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Valuing futures and options on volatility

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Abstract

This paper presents simple closed-form expressions for volatility futures and option prices and examines their implications for the characteristics of these securities. We show that the properties of these volatility derivatives are fundamentally different from those of conventional option and futures contracts. This analysis also provides insights into the role that volatility derivatives may play in managing and hedging volatility risk in financial markets.

JEL classification: G13

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1. Introduction

Few proposed types of derivative securities have attracted as much attention and interest as futures and option contracts on volatility. In April 1993, Reuters began reporting the VIX index which tracks the implied volatility of S&P 100 index calls and puts. The Wall Street Journal recently reported that the Chicago Board Options Exchange plans to unveil options on the VIX index shortly. A recent issue of Futures reports that the American Stock Exchange is also consider-

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ing developing volatility options on the U.S. stock market and that market regulators have privately endorsed the concept. Articles in the November 1993 issue of Futures and Options World and the July 25, 1994 issue of Barrons explain how volatility derivatives could be used to hedge the volatility risk of option portfolios. Volatility swaps have recently begun to trade in the over-the-counter market. In Europe, the German Futures and Options Exchange (DTB) launched a volatility index called VDAX on December 5, 1994. This index tracks the implied volatility of DAX index calls and puts. The DAX index is a value-weighted index of the 30 largest firms traded on the Frankfurt Stock Exchange. The Austrian Futures and Options Exchange (OTOB) announced a volatility index on its Austrian Traded Index (ATX) calls and puts for 1995. Volatility futures and options on volatility indexes are currently being developed by a number of investment banking firms in the U.S. and Europe.

Futures and options on volatility are clearly fundamental types of derivative securities. By allowing investors to hedge directly against shifts in volatility, these securities enable investors to avoid the costs of dynamically adjusting positions for changes in volatility and serve to make the market more complete. [1] One trader stated "by being able to buy and sell volatility, investors would now be able to manage their risk in two dimensions – price risk and volatility risk – an opportunity previously out of reach for all but the large and sophisticated options users."

This paper derives simple closed-form valuation expressions for a variety of volatility derivatives. We then examine the implications of the valuation expressions for the properties of these securities. The objective of this analysis is to understand how these derivative securities differ from more conventional futures and options and to provide some insights into the economic role that volatility derivatives may play. The valuation model we use captures many of the observed properties of volatility. In particular, the model allows volatility to be mean reverting and conditionally heteroskedastic.

We first examine the properties of volatility futures prices. We show that these prices can differ from those implied by the standard cost-of-carry model in a number of important ways. For example, volatility futures prices are bounded above zero and the basis can be either positive or negative. In addition, we show that longer-horizon volatility futures can be virtually useless as hedging vehicles.

We then derive valuation expressions for volatility options. These options have many surprising properties and are fundamentally different from conventional options. For example, the price of a volatility call can be below its intrinsic value, the traditional put–call parity relation does not hold for these options, and the

[1] Brenner and Galai (1989) and Whaley (1993) provide excellent discussions of how volatility derivatives can be used to hedge the volatility risk of portfolios containing options or securities with option-like features.

pattern of time decay is perverse. The underlying reason why these options behave so differently is that volatility is not the price of a traded asset.

As with futures, these results have important implications for the hedging behavior of volatility options. We show that prices of volatility calls and puts become less sensitive to the current level of volatility as their time to expiration increases. This dampening effect implies that the deltas of longer-term options tend toward zero. Furthermore, the absolute value of the delta for a volatility option is always less than one, and is generally less than some fixed number. These features dramatically alter the way in which these options can be used to hedge the volatility risk of option portfolios, and have fundamental implications for the way that volatility option contracts should be designed.

Finally, we focus on the valuation of options on volatility futures. We show that volatility futures options can be valued as if they were simple volatility options by making a transformation of the strike price. A surprising implication of this is that for some strike prices, these options are priced as if guaranteed to be in the money or out of the money at expiration.

