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On Some Parity Conditions Encountered Frequently in International Economics

Interest Rate Parity, Purchasing Power Parity, and the Fisher relation between real and nominal interest are intimately connected consequences of optimal multi-period consumption/investment decisions. These three relations hold in their classic form only under complete certainty. With uncertainty, interest rate parity remains unaltered, but more complex equations involving risk premia obtain for purchasing power parity and the Fisher relation. A more complete market (with commodity futures) simplifies considerably these latter two conditions.

1. Introduction

Three well-known pricing relations of international economics are interest rate parity, purchasing power parity, and the Irving Fisher relation.[1] Table 1 gives their classic definitions, which obtain under the assumption of complete certainty. In this case, the relations are merely three

[1]*Historical/Empirical Note.* All the conditions given in Table 1 have a long and large literature marked by disagreement concerning their empirical validity. Interest Rate Parity (IRP) appears to enjoy a growing consensus about its empirical effectiveness, at least for interest rates that are comparable in risk [Cf. Aliber (1973), Frenkel and Levich (1975), and Herring and Marston (1977, ch. 4)]. The so-called "monetary approach to the balance of payments," [Frenkel and Johnson (1976)] usually assumes that IRP is effective [Cf. Whitman (1975) and Darby (1977)]. A review of earlier literature is given by Officer and Willett (1970). Concerning Purchasing Power Parity (PPP), the prevailing opinion argues for its long-run validity, [Gailliot (1970) and Lee (1975)] along with sizeable short-run deviations [Magee (1976) and Brillembourg (1977)]. According to some theories [Dornbusch (1976)], short-run deviations in PPP are a principal mechanism of international adjustment to monetary disequilibrium. A recent and thorough review of PPP is given by Officer (1976).

Concerning the Irving Fisher Relation, its large literature has been marred by the difficulty in observing real interest rates and anticipated inflation. A review until 1971 is provided in Roll (1972). The best recent evidence is given by Fama (1975) and by Jaffee and Mandelker (1976). This evidence supports the basic validity of the relation but also that it holds with a low degree of explanatory power.

The papers listed contain numerous citations to earlier papers about all three relations. The literature extends back through major figures such as Cassel (1921) and Fisher (1907) to the classical economists of the nineteenth century. A short historical review of the early writers is given in Frenkel and Johnson (1976, Ch. 1).

Journal of Macroeconomics, Summer 1979, Vol. 1, No. 3, pp. 267–283.

applications of the "law of one price" to intertemporal and international exchange.

But given uncertainty about relative prices and about loan repayments, there is no a priori reason that their classic forms should be preserved. The classic certainty conditions need not provide a satisfactory description of *observed* interest, exchange, and inflation rates; indeed, a mixed success in describing data is implied by the very extent of the associated empirical literature.

Our purpose here is to give a unified exposition of the classic conditions and of their analogues under uncertainty. We will show how they all arise as implications of the individual's optimal consumption-investment program and how the presence and type of uncertainty affect their forms. They are strongly interdependent, any one of three conditions being implied by the other two (with or without uncertainty). Their forms under general conditions are affected by many factors including the degree of risk aversion among consumers, the length of the observation period, and the completeness of forward markets. Our analysis, though principally expositional, should help assign reasons to the seeming empirical failures of the classic conditions. We look first at the demand side and the relations implied by the optimality conditions for the consumer. Then we will

TABLE 1. *The Three Parity Conditions Under Certainty*

Condition	*Requires*	*Symbolically**
Interest Rate Parity	The difference between foreign and domestic nominal interest rates →	$r_£ - r_\$$
	equals the difference between current forward and spot exchange →	$=f_{£/\$} = s_{£/\$}$
	rates over the same term.	
Purchasing Power Parity	The difference between foreign and domestic inflation rates →	$i_£ - i_\$$
	equals the change in spot exchange rates over the same period. →	$=\Delta s_{£/\$}$
Irving Fisher Relation	The rate of inflation →	i
	is equal to the difference between nominal and real interest →	$=r_{nominal} - r_{real}$
	rates over the same period.	

