) can be written twice, once for $_A$ and once for $_B$, and the equations can be added to obtain

$$\sigma_A^2\alpha_A + \sigma_B^2\alpha_B \equiv 0 = ZR(\sigma_A^2 + \sigma_B^2) - (\sigma_A^2\gamma_{0A} + \sigma_B^2\gamma_{0B})\iota_k$$
$$- ZV(\gamma_{1A}X_A + \gamma_{1B}X_B) \tag{5}$$

We suppose that there are more contestants (K) than individual assets (N) and that all selected portfolios are different. Under these circumstances $Z'Z$ is non-singular. Furthermore, since each row of Z sums to unity, we must have $Z\iota_N = \iota_k$ and thus $\iota_N = (Z'Z)^{-1}Z'\iota_k$. Multiplying both sides of (5) by Z' and simplifying, we find

$$R(\sigma_A^2 + \sigma_B^2) - (\sigma_A^2\gamma_{0A} + \sigma_B^2\gamma_{0B})\iota_N = V(\gamma_{1A}X_A + \gamma_{1B}X_B) \tag{6}$$

In the equation system (6), there are $N+3$ unknowns: the N elements of X_B plus γ_{0B}, γ_{1B}, and σ_B^2. (Recall that X_A, σ_A^2, γ_{0A}, and γ_{1A} were already known). There are only N equations in (6) but we must also add the portfolio additivity condition, $X_B'\iota_N = 1$, the definition of the variance,

$$\sigma_B^2 = X_B'VX_B \tag{7}$$

and the usual requirement,

$$\gamma_{1B} = X_B'R - \gamma_{0B}, \tag{8}$$

which guarantees that the securities market line passes through the mean return of the index ($X_B'R$) at a beta of unity. Thus, there obtains an identified system of three non-linear equations in three unknowns and at least one solution is guaranteed.

When there are fewer contestants than assets, the system (5) is under-identified and there is an infinity of solutions. *Q.E.D.*[6]

6. A computational problem is present in solving (6) because the system of equations is non-linear. An *approximate* reversal of rankings can be obtained by ignoring the difference in the securities market lines, setting $\gamma_{0A} = \gamma_{0B}$ and $\gamma_{1A} = \gamma_{1B}$. This provides an X_B which approximately reverses the rankings based on X_A. For example, the X_B obtained in this manner corresponding to the index selected by the first judge (which was equally-weighted), is the portfolio $X' = (-.0422, .526, .232, .285)$. It provides the ranking of the 15 selected portfolios from "best" to "worst", of 3, 1, 11, 4, 9, 8, 12, 7/13, 6, 10, 14, 5, 15, 2. Compare judge no. 1's rankings above. The rank correlation between them is $-.961$.

Statements S1 through S6 explain all the facts of the numerical example. For example, Judge no. 3's inability to discriminate among the contestants was caused by his choice of a mean/variance efficient index. As S3 indicates, any such index will cause the scatter of returns and betas to fall exactly along a line.

The portfolios selected by contestants 12 through 15 fall exactly along another line (shown dashed in all three panels of Figure 2). These selections turned out to be mean/variance efficient portfolios which, as S5 indicates, always fall exactly along a line regardless of the index. Note that this line is not necessarily the securities market line. The latter is usually a line of best fit to all selected portfolios and this can deviate from the line through efficient portfolios when the index itself is not efficient.[7]

Finally, the deviations of portfolios from the securities market lines used by Judge no. 1 and Judge no. 2 are a consequence of S4. These judges did not use efficient portfolios as their indices so the *individual* asset returns could not be exactly linear in the individual betas. Of course, contestants could have made choices such that their portfolio's betas and returns were linear; but this was not the case for the example nor is it likely in general. S6 shows that these judges might even have selected indices which reversed the rankings of contestants.