The remainder of this paper is organized as follows. Section 2 presents the basic valuation framework for volatility derivatives. Section 3 applies the model to volatility futures contracts and examines their properties. Section 4 considers the properties of volatility option prices. Section 5 addresses the valuation of volatility futures options. Section 6 summarizes the results and makes concluding remarks.

2. The valuation framework

In this section, we present the basic valuation framework within which specific expressions for volatility derivative securities are derived. Although the discussion is couched in terms of stock index volatility, the framework could also be applied to derivatives on other types of volatility such as currency or interest-rate volatility. In fact, these results could easily be extended to include derivatives on such diverse non-price state variables as inflation rates, casualty insurance claims, or health care cost indexes.

Let V denote the current value of the standard deviation of returns for the stock index. This standard deviation can be either the instantaneous volatility or the volatility implied from some option pricing model – the distinction does not affect the form of the resulting valuation expressions. [2] Let the dynamics of V be given by

$$dV = (\alpha - \kappa V)dt + \sigma\sqrt{V}\, dZ, \tag{1}$$

[2] For example, Taylor and Xu (1994) show that when volatility is stochastic, the volatility estimate implied by inverting the Black–Scholes model is nearly a linear function of the actual instantaneous volatility.

where α, κ, and σ are constants, and Z is a standard Wiener process. This framework is conceptually similar to that used by Wiggins (1987), Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Stein and Stein (1991), and Heston (1993).

This specification of the volatility dynamics is also consistent with many of the observed properties of stock index volatility. Empirical evidence by French et al. (1987), Harvey and Whaley (1992), and Sheikh (1993) suggests that index volatility follows a mean-reverting AR(1) process. Regressions of squared changes in implied volatility on volatility levels indicate that the variance of changes in implied volatility is not constant, but increases with the level of volatility. The dynamics for V given in (1) capture both of these features. In addition, these dynamics imply that V is always positive and has a long-run stationary gamma-distribution. [3] The mean and variance of the stationary distribution are α/κ and $\alpha\sigma^2/2\kappa^2$. Similarly, the first-order serial correlation for V is $e^{-\kappa\Delta t}$. Given estimates of the mean, variance, and serial correlation of V from indexes such as the VIX or VDAX, the parameters α, κ, and σ^2 can easily be obtained by inverting the analytical expressions for the mean, variance, and serial correlation.

Now consider the valuation at time zero of a contingent claim with a payoff $B(V_T)$ at time T depending only on V_T, the time-T value of V. Since V is not the price of a traded asset, we allow for the possibility that volatility risk is priced by the market. Consistent with Wiggins (1987), Stein and Stein (1991), and others, we make the assumption that the expected premium for volatility risk is proportional to the level of volatility, ζV. We note that this assumption is similar to the implications of general equilibrium models such as Cox et al. (1985), Hemler and Longstaff (1991), and Longstaff and Schwartz (1992), in which risk premia in security returns are proportional to the level of volatility. We also make the usual assumptions that markets for securities are perfect, frictionless, and are available for continuous trading. Furthermore, we assume that the riskless interest rate r is constant.

It is important to note that the basic nature of our results is unaffected by whether V can be expressed as a non-linear function of other security prices as in the case where V is the implied volatility of an option. This is because V cannot be replicated by a self-financing portfolio of the option and the index, even though a non-self-financing portfolio can be constructed which replicates V exactly. The intuition for why this is true is related to the fact that all securities must earn the riskless rate of return in the risk-neutral economy. Thus, all self-financing portfolios must earn the riskless rate. As implied by (1), however, the 'expected

[3] The geometric Brownian motion process for V used by Hull and White (1987) implies that V is non-stationary. Similarly with the constant elasticity of variance process assumed by Johnson and Shanno (1987). The Ornstein–Uhlenback process used by Scott (1987) and Stein and Stein (1991) implies that negative values of V are possible.

return' on V, can be either positive or negative and generally will not equal the riskless rate. This means V cannot be the value of a self-financing portfolio of securities and that standard hedging arguments are not applicable. Note that this issue is different from the issue of whether we can substitute out the value of V in the pricing expressions using the non-linear mapping between V and observed market prices. This latter issue is simply whether it is possible to make a change of variables. If so, the change of variables is mathematically neutral and does not affect any of the properties of the pricing expressions.