*Strictly speaking, these relations hold in difference form only for continuously compounded rates of interest and inflation and for logarithms of exchange rates. Thus, if R is a discretely compounded rate, the symbolic descriptions given here refer to the continuously compounded rates, $r = \log_e(R)$. (R is either the raw exchange rate or one plus the rate of interest or inflation.)

turn to the supply side, i.e., to the technical relation among prices due to the economics of international trade and production.

2. The Consumer's Program

Begin by assuming a perfect and complete world with the existence in each country of (1) a capital market, (2) spot and forward exchange markets, and (3) a futures market for each commodity. In this microeconomic model, the consumer allocates his resources among domestic and foreign commodities and assets according to the following rule; he strives to maximize expected utility defined over the consumption of goods originating in all countries in two periods:

$$max\ E\ [U(C_1,\ \tilde{C}_2)],$$

where $C_t' = (C_{t0}, C_{t1}, \ldots, C_N)$ and C_{tj} denotes[2] the quantity consumed of a composite good produced in country j in period t. As the "~" indicates, period one choices are made immediately while period two choices are entirely stochastic, since they are made only after all uncertainty is resolved.

For period one, labor income, y_1, and initial capital wealth, w, are partitioned into consumption and investment as follows:

$$y_1 + w \ = \ \underbrace{S_1' \, p \, L}_{\text{investment}} \ + \ \underbrace{S_1' \, P_1 \, C_1}_{\text{consumption}} \ . \tag{1}$$

(The symbols are defined in Table 2.)

For period two, labor income and receipts from previous investments are completely exhausted on consumption expenditure as follows:

$$\underbrace{\tilde{S}_2' \, \tilde{P}_2 \, \tilde{C}_2}_{\text{consumption}} \ = \ \tilde{y}_2 + \underbrace{\tilde{S}_2 \, L}_{\text{loans}} \ + \ \underbrace{(F - \tilde{S}_2)' \, X}_{\text{Forward market return}} \ + \ \underbrace{\tilde{S}_2'(\tilde{P}_2 - \Pi_2)K}_{\text{Futures market return}} \tag{2}$$

Constraint (2) states that income available for consumption in period two comes from labor income, $\tilde{y}_2$, and from investments. These include loans plus returns on forward exchange and on commodity futures contracts.

The vector of choice variables over which the consumer must maximize is $(C_1, L, X, \tilde{C}_2, K)$. Since the elements of $\tilde{C}_2$ are not under the

[2]The symbol "′" denotes matrix transpose, "~" denotes random variable, and E is mathematical expectation.

TABLE 2. *Definitions of Variables*

w	is initial wealth	expressed in currency units of the home country (country 0).
y_t	is labor income in period t	

S_t, F, C_t, L, X, K are each $(N+1)$ column vectors whose j^{th} elements are; for country j:

Variable	*Definition*	*Units*
S_{tj}	Rate of exchange in period t	$\frac{\text{currency of } 0}{\text{currency of } j}$
F_j	Forward exchange rate	$\frac{\text{currency of } 0}{\text{currency of } j}$
C_{tj}	Consumption in t of composite good purchased in country j	quantity
L_j	Nominal value of loan repayments in country j	currency of j
X_j	Forward exchange contracted in period 1 for delivery in period 2	currency of j
K_j	Goods produced in country j but purchased in futures market in period 1 for delivery in period 2	quantity

p, P_t, Π_2 are diagonal matrices whose j^{th} diagonal elements are:

Variable	*Definition*	*Units*
p_j	Discounted price in period 1 of one unit of currency j loaned until period 2	pure number
P_{tj}	Per unit price of C_{tj}	currency of j
Π_{2j}	Per unit futures price of K_j	currency of j
$i_j = P_{2j}/P_{1j} - 1$	is the rate of commodity price inflation	
$R_j = \frac{1}{p_j} - 1$	is the nominal interest rate in country j	

individual's control in period one, a stochastic calculus must be used [see Kushner (1965)]. This method provides the first-order conditions displayed in Table 3, where the new symbols, J_1 and J_2, are random (scalar) Lagrange multipliers associated with constraints (1) and (2) respectively; U_1 and U_2 are vectors of marginal utilities,[3] and 0 is the $(N+1)$ zero vector.

[3]I.e., $U_t = (U_{t,0} : U_{t,1} : \ldots U_{t,N})$, where $U_{tj} = \dfrac{\partial U}{\partial C_{tj}}$.