It is quite easy to show how the beta of an arbitrary portfolio varies with the index. Suppose we simplify matters by assuming that all the indices being considered have the same return but different compositions (as in the example). Of course, only one will be mean/variance efficient. If I^* is the index that is mean/variance efficient and I is another index with the same return, the difference in the beta of p from using I^* rather than I is

$$\Delta\beta_p = X_p V\left[(\sigma_I^2/\sigma_{I^*}^2)X_{I^*} - X_I\right]/\sigma_I^2 \tag{9}$$

The sum of elements in the bracketed vector of (9) is equal to $\sigma_I^2/\sigma_{I^*}^2 - 1$, which is positive (because $\sigma_{I^*}^2 < \sigma_I^2$). Thus, the numerator of (9) is proportional to the covariance between p and a hybrid portfolio whose investment proportions vector is $[(\sigma_I^2/\sigma_{I^*}^2)X_{I^*} - X_I]/(\sigma_I^2/\sigma_{I^*}^2 - 1)$. The closer the pattern of p's weights to the pattern of weights in this hybrid portfolio, the larger is beta. Roughly speaking, if the weights of p match the difference in weights between an efficient index and the index actually employed, the observed beta will be "small" relative to the magnitude of beta had the efficient index been employed. Since the return of p is invariant to the index employed, a relatively smaller beta corresponds to better measured performance and vice versa.

As an example, consider contestant no. 2, who was ranked first by Judge no. 1 and 13th by Judge no. 2. His portfolio and the vectors $[(\sigma_I^2/\sigma_{I^*}^2)X_{I^*} - X_I]/(\sigma_I^2/\sigma_{I^*}^2 - 1)$ for the two judges are given in table 4. Clearly, contestant no. 2 selected a portfolio that was "different" from the first judge's index.[8] The beta perceived by

7. Portfolio 12 was selected to be the global minimum variance portfolio and its return was 6.58 percent. Since portfolios 13, 14, and 15 all had returns of at least 7 percent and since they were mean-variance efficient, they were all located on the positively-sloped segment of the ex-post efficient frontier.

8. Different in the sense that the largest weight of contestant no. 2's portfolio corresponded to the (algebraically) smallest value of $X_I - X_{I^*}(\sigma_I^2/\sigma_{I^*}^2)$ for judge no. 1; and thus the largest weight of the hybrid portfolio for Judge no. 1 matched the largest weight for contestant no. 2.

this judge was correspondingly lower and the performance was highly rated. The opposite was the case for this contestant as perceived by Judge no. 2.[9] Similar situations for the other selected portfolios are evident in the example. Roughly speaking, contestants' hybrid portfolios are ranked low when their weight patterns are "close" to the judge's hybrid portfolio weight patterns, and vice versa.

TABLE 4

AN EXAMPLE OF THE RELATIONSHIP BETWEEN A SELECTED PORTFOLIO AND THE INDEX' DEVIATION FROM MEAN/VARIANCE EFFICIENCY

	Weights		
		Hybrid Portfolios Which Represent Deviations of Index from Efficiency	
Asset	Selected Portfolio no. 2	Judge no. 1	Judge no. 2
1	0.	−1.77	.622
2	1.00	3.82	.209
3	0.	−.79	−.778
4	0.	−.26	.947

Another measure of the alteration in beta caused by using an inefficient index is the displacement of portfolio p's beta from the line connecting all efficient portfolios (the dashed line of Figure 2). Note that all assets and portfolios would be situated on such a line if an efficient portfolio had been employed as an index. Formally,

S7: the displacement, from the line connecting efficient portfolios, induced in p's beta by using an inefficient index is $X_p V(X_I - X_{I^*})/\sigma_I^2$, where I^* is the efficient portfolio whose return is the same as the return of I, the index.

Proof: Let p^* be the efficient portfolio whose return is the same as p. Then the displacement of p's beta from the dashed line of efficient portfolios is $\beta_{p,I} - \beta_{p^*,I} = (X_p' - X_{p^*}')VX_I/\sigma_I^2$. Since the information matrix A in (3) is symmetric, the covariance between any efficient portfolio and any arbitrary asset or portfolio depends only on their returns. Thus $X_{p^*}'VX_I = X_p'VX_{I^*}$ and $\beta_{p,I} - \beta_{p^*,I} = X_p'V(X_I - X_{I^*})/\sigma_I^2$. *Q.E.D.*

Still a third way to measure alteration in beta with alteration of the index is provided in the appendix.