Given this framework, the current value of this claim, $A(V,T)$, satisfies the fundamental valuation equation

$$\frac{\sigma^2}{2} VA_{VV} + (\alpha - \beta V) A_V - rA = A_T, \tag{2}$$

where $\beta = \kappa + \zeta$, subject to the expiration-date condition

$$A(V_T, 0) = B(V_T). \tag{3}$$

Let $D(T)$ denote the current price of a T-maturity riskless unit discount bond. The solution to this partial differential equation can be expressed as

$$A(V,T) = D(T) E[B(V_T)], \tag{4}$$

where the expectation is taken with respect to the risk-adjusted process for V

$$\mathrm{d}V = (\alpha - \beta V)\mathrm{d}t + \sigma\sqrt{V}\,\mathrm{d}Z. \tag{5}$$

From Feller (1951) and Cox et al. (1985), this risk-adjusted process implies that γV_T is distributed as a non-central chi-squared variate with ν degrees of freedom and non-centrality parameter λ, where

$$\gamma = \frac{4\beta}{\sigma^2(1 - \exp(-\beta T))},$$

$$\nu = \frac{4\alpha}{\sigma^2}, \tag{6}$$

$$\lambda = \gamma \exp(-\beta T)\, V.$$

The non-central chi-squared distribution is described in Ch. 28 of Johnson and Kotz (1970). With this representation of solutions to the partial differential equation in (2), valuation expressions for volatility derivatives can be obtained by directly evaluating the expectation in (4).

3. Volatility futures

In this section, we derive futures prices for futures contracts on volatility and examine some of the properties of these prices. [4]

[4] Since the riskless interest rate is not stochastic, volatility futures and forward prices are identical.

3.1. The volatility futures model

Let $F(V,T)$ denote the futures price for a futures contract on V with maturity T. Following (4) and Eq. (46) of Cox et al. (1981), the futures price can be expressed as the expected value of V at time T

$$F(V,T) = E[V_T], \tag{7}$$

where the expectation is taken with respect to the risk-adjusted process for V. Evaluating this expectation gives the following expression for the volatility futures price

$$F(V,T) = (\alpha/\beta)(1 - \exp(-\beta T)) + \exp(-\beta T)V. \tag{8}$$

In this model, volatility futures prices are exponentially weighted averages of the current value of V and the long-run mean α/β of the risk-adjusted process. As $T \to 0$, the futures price converges to the current value of V. As $T \to \infty$, the futures price converges to α/β. Note that since the futures price is the expected value of V taken with respect to the risk-adjusted process for V, the futures price generally will be a biased estimate of the actual expected future spot value of V.

3.2. Properties of volatility futures prices

Volatility futures prices have a number of interesting properties. For example, as $V \to 0$, the volatility futures price does not converge to zero. Thus, volatility futures prices are bounded above zero. The intuition for this is related to the mean reversion of the volatility process. When V reaches zero, V immediately returns to positive values. Thus, the expected value of V_T is strictly greater than zero even when the current value of V is zero. This feature of volatility futures prices contrasts with those of futures prices on traded assets. The lower bound on futures prices also has important implications for pricing futures options which is discussed later.

Another important property of volatility futures prices is that their hedging effectiveness is a function of their maturity. In particular, the partial derivative of $F(V,T)$ with respect to V is $\exp(-\beta T)$. This means that volatility futures prices do not move in a one-to-one ratio with changes in V. Furthermore, as T increases, a change in V has less of an effect on the futures price. In the limit as $T \to \infty$, futures prices approach the long-run mean α/β of the volatility process and are unaffected by the current value of V. For this reason, longer-term futures contracts may not be effective instruments for hedging volatility risk. The intuition for this is again related to the mean reversion of V. Any change in the current value of V is expected to be partially reversed prior to the expiration of the contract. Hence, volatility futures prices move sluggishly in response to volatility shocks. [5]

[5] Brenner and Galai (1989) and Whaley (1993) describe how volatility futures can be used to hedge volatility risk, but do not provide explicit solutions for volatility futures prices.