TABLE 3. *First-Order Optimality Conditions for the Consumer's Maximization Problem*

Derivative taken with respect to	First-order condition
C_1	$E(\tilde{U}_1 - \tilde{J}_1 P_1 S_1) = 0$
$\tilde{C}_2$	$\tilde{U}_2 = \tilde{J}_2 \tilde{P}_2 \tilde{S}_2$
L	$E(\tilde{J}_1 p S_1 - \tilde{J}_2 S_2) = 0$
X	$E[\tilde{J}_2 (F - S_2)] = 0$
K	$E[\tilde{J}_2 (\tilde{P}_2 - \Pi_2) \tilde{S}_2] = 0$

The conditions listed in Table 3 determine jointly the individual's optimal portfolio, which is composed of varying amounts of loans in $N+1$ different currencies, of forward exchange in N currencies, and of commodity futures contracts in N goods. In principle, these conditions can be solved for individual demand equations that in turn provide market equilibria relations (after aggregation). The explicit market equilibrium equations cannot be deduced without further assumptions about the form of the utility function; nevertheless, some information about the quality of the equilibrium can be obtained directly from the individual's optimality conditions. This information turns out to be representable by the parity relations. We now show that these are implied in every individual's solution whatever set of prices he faces. It follows that they must also hold at the equilibrium set of interest and exchange rates.

3. Interest Rate Parity

The first result implied by the optimality conditions is interest rate parity. Note that constraints associated with X and L together imply the vector equation:

$$pS_1 \, E(\tilde{J}_1) = E(\tilde{J}_2) \, F \ ,$$

where $\tilde{J}_1$ $\tilde{J}_2$ are scalars, and the other variables have been factored out of the expectation, since they are non-stochastic. Any two elements, say j and h, of this vector equation can be used to eliminate $E(\tilde{J}_1)$ and $E(\tilde{J}_2)$ and to obtain the scalar relation:

$$\frac{S_{1,j}\, p_j}{S_{1,h}\, p_h} = \frac{F_j}{F_h} \ .$$

Recall that p_j is the discounted price of one loaned unit of currency j. Thus, its reciprocal, $1/p_j = 1 + R_j$, is unity plus the nominal interest rate, R_j, in country j, giving the interest rate parity formula directly,

$$\frac{1 + R_h}{1 + R_j} = \frac{F_j}{F_h} \frac{S_{1,h}}{S_{1,j}} \text{ ,[4]} \tag{3}$$

and for the "home" country ($h=0$),

$$\frac{1 + R_0}{1 + R_j} = \frac{F_j}{S_{1,j}}, \ j=1, \ldots , N \ . \qquad \textit{Interest Rate Parity} \ \text{(I)}$$

The fact that this expression does not involve variables specific to the investor shows that it is an arbitrage condition. Furthermore, recent empirical evidence [Aliber (1973) and Frenkel and Levich (1974)] shows that interest rate parity is within the permissible deviations induced by trading costs when the variables are properly measured.[5] Indeed a forward exchange market is redundant with an efficient international credit market.

The Interest Rate Parity relation (I) is the only pure arbitrage condition in international financial transactions. It is the only relation that is independent of investor preferences and that holds regardless of the uncertainty in consumption prices and exchange rates. The other two parity conditions hold in their classic forms (of Table 1) only under perfect certainty, as we now demonstrate.

4. The Certainty Case

Given perfect foresight, the first three conditions of Table 3 can be solved to obtain:

$$(1 + R_j) = (U_{1,j}/U_{2,j})(P_{2,j}/P_{1,j}), \ (j=0, \ 1, \ldots , N). \tag{4}$$

$r_j \equiv U_{1,j}/U_{2,j} - 1$ is recognized as the rate of marginal time preference, or the "real" rate of interest, and $i_j \equiv P_{2,j}/P_{1,j} - 1$ is the rate of inflation. Thus (4) is the Irving Fisher (1907) relation between the rate of nominal

[4]Note that F_j/F_h has the dimensions of currency units of country h per unit of country j (and similarly for the spot rates of exchange).