III. MEASUREMENT OF ABILITY, MARKET EFFICIENCY AND INDEX FUNDS

Individual *differences* in portfolio selection ability cannot be measured by the securities market line criterion. This was the general thrust of the preceding argument. If the index is ex ante mean/variance efficient, the criterion will be unable to discriminate between winners and losers. If the index is not ex ante efficient, the criterion will designate winners and losers; but another index could

9. Correlations between this contestant's weights and the judges' hybrid portfolio weights are .73 for judge no. 1 and −.11 for judge no. 2.

cause the criterion to designate different winners and losers and there is no objective way to ascertain which index is correct.

Although the securities market line cannot distinguish between superior and inferior selected portfolios, it does contain information about the quality of one portfolio, the index itself. In fact, a repeated running of a contest such as the one described on page 1051 will, in principle, provide the necessary information to infer whether the *index* is ex ante mean/variance efficient. It will provide no information about any other portfolio.

We must be very careful to delineate the difference between running the contest, (or conducting an experiment with actual returns), in order to measure relative abilities of different individuals and in order to test the ex ante efficiency of the particular portfolio employed as an index. These two purposes are not the same.

To clarify the difference in measuring ability and in testing for ex ante efficiency of the index, it will be convenient to make the concept of ability more precise. Begin with an *objective* ex ante expected return vector and covariance matrix, the "state of nature," perhaps unknown to anyone. Perfect ability should be defined as selecting a portfolio located on the ex ante mean/variance efficient set computed with these objective parameters.[10] Less than perfect ability would be revealed by the selection of an ex ante non-efficient portfolio. Such a lapse might be caused by incorrect estimates of the true objective means, variances, and covariances, by an inability to compute the efficient set, given correct estimates, by transaction's costs, etc. Now consider the task of separating individuals with perfect and imperfect abilities. If we employ the securities market line criterion and choose *any* index that is itself ex ante mean/variance efficient, and such indices will always be present, we would be unable to ascertain which individuals have imperfect ability. All portfolios would be *on* the securities market line. This would be exactly true ex ante, using the true objective expected returns and betas, and it would be true on average ex post. (There would be no observed ex post consistency in winning.)

On the other hand, if we choose an inefficient index, we would judge some of the individuals with perfect ability to be inferior and some individuals with less than perfect ability to be superior. In the numerical example, for instance, both judges no. 1 and no. 2, who used non-mean/variance efficient indices, concluded that contestants 12 and 13 were losers. Actually, 12 and 13 selected portfolios that turned out to be exactly efficient, thus indicating perfect ability (ex post).

Considering the second purpose, suppose we wish to infer whether a particular portfolio is ex ante mean/variance efficient. This is a statistical problem since we could make the inference directly if the ex ante probability parameters were known. Given uncertainty about means, variances, and covariances, however, it becomes necessary to employ the portfolio in question as the index in a selection contest (or experiment). From mathematical principles S3 and S4, we see that repeated calculations of the securities market line over many intervals will discover *no* consistent winners *or* losers if and only if the index is ex ante efficient. In other and more precise words, if no measured deviation from the line is statistically significant, the hypothesis of the index' ex ante efficiency cannot be rejected. Some

10. Note that there is no use discussing this topic at all if individuals do not have utility functions that depend on the expected return and variance of return of their portfolios.

contestants will be adjudged winners and some losers at every intermediate stage, step (d) of the contest, but if these judgements can be attributed to chance alone, the index is efficient.

Suppose that such contests or experiments are conducted and the hypothesis of ex ante efficiency for the index is not rejected. The particular collection of individual assets, the index, would then be certified as a non-dominated portfolio in the ex ante mean/variance space. No portfolio manager, regardless of his skill, could possibly pick a superior portfolio. Notice, however, that many observed, managed portfolios might also be ex ante efficient and thus be just as good as the index (gross of fees). This fact could not be determined by using the securities market line criterion with the original index, of course, but it could be determined by employing each managed portfolio as the index in another contest.