The basis for volatility futures is given by

$$(\alpha/\beta - V)(1 - \exp(-\beta T)). \tag{9}$$

The basis converges to zero as $T \to 0$, and converges to the difference between the current value of V and the long-run mean α/β as $T \to \infty$. An implication of this is that the volatility futures basis can be either negative or positive, depending on whether the current value of V is above or below its long-run mean α/β. This again contrasts with the properties of futures on traded commodities or assets.

Finally, it is also useful to derive the dynamics of the futures price. This is important since if volatility futures prices were to become observable, one might suspect that it would be easier to value derivatives on volatility in terms of the futures price rather than V. The dynamics for the futures price are given by a simple application of Ito's Lemma to (8),

$$\mathrm{d}F = \sigma \exp(-\beta(T-t)) \sqrt{V}\, \mathrm{d}Z. \tag{10}$$

Not surprisingly, since (7) implies that $F(V,T)$ is an expectation, the futures price is a martingale. Inverting (8) and substituting in for V gives

$$\mathrm{d}F = \sigma \exp(-\beta(T-t)) \times \sqrt{F \exp(\beta(T-t)) - (\alpha/\beta)(\exp(\beta(T-t)) - 1)}\, \mathrm{d}Z. \tag{11}$$

Since F is a Markov process, derivatives on V can be valued using the futures price as the underlying state variable. Note, however, that the market price of risk still appears in the dynamics for F via the β parameter. Thus, using the futures price rather than V does not eliminate the need to use risk-adjusted parameters in the model. In addition, the dynamics for F are now time dependent and more complex than the dynamics of V given in (5). Furthermore, since F is not lognormally distributed, the Black (1976) framework is not applicable in pricing derivatives. These considerations suggest that there may be limited benefits to using the futures price rather than the volatility index in valuing volatility derivatives.

4. Volatility options

In this section, we derive valuation expressions for volatility options and examine their implications for the properties of these securities. We focus first on European options since current proposals for volatility option prices have European exercise features.

4.1. Valuation expressions

Let $C(V,K,T)$ denote the current value of a call option on V, where K is the strike price of the option and T is the time until expiration. From (4), the value of the call can be expressed as

$$C(V,K,T) = D(T)\,E\big[\max(0,V_T - K)\big]. \tag{12}$$

Evaluating this expectation gives the following closed-form expression for the value of a volatility call option

$$\begin{aligned} C(V,K,T) = {} & D(T)\exp(-\beta T)\,V Q(\gamma K|\nu+4,\lambda) \\ & + D(T)(\alpha/\beta)(1-\exp(-\beta T))\,Q(\gamma K|\nu+2,\lambda) \\ & - D(T)\,K Q(\gamma K|\nu,\lambda), \end{aligned} \tag{13}$$

where $Q(\cdot|\nu+i,\lambda)$ is the complementary distribution function for the non-central chi-squared density with $\nu + i$ degrees of freedom and non-centrality parameter λ. [6] The volatility call price is an explicit function of V and T, and depends on the exercise price K, the riskless interest rate r through $D(T)$, and the parameters of the risk-adjusted volatility process α, β, and σ.

In some ways, this expression for the value of a volatility call resembles the yield option formula derived in Longstaff (1990). This is because Longstaff assumes that the short-term interest rate follows a square-root process similar to (1). Despite this, however, there are many differences between the two models. In particular, the discount factor is uncorrelated with the expected payoff of the option in this model. In contrast, the discount factor and the expected payoff of the option are perfectly negatively correlated in the Longstaff yield option model. This distinction leads to significant differences between the pricing behavior of volatility options and yield options. For example, the delta of a volatility call is always positive while the delta of a yield call can be negative. Thus, the similarity between this model and the Longstaff yield option model is largely superficial.