[5]Earlier tests of interest rate parity often suffered from measurement problems. For example, if the forward rate is not for exactly the same term as the interest rates, the relation will not hold (and it should not be expected to hold). Similarly, if the interest rates are for loans that have default risk, there will be deviations from the computed parity. In this article, we only deal with risk in the form of commodity price and exchange rate uncertainty. In the rare cases where uncertainty exists as to the negotiation of arbitrage funds prior to maturity or the imposition of constraints over capital flows, it would create deviations from interest rate parity. This case is not treated here but can be found in Adler and Dumas (1976).

interest expressed in currency of country j and the rate of commodity price inflation for goods produced in the same country:

$$(1 + R_j) = (1 + r_j)(1 + i_j), \ (j = 0, \ldots, N). \qquad \textit{Fisher Relation (II)}$$

Complete certainty is required to obtain this relation, but certainty also implies that forward and *future* spot exchange rates are equal:[6]

$$F_j = S_{2j} \,. \qquad \textit{Forward Relation (III)}$$

If real interest rates were equal in these countries (if marginal intertemporal tastes were identical for the goods of the two countries), we could combine I, II, and III to obtain

$$\frac{S_{1,h}/S_{2,h}}{S_{1,j}/S_{2,j}} = \frac{1 + i_h}{1 + i_j} \qquad (5)$$

and thus for the home country:

$$\frac{S_{1j}}{S_{2j}} = \frac{1 + i_j}{1 + i_0}, \ j = 1, \ldots, N \,. \qquad \textit{Purchasing Power Parity (IV)}$$

Relation (IV) is the Purchasing Power Parity relation: the relative change in exchange rates is equal to the relative inflation.[7] Notice that this relation is verified for any pair of consumption goods (in this case, goods produced in countries h and j). It will also hold, of course, for any collection of goods (price indices) with identical proportions.

Finally, the constraint associated with K implies that commodity prices in the futures market have to be equal to the (future but certain) spot price at the delivery date:

$$\Pi_2 = P_2 \,. \qquad \textit{Futures Relation (V)}$$

5. The Impact of Uncertainty

When the assumption of perfect certainty is dropped, the Fisher and Purchasing Power Parity relations will no longer be valid in their classic forms. There will be corresponding equilibrium relations, of

[6]If the forward rate were not equal to the future spot rate (which is assumed known with certainty), an arbitrage profit could be generated by selling (purchasing) forward exchange and then later by purchasing (selling) spot exchange.

[7]Recall that $S_{1,j}/S_{1,h}$ and $S_{2,j}/S_{2,h}$ have currency units of h per currency unit of country j.

course, but they will involve the subjective estimates of all investors (the moments of their subjective probability distribution) and their attitudes toward uncertainty (utility function curvatures). Whereas under certainty all investors would have the same rate of marginal time preference at equilibrium (and thus individual optimum conditions and the market equilibrium relation would be identical), under uncertainty, every investor might choose an investment allocation that gives a different *anticipated* rate of marginal time preference. We will first assume that commodity futures markets do not exist and then study their influence.

No Commodity Futures Markets

Given uncertainty about exchange rates and inflation, an individual optimum relation similar to the basic Irving Fisher relation (4) is obtained by eliminating J_1, J_2, and S_1 from the first three first order conditions of Table 3. The result is:

$$(1 + R_j) = E\ (\tilde{U}_{1,j}/E\ [\tilde{U}_{2,j}/(1 + \tilde{i}_j)]\ (j=0, \ldots, N). \tag{6}$$

Fisher (IIa)

The similarity in structure between (6) and the corresponding certainty condition (4) permits the deduction of uncertainty analogues to the classic parity relations. In fact, (6) is a direct counterpart to the Fisherian relation, although it cannot be written any more simply without further assumptions. One assumption might be, for example, mutual statistical independence of marginal utility and inflation rate. In this special case, the Fisher relation (II) would remain valid but with i_j being replaced by its expectation and with $1 + r_j$ being defined as the ratio of expected marginal utilities.

Notice, however, that although the stochastic variables might be assessed as mutually independent by one individual, others might perceive some correlation and thus demand risk premia, which means that this revised version of (II) would not necessarily be a market equilibrium condition (unless all individuals perceived mutual stochastic independence).