If managers charge fees, and if the original index were found to be ex ante efficient, investors could select one point on the ex ante efficient frontier very cheaply. This seems to be the rationale for index funds. To the extent that the empirical articles on performance measurement (Jensen [1968, 1969]) present evidence that the indices employed were ex ante mean/variance efficient, a free management consulting service was provided to all investors.[11]

In order to know whether index funds make sense for a particular investor, two questions must be answered affirmatively:

(a) Was the empirical judgement correct about the index' ex ante efficiency?
(b) Was the index positioned at an efficient frontier tangency point with respect to a line drawn from the measured riskless asset?

The answer to both questions is no.

Question b's answer can be obtained by noting the wide-spread Friend/Blume [1970] phenomenon—that securities market lines, computed with value-weighted market indices consisting of equities only, produce positive performance measures for low-beta portfolios and negative measures for high-beta portfolios. This is equivalent to the index having a larger return than the tangency position of a line drawn from the riskless asset.[12]

Question (a) requires a critical evaluation of evidence offered in the literature, which consists of tests on the deviations of selected portfolios about measured securities market lines. All of these tests presume that such deviations are cross-sectionally independent. *This presumption is false and the tests are therefore inconclusive.*

11. There was another important product of these empirical papers, a test of the generalized (by Black [1972]), capital asset pricing theory originally due to Sharpe [1964] and Lintner [1965]. The principal implication of this theory is that the market value-weighted portfolio of all assets is ex ante mean/variance efficient. A proxy for the market portfolio is employed as an index in the empirical performance measurement literature. To the extent that deviations about the securities market line computed with such an index are non-systematic, and to the extent that the index is an accurate measure of the true value-weighted market portfolio, a test of the theory is conducted—and the theory is not rejected. Interestingly, the original and best empirical papers (Jensen [1968, 1969]) were finished *before* the Black version of the theory was developed. A more detailed examination of the theory's testability is given in Roll [1977].

12. See Roll [1977, pp. 140–144].

The cross-sectional statistical dependence of performance measures (deviations about the securities market line) is easily recognized by noting their complete functional dependence on the index. If the index happened to be *exactly* on the ex post efficient frontier during a given sample period, all deviations would be identically zero. If, as usual, the index does not turn out to be exactly ex post efficient, each observed deviation is determined analytically by the distance of the index' position from the ex post efficient frontier. This is obvious from equation (4). If betas are recomputed each period, the deviations (α's in 4) are linear functions of the observed return vector (R) and observed covariance matrix (V). Even if the betas are not recomputed, the α's are still functions of these two sampling statistics (R and V) because the index' position, the sample efficient set, and the sample securities market line depend only on R and V.[13] Thus, the sampling variation of the deviations is much more complex than might at first be imagined. A proper test of the index' efficiency must regard the entire set of deviations as a single sample statistic. Thus, a multinomial type test based on their presumed independence will produce incorrect significance levels and incorrect inferences.[14]

Even the best empirical literature does not take this problem into account. For example, in Jensen [1968], a scatter diagram of t-ratios of the deviations is presented and an inference is drawn about the average t-ratios. Of course, t-ratio tests rely on independence among observations.

More sophisticated tests are presented in Jensen [1969, Fig. 19] which depicts a highly significant positive relation between deviations, net of management expenses, in two successive periods. If truly significant, this result would be inconsistent with the index' ex ante efficiency. However, Jensen argued that the observed relation was caused by the high fees charged by some managers in both periods and he provided a still more sophisticated test in Table 9, p. 239. In this test, a run of k years of positive deviations from the securities market line, *gross* of expenses, is used as a predictor for the deviation in the subsequent year. For $1 \leqslant k \leqslant 4$, where sample sizes are large, the predictor does better than average but, it was claimed, not significantly better. The computed significance level was based on the supposition of cross-sectional independence. Some doubt remains about the actual level of statistical significance.