The price of a European put $P(V,K,T)$ is given by the following put–call parity relation for volatility options

$$P(V,K,T) = C(V,K,T) - D(T)F(V,T) + D(T)K. \tag{14}$$

This put–call parity relation differs from the put–call parity relation for options on traded assets. The reason for this is that the present value of a portfolio that pays V at time T is not equal to the current value of V. Rather, it equals the present value of the futures price. Thus, the put–call parity relation for volatility options differs from the usual put–call parity relation by the substitution of $D(T)F(V,T)$ for V on the right-hand side of (14).

Computing call and put prices requires calculating the complementary non-central chi-squared distribution. While this function is straightforward to evaluate, it is often more convenient to calculate its value using the highly accurate normal approximation suggested by Sankaran (1963). This approximation is

$$Q(\gamma K|\nu,\lambda) = 1 - N(d), \tag{15}$$

[6] The derivation is available upon request from the authors.

where $N(\cdot)$ is the cumulative standard normal distribution function and

$$d = k\left(\left(\frac{\gamma K}{\nu+\lambda}\right)^{h} - l\right),$$

$$h = 1 - \tfrac{2}{3}(\nu+\lambda)(\nu+3\lambda)(\nu+2\lambda)^{-2},$$

$$k = \left\{h^2 \frac{2(\nu+2\lambda)}{(\nu+\lambda)^2}\left[1-(1-h)(1-3h)(\nu+2\lambda)(\nu+\lambda)^{-2}\right]\right\}^{-1/2},$$

$$l = 1 + h(h-1)\frac{\nu+2\lambda}{(\nu+\lambda)^2} - h(h-1)(2-h)(1-3h)\frac{(\nu+2\lambda)^2}{2(\nu+\lambda)^4}.$$

An advantage of this approximation is that call and put prices can be calculated using the same types of programming routines used to compute Black and Scholes (1973) option prices. Examples of the accuracy of this algorithm are presented in Ch. 28 of Johnson and Kotz (1970).

4.2. Properties of volatility calls

Taking the limit of the volatility call valuation expression in (13) shows that the solution satisfies the expiration date condition $C(V_T, K, 0) = \max(0, V_T - K)$. This follows since $Q(\gamma K|\nu,\lambda)$ converges to zero when $K < V$, and converges to one when $K \geq V$.

The first major difference between the properties of volatility calls and calls on traded assets is that $C(V,K,T)$ does not converge to zero as $V \to 0$. This is shown in Fig. 1, which graphs the values of volatility calls as a function of V for various values of T. The parameters used in Fig. 1 imply a long-run mean and standard deviation for V of 0.15 and 0.05, respectively.[7] The reason why the call price does not converge to zero as $V \to 0$ is related to the mean-reverting behavior of V. If the value of V ever reaches zero, the process for V immediately returns to non-zero values. Consequently, the value of a volatility call is not zero when $V = 0$ since the call could still be in the money on its expiration date. In contrast, if the price of a traded asset equals zero, there is no possibility that the price of the asset will eventually become non-zero (otherwise the present value of the asset could not be zero). This is also the reason why there are three terms in (13) rather than the usual two terms in models such as the Black and Scholes (1973) equation.

[7] The parameters used in the figures are consistent with findings in the literature. We assume a long-term volatility of 0.15, which is similar to estimates reported by Harvey and Whaley (1992) for implied volatilities on the OEX. The same holds for σ^2 which is equal to 0.133 in the figures. We assume a half-life of 90 trading days for volatility shocks. This is consistent with many recent papers which find that shocks in volatility are highly transitory.