The problem in interpreting condition (6) for arbitrary utilities stems from the right-hand denominator. It portrays possible interactions between the rate of inflation, i_j, and the marginal utility, $U_{2,j}$, which in principle could depend on first and second period consumption levels of commodities from every country. If we tried to convert (6) into an equation involving *anticipated* values of real interest, exchange rates, and inflation, it would be necessary to append risk premia involving individual investor preferences over the entire range of possible outcomes. Furthermore, individuals might differ in their anticipations (including higher moments of their subjective probability distributions). This shows why aggregation and the depiction of an explicit equilibrium position is not possible for arbitrary utility functions and probability distributions.

An uncertainty analogue for the forward relation (III) is obtained by combining the constraints associated with $\tilde{C}_2$ and X:

$$F_j \cdot E\left[\frac{\tilde{U}_{2j}}{\tilde{P}_{2j} \cdot \tilde{S}_{2j}}\right] = E\left[\frac{\tilde{U}_{2j}}{\tilde{P}_{2j}}\right]. \qquad \textit{Forward Relation} \text{ (IIIa) (7)}$$

Generally the forward exchange rate cannot be equal to the expected (future) spot exchange rate and a risk premium will be appended.

An uncertainty analogue to Purchasing Power Parity (IV) is obtained by combining (6) and (7) and then eliminating the forward exchange rates and interest rates using the Interest Rate Parity arbitrage condition. This procedure gives:

$$\frac{E\ \{\tilde{U}_{2,h}/[(\tilde{S}_{2,h}/S_{1,h})\ (1 + \tilde{i}_h)]\}}{E\ \{\tilde{U}_{2,j}/[(\tilde{S}_{2,j}/S_{1,j})\ (1 + \tilde{i}_j)]\}} = \frac{E\ (\tilde{U}_{1,h})}{E\ (\tilde{U}_{1,j})},\ (j \neq h) \qquad (8)$$

and for the home country:

$$\frac{E\ [\tilde{U}_{20}/(1 + \tilde{i}_0)]}{S_{1j}\ E\ \{\tilde{U}_{2j}/[\tilde{S}_{2j}\ (1 + \tilde{i}_j)]\}} = \frac{E\ (\tilde{U}_{10})}{E\ (\tilde{U}_{1j})}. \qquad \textit{PPP} \text{ (IVa)}$$

Recall that the certainty version of purchasing power parity was obtained by assuming that real interest rates were equal in both countries j and h. (This was equivalent to assuming identical rates of marginal time preference). As the equation above makes evident, the corresponding simplifying assumption with uncertainty is that anticipated marginal utilities for period one consumption are equal for goods of both countries and that period two marginal utilities are equal *and* uncorrelated with exchange rates and inflation rates. Even with these strong requirements, the modified purchasing power parity relation could not be written in terms of *anticipated* exchange and inflation rates alone but would have to involve the higher order subjective movements on future spot exchange rates and inflation rates.

With Commodity Futures Markets

The existence of complete markets brings important simplifications to the parity relations under uncertainty. The first order optimality condition associated with K (futures contracts) can be combined with the condition associated with $\tilde{C}_2$ to give:

$$\frac{1}{\Pi_{2j}} E\,(\tilde{U}_{2j}) = E\left[\frac{\tilde{U}_{2j}}{\tilde{P}_{2j}}\right] \qquad \textit{Futures Relation (Vb)}$$

Thus the reciprocals of commodity futures prices will be equal to expected reciprocals of spot commodity prices if marginal utilities and commodity prices are uncorrelated. Otherwise there will be a bias that depends on the degree of comovement between prices and marginal utilities.

The existence of complete futures markets allows us to simplify the uncertainty analogs of the Fisher and Purchasing Power Parity relations.

The Fisher relation looks like the traditional certainty case:

$$1 + R_j = \frac{E\,(\tilde{U}_{1j})}{E\,(\tilde{U}_{2j})} \cdot \frac{\Pi_{2j}}{P_{1j}}\,, \qquad \textit{Fisher (IIb)}$$

where the real rate may be interpreted as the ratio of expected marginal utilities, and the inflation rate is replaced by the rate implicit in futures prices, Π_{2j}/P_{1j}. If a future market exists, this forward inflation rate is directly measurable, but note that it will not necessarily equal the expected inflation rate, $[\Pi_{2j} \neq E(\tilde{P}_{2j})]$.