Based on these facts, might it still be rational for investors to incur positive management costs to achieve a portfolio on or close to the objective ex ante efficient frontier? No direct evidence has appeared to indicate that some portfolio selectors do indeed have ability; but there seems to be plenty of indirect evidence: the continued existence of professional money managers. The existence of managers who collect fees would itself be a violation of the theory of efficient markets if publicly-available information implied that no fees need be expended to select an optimal portfolio. If index funds were optimal, the aggregate market value for the

13. Note that the observed return vector is cross-sectionally dependent unless the population covariance matrix is diagonal. Furthermore, portfolios would be cross-sectionally dependent even if individual assets were not.

14. In Roll [1977b] the econometric problems of testing a given portfolio's efficiency are discussed in more detail.

portfolio advisory industry would be zero. Clearly, the market wage per advisor does not contain all relevant information (or does it?)

IV. Changing-Composition Portfolios

As the portfolio contest is repeated, players are allowed to select different portfolios during each trial or series of trials. This mimics real-world portfolio managers, who seem inclined to alter their portfolios' compositions rather frequently. Does the very fact that selected portfolios change composition over time have any implications for the conclusions drawn so far?

In looking into this question, it will be convenient to distinguish two cases. First, the case of stationarity in which the true objective (and perhaps unknown) ex ante mean return vector and covariance matrix are constant. If we denote these parameters by R_t^* and V_t^*, then we require $R_t^* = R_\tau^*$, $V_t^* = V_\tau^*$ for all t and τ. In this case, none of the conclusions can possibly be altered.

For example, suppose that an index is employed which is mean/variance efficient with respect to R_t^* and V_t^*. (It is on the ex ante objective efficient frontier). Then the deviations of every individual asset about the securities market line will be zero ex ante and zero on average ex post. The manager can change compositions as often as he likes, but since every asset is neither a consistent winner nor loser, *no* portfolio of assets that he selects can *significantly* win or lose in any trial. It follows that changing compositions cannot matter for assessing ability.[15]

Similarly, if the index is inefficient (with respect to R_t^* and V_t^*), the manager can consistently "win." As we have seen, he can do so even with a constant composition portfolio. But since "winning" with respect to the securities market line does not imply selection ability (or vice versa) again the fact of changing composition does not alter the basic conclusion that the criterion cannot distinguish good managers from bad ones.

The second case of non-stationarity is probably the situation most people have in mind when they think of the manager possibly "outperforming" the index by altering his portfolio's composition over time. The intuitive notion seems to be that a manager might not do well if he is required to pick a portfolio and stick with it but that he might be able to predict when general market conditions are going to turn out more or less favorably than normal. Alternatively he might have privileged information about certain assets during some periods.

The intuition is certainly correct in the sense that clairvoyance of one kind or another will yield "superior" performance. The question here, however, is whether that superiority can be detected by the securities market line. Imagine that the objective ex ante mean return vector and covariance matrix R_t^* and V_t^* are not constant, but depend on t. In addition, suppose that the manager knows their time paths perfectly. Each period, he selects an ex ante mean/variance efficient portfolio with respect to the R_t^* and V_t^* prevailing at that time. Suppose also that

15. Changing compositions will matter, however, for the appropriate statistical test to use in determining whether the index is significantly nonefficient ex post. The statistical sampling properties of the fitted securities market line are obviously more complex when the selected portfolios change compositions within the sample period.

the index used to assess his performance has a constant composition. It may have been ex ante mean/variance efficient for some period, say against R_1^*, and V_1^*, but it need not be ex ante efficient at every period. In fact, there may be no constant composition portfolio which is efficient for all t.

The sample observed covariance matrix and mean return vector, $\hat{V}$ and $\hat{R}$, are convolutions of the sequences $\{V_1^*,\ldots,V_T^*\}$ and $\{R_1^*,\ldots,R_T^*\}$ actually followed by the objective parameters during the sample, (which covers, say, periods 1 to T). Thus, with respect to the sample, the manager's portfolio may turn out to be either above or below the securities market line since the index is mean/variance inefficient, and statistically significantly inefficient. Needless to say, another manager who does not know the sequences $\{V^*\}$ and $\{R^*\}$, and thus may have selected an inefficient portfolio, can find his "performance" rated highly by the securities market line, and higher than the manager with perfect information.