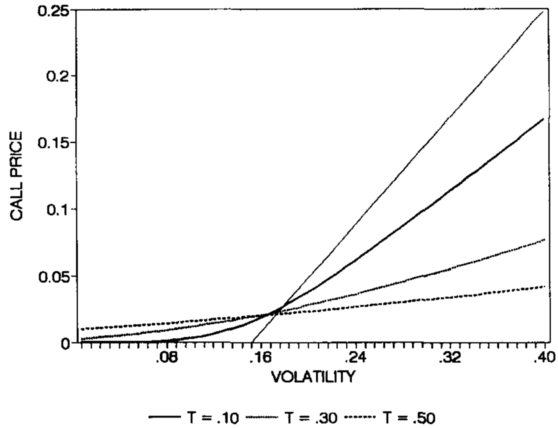

Fig. 1. The value of a volatility call graphed as a function of the underlying volatility. T denotes the maturity of the option in years. The parameters used are $\alpha = 0.60$, $\beta = 4.00$, $\sigma^2 = 0.133$, $r = 0.05$, and $K = 0.15$.

The extra term reflects that the option still has value even when $V = 0$ (when $V = 0$, the first term in (13) disappears).

Because the value of a volatility call is greater than zero when $V = 0$, the volatility call does not satisfy the upper boundary restriction derived by Merton (1973) for calls on traded assets. In particular, the value of a volatility call exceeds the numerical value of the underlying variable V for sufficiently small V. Intuitively, the reason for this property is that the expected payoff for a volatility call is higher than the current value of V when V is small because of the mean-reverting behavior of volatility. Note that since V is not the price of a traded asset, the violation of the Merton (1973) upper boundary does not imply the existence of arbitrage opportunities.

Differentiation shows that volatility calls are increasing functions of V. In the limit, the value of a volatility call approaches $D(T)\exp(-\beta T)V$, which becomes infinite as $V \to \infty$. An important property of this limit, however, is that the value of a volatility call becomes less than its intrinsic value for some value of V. This is also illustrated in Fig. 1, which shows that the price of a volatility call can be less that its intrinsic value when the call is only slightly in the money. The reason for this property again follows from the mean reversion of volatility. When V is above its long-run mean, mean reversion implies that the expected future value of V will be lower than its current value. This implies that the expected payoff for a volatility call can be less than its current intrinsic value – the expected change in V is negative. This could not occur if V were the price of a traded asset since negative returns would not be consistent with the absence of arbitrage. As before, the violation of the Merton (1973) lower boundary restriction by volatility call options does not imply the existence of arbitrage opportunities.

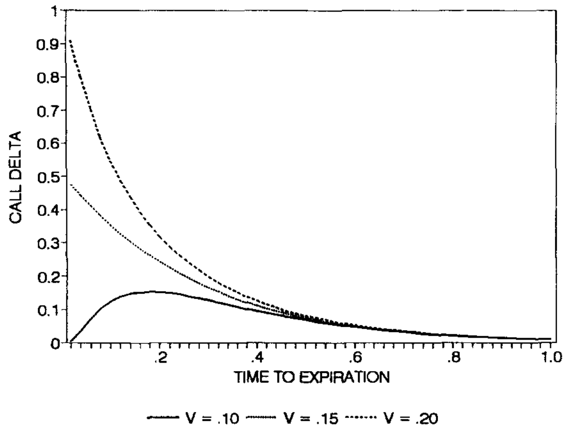

Fig. 2. The delta of a volatility call graphed as a function of its time until expiration measured in years. The parameters used are $\alpha = 0.60$, $\beta = 4.00$, $\sigma^2 = 0.133$, $r = 0.05$, and $K = 0.15$.

As with volatility futures prices, the pricing expression for volatility calls has implications for the hedging behavior of these options. By inspection of (13), the volatility call price depends on V only through the term $\exp(-\beta T)V$. This means that when $\exp(-\beta T)$ is small, V has little influence of the current value of the call option. In other words, as T increases, the delta of the call approaches zero and the graph of the call value is essentially flat for relevant ranges of V. This can be seen in Fig. 1, where the call price flattens out as T increases. Similarly, Fig. 2 shows that the deltas of at-the-money and in-the-money calls can decrease as T increases. For large enough T, the deltas of all calls approach zero.