Similarly the Purchasing Power Parity relation has an uncertainty analog:

$$\frac{S_{1j}}{F_j} = \frac{E\,(\tilde{U}_{1j})}{E\,(\tilde{U}_{2j})} \cdot \frac{E\,(\tilde{U}_{20})}{E\,(\tilde{U}_{10})} \cdot \frac{\Pi_{2j}}{P_{1j}} \cdot \frac{P_{10}}{\Pi_{20}} \qquad \textit{PPP (IVb)}$$

If real rates of return as measured by the ratio of expected marginal utilities are equal, this relation is similar to the certainty case where futures prices replace expected commodity prices. What is happening is that $E\,(\tilde{U}_{2j}/\tilde{S}_{2j}\,\tilde{P}_{2j})$ in relation (IVa) is replaced by $E\,(\tilde{U}_{2j})/F_j\,\Pi_{2j}$. F_j and Π_{2j} are evidently the "certainty equivalents" of $\tilde{S}_{2j}$ and $\tilde{P}_{2j}$. The existence of forward and futures markets enables us to write the ex ante PPP in a form identical to the certainty case (IV) but where uncertainty and risk aversion parameters are included in the forward and futures prices. This relation would hold ex ante even if goods markets were not perfect (slow shipping etc.).

All these relations are summarized in Table 4.

6. Supply Side

So far, we have considered only the demand side and have implicitly taken commodity prices as exogeneous. We now have to specify more clearly the supply side and consider the relationship between prices of commodities produced in various countries. The supply side is frequently analyzed via the Law of One Price or Purchasing Power Parity as formulated in terms of national price indices based on national "consumption baskets." As long as the baskets (tastes) differ among countries, PPP need not be verified *even in a world of certainty and perfect goods markets* [Cf. Samuelson (1964)]. One must distinguish between nominal variation in the indices (monetary inflation), to which PPP would apply, from relative price variation. Furthermore, the existence of many non-traded (internationally) goods implies that relative price variation may loom as important as nominal price charges, at least for moderate rates of inflation. The long literature of Purchasing Power Parity has thoroughly investigated this possibility. In addition, national differences in tastes and in relative prices can be responsible for exchange risk in foreign asset positions [Cf. Solnik (1974)].

If all commodities were freely and instantaneously transportable, the classic expression for purchasing power parity would hold a posteriori and indeed, would be valid at every moment of time. This reasoning follows from the observation that producers (or holders of commodity inventory) could direct their shipments toward locations that pay the highest prices (expressed in the seller's currency). In the first period, during which an equilibrium is presumed, the ratios of currency prices must be equal to the exchange rate; i.e., $P_{1j}/P_{2h} = S_{1h}/S_{1j}$. If shipments are free and can be made without any delay, the same relation must also hold at any later instant, say period two. The combination of these two equations gives the classic Purchasing Power Parity relation (5).[8] Notice two facts: (1) the assumption of free and immediate shipment is essentially an assumption of a continuous state of equilibrium in the international commodity markets; (2) relative prices of different commodities can diverge from each other, but the divergence must be equal in every country (if the free shipment presumption is satisfied). This statement implies not only that the average inflations of two countries must a posteriori be equal to the change in their exchange rates expressed as a ratio but also that the relative change in nominal price for every individual commodity must adhere to the same relation. A test of this basic idea could be made quite

[8]However, the inflation rates and period two exchange rates in (5) would be random (and thus surmounted by "~"s).

simply: Nominal price changes in individual commodities should follow exchange rate changes more closely the lower their shipping costs (perhaps as measured by geographical distance or by direction information on transportation rates).