Finally, an index could conceivably be ex ante mean/variance efficient with respect to every V_t^* and R_t^* even though it has constant composition. The index might also remain efficient because it changes composition along a simple path which happens to match the paths $\{V^*\}$ and $\{R^*\}$. For example a buy-and-hold index, (not rebalanced), would be a feasible choice for many investors because it is even easier to monitor than a constant composition index. In either case, investors would not need to pay for the services of professional portfolio managers if the index were always efficient. If a simple index possesses mean/variance efficiency, no portfolio manager could be found who consistently placed a portfolio above or below the securities market line in repeated samples. This lack of consistency would constitute evidence that the index itself is efficient with respect to the sequences $\{V^*\}$ and $\{R^*\}$. However, it would constitute no information about the efficiency of the selected portfolios. Selected portfolios which were also mean/variance efficient with respect to $\{V^*\}$ and $\{R^*\}$ *and* selected portfolios which were not efficient would both show no consistency in their positions with respect to the securities market line.

Of course, if the scenario above described the state of nature, investors would be advised to select the simple index and pocket the management fees. We have not yet determined empirically, however, whether the scenario contains even the slightest element of realism.

V. The Securities Market Line and The Return Generating Process: Resolution of a Paradox[16]

This paper has presented some negative aspects of the securities market line as a performance measuring device. Yet the theory of portfolio diversification and the concept of "beta" ($\beta_{jp}=\sigma_{jp}/\sigma_p^2$) as a systematic risk measure constitute a pervasive paradigm among financial economists. Is the paradigm without merit? I hope to argue persuasively in this section that the paradigm has some merit but that the usual implementation of the paradigm leads to the problems discussed in the preceding sections.

16. I am greatly indebted to Stephen A. Ross for many conversations about the material in this section.

The argument is: when discussing the securities market line and beta, most people are actually thinking intuitively about a one-factor linear return generating process of the form

$$R_{jt} = \alpha_j + \rho + \beta_j(R_{Gt} - \rho) + \zeta_{jt}; \quad j = 1, \ldots N \tag{10}$$

The t subscript indicates time period; ζ_{jt} is a stochastic error with zero mean and is unrelated to R_G or to $\zeta_{i\tau}$ for $i \neq j$ or $t \neq \tau$; R_{Gt} is a common generating factor and ρ is a constant across assets.

Process (10) looks similar to a securities market line. Indeed, by taking expectations of both sides of (10), a system of equations is obtained which is indistinguishable from (4) with $\gamma_0 = \rho$ and $\gamma_1 = E(R_{Gt} - \rho)$. However, it is a mistake to interpret the expection of (10) as a securities market line because G is not a portfolio. Instead, G is the ***unique*** source of common variation in the ensemble of asset returns. This seemingly innocuous distinction is actually critical.

The interpretation usually given to deviations about the securities market line is validly applicable to the deviations, (α_j's), about the return generating process (10).[17] As Ross [1976] showed in the development of his theory of asset pricing, a riskless profit can be obtained when there is at least one asset j for which α_j is not zero, provided that (10) describes the return generating process and that there exists a "sufficiently" large number of individual assets. Furthermore, the hedge portfolio formed to obtain this profit should contain long positions in securities with positive α_j's. Clearly α_j in (10) could be interpreted as an ex ante measure of an asset's desirability and ex post as a measure of performance.

But a problem arises because G is not necessarily observable and because replacing G by some portfolio index can result in the problems described earlier in this paper. For example, suppose that there are truly some non-zero α_j's in (10). In this case, a riskless profit is possible. But the portfolio used to replace G might be mean/variance efficient. As S3 shows, the "betas" computed against this index would be exactly linear in the mean returns. Of course, these computed betas would not be equal in value to the slope coefficients in the generating process (10).

Alternatively, the selected index might not be mean/variance efficient. However, the $\hat{\alpha}_j$'s it produces would be equal in value to the α_j's of (10) if and only if the index is perfectly positively correlated with G. One would be fortunate indeed to happen upon such a perfect index. As we have seen, imperfect correlation could cause the computed performance measures to be of different magnitudes and even of opposite sign from the α_j's of (10). Incorrect inferences about performance would thereby be obtained.