Since the purchasing power parity relation must hold a posteriori over a sufficiently long interval to allow free shipping, rational producers must *anticipate* that it will hold, implying that a second uncertainty analogue of the relation can be written as

$$E\left[\frac{(\tilde{S}_{2,h})(1 + \tilde{i}_h)}{(\tilde{S}_{2,j})(1 + \tilde{i}_j)}\right] = \frac{S_{1,h}}{S_{1,j}}. \tag{9}$$

Equation (9) portrays the supply side of international trade in commodities, whereas equation (8) portrayed the demand side. Given free and non-instantaneous shipping, the supply side (9) is strictly analogous to the efficient markets hypothesis so familiar from capital market theory. To understand the analogy, note that if the current spot rate of exchange, $S_{1,h}/S_{1,j}$, did not contain all relevant information about the expected future *real* rate of exchange, a commodity profit could be obtained from shipping goods. By taking account, for example, of any serial dependence in changes in the variable $(S_{t,h}P_{t,h})/(S_{t,j}P_{t,j})$, a speculator could generate extraordinary trading profits. In durable goods, these profits would be attained by buying previous production in one country and selling it in a different country the following period. For a producer of non-durable goods, abnormal profits would be obtained by the shipment of productive factors in advance to those location whose prices are expected to be the highest at the end of the production cycle. Effectively, (9) describes the equilibrium when there is delay in the costless shipment of goods, and when producers must make shipment and production decisions without exact knowledge of their selling price in each market.[9]

The interaction of supply and demand (9) and (8) determine how closely the classic purchasing power parity will be observed. It is interesting to note, however, that some information about equilibrium can be deduced directly. The shorter the observation interval, the more likely demand will be the principle determinant of the empirical data. Thus, for very short intervals, we would expect to observe only a biased purchasing power parity because personal preferences acting through (8) would

[9]If, in particular, producers were to decide in each period where goods were to be allocated the next period, PPP would hold ex ante; i.e., the expected real prices of goods would be equal everywhere as described in (9).

dominate. As the measurement interval becomes longer, however, supply considerations should predominate, and something akin to the classic parity should obtain.

It is important to realize, however, that "longer" is different for each commodity. For commodities such as gold with little direct utility in use, no shipping need take place when ownership is transferred, and the purchasing power parity must be nearly valid daily. For mangoes, the destruction of supply by a single truck accident in Belgium might create a 4-month disparity between Belgian and Australian mango prices. The price differential for butlers might differ for decades by as much as the first class air fare plus the cost of customs harassment.

There are numerous imperfections in the international supply of commodities which will cause deviations from PPP (non-proportional transportation costs, export-import taxes, quotas and particularly non-traded commodities). With few exceptions,[10] these imperfections are difficult to incorporate in a simple analytical framework, and only some heuristic conclusions have been given here.

It might be interesting to compare these results with the monetarist approach to international finance. Following Mundell (1968), Dornbusch (1976), for example, stresses the instantaneity of adjustment in financial markets as opposed to the stickiness of adjustment in the real sector. His macroeconomic, certainty-world model assumes that all five relations of parity hold in the long run in their classic forms (cf. the certainty column of Table 4). Over the short run, relations involving only financial variables, (i.e., I and III) will be valid, while the Fisherian relation (II) and PPP (IV) will generally be in disequilibrium (because of the slow adjustment of commodity prices). Such an approach based on the convergence towards a long-term equilibrium, whose conditions cannot be regarded as certain, lacks the explicit incorporation of uncertainty and ignores risk aversion. Furthermore, as we have shown, the existence of complete markets effectively makes all five relationships mixed; i.e., they all contain both financial and real components. As seen in the last column of Table 4, even Purchasing Power Parity (IV) contains a commodity future price (and a commodity future is a financial contract, not a real quantity). It would therefore appear that the assumptions of disequilibrium in some of the parity relations is inconsistent with equilibrium in all financial markets when futures contracts are available for all commodities.

[10]One exception would be ad valorem or proportional costs. If commodities are exclusively produced in one country and shipped to others, PPP would hold for that commodity because PPP is a relative price formula. Proportional costs will have the same influence in period one and two and therefore will cancel [cf. Grauer, Litzenberger and Stehle (1976)].