The same problems arise when there are no available arbitrage profits. In this case, all the true α_j's in (10) are zero. Furthermore, G *does* have the same mean and variance as some mean/variance efficient portfolio. But if the index employed is not efficient, non-zero $\hat{\alpha}_j$'s will be found and non-neutral performance will be falsely indicated.

17. This argument is presented in a somewhat different form by Mayers and Rice [1977].

The analytic components of the problem can be examined by selecting an index, say with investment proportions vector X_I, and then using it as a replacement for G. From (10), the index' return is

$$R_{It} = X_I' R_t = X_I'[\alpha + \rho + \beta(R_{Gt} - \rho) + \zeta_t]$$

where the absence of j subscript indicates an $N \times 1$ column vector. Thus,

$$R_{It} = \alpha_I + \rho + \beta_I(R_{Gt} - \rho) + \zeta_{It} \tag{11}$$

Solving (11) for $R_{Gt} - \rho$ and substituting back into (10), we have

$$R_{jt} = \alpha_j - \frac{\beta_j}{\beta_I}\alpha_I + \rho + \frac{\beta_j}{\beta_I}(R_{It} - \rho) + \zeta_{jt} - \frac{\beta_j}{\beta_I}\zeta_{It} \quad j = 1, \dots N \tag{12}$$

Equation (12) is now in a form similar to the true generating process, but with the common factor, R_G, replaced by the return on the operational index, R_I. We can observe that there are basically two differences between (12) and (10) and these must be responsible for the problems encountered previously.

First, the index may be imperfectly diversified; i.e., ζ_{It} may not be identically zero in every period. This is basically an econometric problem which will cause mis-estimation of the slope and intercept of (12). Conceivably, it alone could result in incorrect measures of performance. Second, the index may have its own non-zero deviation α_I about the true generating process. If there are arbitrage opportunities, there would be no a priori reason that $\alpha_I = X_I'\alpha$ should be zero. Unfortunately, if α_I is non-zero, the performance measure for asset or portfolio j becomes $\alpha_j - \alpha_I(\beta_j/\beta_I)$. Since it depends now also on β_j, assets with large slope coefficients in the true process (10) will have positively (negatively) biased performance as the true but unknowable performance of the index is itself negative (positive).

If and only if the index has no residual variation (is perfectly correlated with the common factor), *and has exactly neutral performance* ex post will the observed securities market line generate the same performance measures as the true generating process. In this unlikely circumstance, $\zeta_{It} \equiv 0$ and $\alpha_I = 0$ and thus

$$R_{jt} = \alpha_j + \rho + (\beta_j/\beta_I)(R_{It} - \rho) + \zeta_{jt}, \quad j = 1, \dots N \tag{13}$$

Of course, we have no way to ascertain, either ex ante or for a given sample, whether such conditions are satisfied.

Finally, there remains the possibility that (10) is false and that more than a single common factor affects asset returns. Using a single index securities market line when the true process contains several factors will give incorrect measures of performance. For example, riskless and profitable hedge portfolios may be completely unavailable. Yet a single index that is perfectly diversified in the sense of exact linear dependence on the factors will not be mean/variance efficient. It will, therefore, indicate the presence of non-zero deviations about a computed securities market line.

Appendix

Alteration in Measured Beta with Changes in the Index

As an illustration of some possible beta pitfalls, consider changing the index against which beta is computed while retaining the index' position *on* the mean/variance efficient frontier. There is a functional relation between β_{jp} for individual asset j and the mean return r_p on a mean/variance efficient portfolio p.[18] As illustrated in Figure 3, the relation is nonmonotonic and has a unique maximum and minimum. It follows that a β_{ip} for asset i can be increasing with respect to a movement along the efficient set, (a change in the efficient index), while $\beta_{jp}(j \neq i)$ is at the same time decreasing. In particular, if we begin at the global minimum variance portfolio, all β's are equal to 1.0. Then moving upward toward higher efficient mean returns, some β's will decrease and some will increase, and for those that move in the same direction, the rates of change will generally differ.