TABLE 4. *Parity Conditions Under Three Views of the World*

	Certainty	*Uncertainty* *No Futures Market*	*Uncertainty* *Futures Market*
I. Interest Rate Parity	$\frac{1+R_0}{1+R_j} = \frac{F_j}{S_{1j}}$	$\frac{1+R_0}{1+R_j} = \frac{F_j}{S_{1j}}$	$\frac{1+R_0}{1+R_j} = \frac{F_j}{S_{1j}}$
II. Fisher Relation	$(1+R_j) = \frac{U_{1j}}{U_{2j}} \cdot \frac{P_{2j}}{P_{1j}}$	$(1+R_j) = \frac{1}{P_{1j}} \cdot \frac{E(\tilde{U}_{1j})}{E(\tilde{U}_{2j}/\tilde{P}_{2j})}$	$1+R_j = \frac{E(\tilde{U}_1)}{E(\tilde{U}_{2j})} \cdot \frac{\Pi_{2j}}{P_{1j}}$
III. Forward Relation	$F_j = S_{2j}$	$F_j = \frac{E(\tilde{U}_{2j}/\tilde{P}_{2j})}{E(\tilde{U}_{2j}/\tilde{P}_{2j}\tilde{S}_{2j})}$	$F_j = \frac{1}{\Pi_{2j}} \cdot \frac{E(\tilde{U}_{2j})}{E(\tilde{U}_{2j}/\tilde{P}_{2j}\tilde{S}_{2j})}$
IV. Purchasing Power Parity	$\frac{S_{1j}}{S_{2j}} = \frac{P_{2j}}{P_{1j}} \cdot \frac{P_{10}}{P_{20}}$	$S_{1j}\frac{E(\tilde{U}_{2j}/\tilde{S}_{2j}\tilde{P}_{2j})}{E(\tilde{U}_{20}/\tilde{P}_{20})} = \frac{E(\tilde{U}_{1j})}{E(\tilde{U}_{10})} \cdot \frac{P_{10}}{P_{1j}}$	$\frac{S_{1j}}{F_j} = \frac{E(\tilde{U}_{1j})}{E(\tilde{U}_{2j})} \cdot \frac{E(\tilde{U}_{20})}{E(\tilde{U}_{10})} \cdot \frac{\Pi_{2j}}{P_{1j}} \cdot \frac{P_{10}}{\Pi_{20}}$
V. Futures Relation	$\Pi_{2j} = P_{2j}$		$\Pi_{2j} = \frac{E(\tilde{U}_{2j})}{E(\tilde{U}_2/\tilde{P}_{2j})}$

Here we are speaking only of traded commodities for which futures markets exist. PPP would not be expected to hold for perishable and non-traded goods, not necessarily indicating a disequilibrium. As a practical matter, futures markets are not complete, even for durable goods, so an interesting avenue for future research might be the measurement of PPP deviations for commodities or productive factors which have no futures markets. Presumably, these deviations could be effective predictors of payment imbalances and capital flows.[11]

7. Conclusion

Three classic conditions of international equilibrium are the Interest Rate Parity, the Purchasing Power Parity, and the Irving Fisher relation between nominal and real interest. We have shown how all three are implied by the optimal consumption-investment program of an internationally-oriented consumer, i.e., a consumer who can invest both at home and abroad and who can import goods for his own consumption.

Only the first condition, Interest Rate Parity, survives in the same form as a strict arbitrage relation under uncertainty. Analogues to the other two conditions exist under uncertainty, but their algebraic forms are altered. They might contain biases that reflect individual attitudes toward risk.

The Purchasing Power Parity condition is particularly interesting because there is a corresponding supply side. Under certainty, both consumers and producers have incentives to act in such a way that the classic purchasing power parity obtains. Under conditions of uncertainty, however, their behavior can be somewhat divergent. If the consumer side dominates, as it would if there were absolute restrictions on the international transfer of either a finished product or its constituent factors, then the purchasing power relation would probably be biased—again to reflect risk preferences of consumers. We argue, however, that the longer the estimation interval, the more likely the observation of a classic purchasing power parity, as enterprising producers seek out and take advantage of internationally divergent real prices.

The existence of complete commodity futures markets simplifies the forms of the parity relations. With complete markets, the commodity

[11]This empirical methodology has already been implemented [e.g., by Brillembourg (1977)] but only with broad price indexes. The results were not satisfactory for the hypothesis that PPP deviations are predictors of capital flows but perhaps the failure was due to the implicit measurement error introduced by using price indexes instead of a single good.

futures price replaces the future (unknown) spot price, so that all the parity relations contain at least one financial variable (as well as real variables).

Received: August, 1978

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