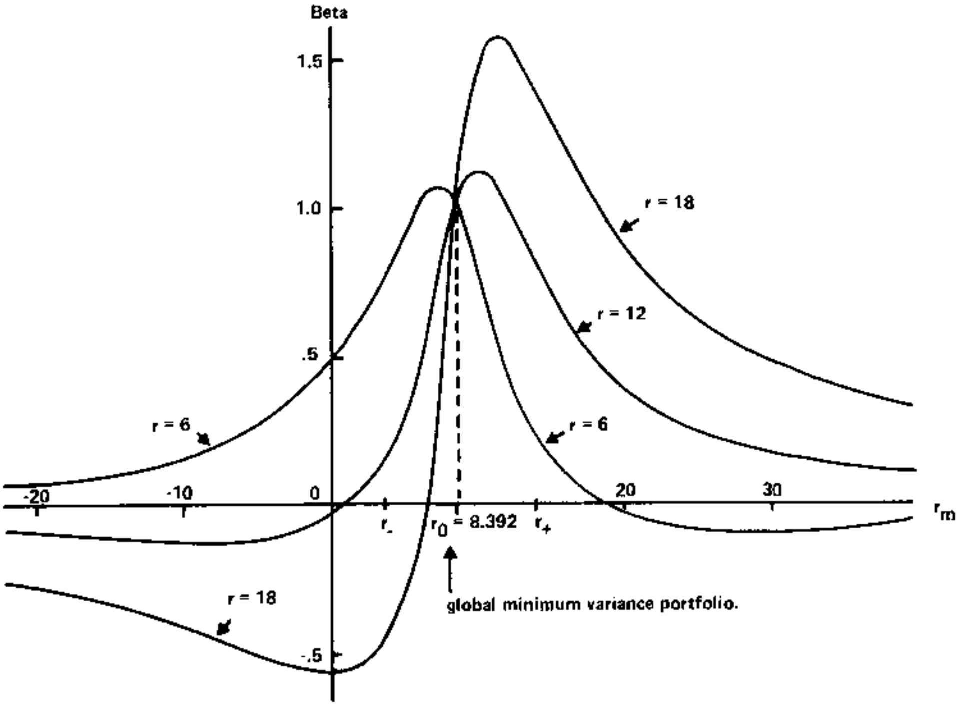

Figure 3. The Variation of Individual Betas According to the Efficient Portfolio Used in Their Calculation

The numbers used to construct this figure were obtained from Black, Jensen, and Scholes [1972]. Using a method described in Roll [1977], the equation of the ex post efficient set was estimated as $\sigma_p^2 = 7993 - 1398\, r_p + 83.3\, r_p^2$, where returns are measured in percent per annum. Movements along this equation were then linked to changes in the betas of three illustrative securities, with mean returns of 6., 12., and 18. percent per annum. Points $r+$ and $r-$ indicate the returns of efficient portfolios which maximized the cross-sectional variance of all individual betas. See Roll [1977, p. 170].

18. The function is derived in Roll [1977, pp. 169–171].

The individual asset's *weight* in the efficient portfolio changes monotonically along the efficient frontier.[19] If the weight decreases (increases) at any point, it decreases (increases) at all points. Suppose it increases with the return. Then it represents an ever larger proportion of efficient portfolios as we move upward in return. And yet, its beta will decrease after a certain point, which should mean, given the usual interpretation, that its "contribution" toward the portfolio's total risk declines. The variance of the efficient index grows without limit as its return increases, yet the betas of *all* individual assets converge to zero. See Figure 3.

There is only one point of intersection for the betas of all assets—at the global minimum variance portfolio, r_0 as shown in Figure 3. This implies that betas give an unambiguous ranking of assets which is the same for all efficient portfolios with return larger than r_0. (It is reversed for portfolios below r_0). Thus, β has a desirable property which admits the following interpretation: "If for any mean/variance efficient index I with $r_I > r_0$ a given pair of individual securities has $\beta_{iI} > \beta_{jI}$ then $\beta_{ip} > \beta_{jp}$ for all p with $r_p > r_0$. Unfortunately, no similar statement is true for non-efficient indices.

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19. For a proof of this assertion, see Roll [1977, pp. 168–169].