An immediate implication of these results is that longer-maturity call options have little or no value as hedging instruments since their prices are not affected by changes in V. The intuition for why this property holds is related to the half-life of deviations in V from its long-term mean. Assume that a sudden increase in V occurs. The dynamics of V imply that roughly one-half of the deviation will be eliminated in $1/\beta$ periods. If the horizon of the option T is many times that of the half-life, then an increase in V will have little effect of the expected payoff of the options and, therefore, on the current price of the call. Thus, the sensitivity of call prices to changes in the current value of V is dampened by the effects of mean reversion.

Although longer-maturity calls have deltas near zero, shorter-maturity calls can be used to hedge. An important feature of volatility calls, however, is that their deltas are bounded above by $D(T)\exp(-\beta T)$. This upper bound on the delta of the call can be a significant restriction on the hedging properties of even short-term options. This is illustrated in Fig. 3, which plots the delta of a call as a function of V for various values of T. For example, the short-term call with $T = 0.10$ has a delta that is never greater than 0.70. The deltas for the longer-ma-

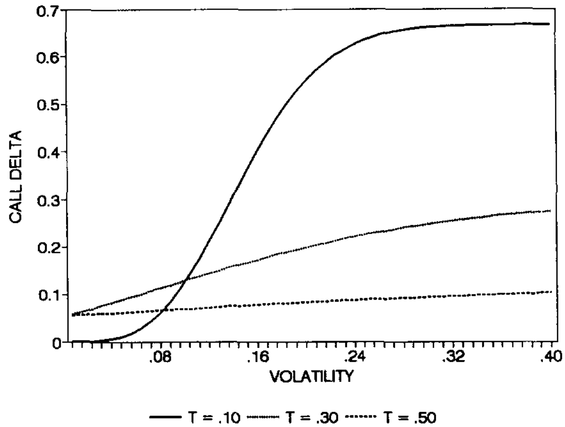

Fig. 3. The delta of a volatility call graphed as a function of the underlying volatility. T denotes the maturity of the option in years. The parameters used are $\alpha = 0.60$, $\beta = 4.00$, $\sigma^2 = 0.133$, $r = 0.05$, and $K = 0.15$.

turity calls never exceed 0.30. Note that the deltas for the longer-maturity calls are also bounded above zero as $V \to 0$.

The second derivative of C with respect to V, the gamma of the call, is positive. As with options on traded assets, the gamma of a volatility call is highest for near-expiration at-the-money options. Consequently, this feature makes it clear that hedgers need to monitor the delta of these types of option positions carefully since the delta of the position can change significantly in response to small changes in V. In contrast, the gamma for a longer-maturity volatility call is near zero for all values of V.

In contrast to the Black–Scholes formula, the value of a volatility call option is not always an increasing function of T. In fact, the limit of $C(V,K,T)$ as $T \to \infty$ equals zero. This can be seen in Fig. 4, which graphs volatility call prices as a function of T. Intuitively, the reason for this property is that V has a long-run stationary distribution. Consequently, as T increases, the expected payoff for the call option is bounded. However, as T increases, the value of $D(T)$ used to discount the expected payoff approaches zero. As a result, the product of $D(T)$ and the expected payoff converges to zero. Because of this property, the sign of the derivative C_T, the theta of the call, is ambiguous. For small T, theta can be greater than zero. As T increases, theta ultimately becomes negative. The theta of volatility calls is greatest in absolute terms for short-term in-the-money options.

The effect of an increase in the strike price of a volatility call is always negative. Interestingly, an increase in K does not have an effect symmetric to a decrease in V. An increase in K has a significant effect on the prices of both short-term and long-term calls. In contrast, a decrease in V has little effect on the value of a long-term call. Thus, the notion of 'moneyness' is subtly different for

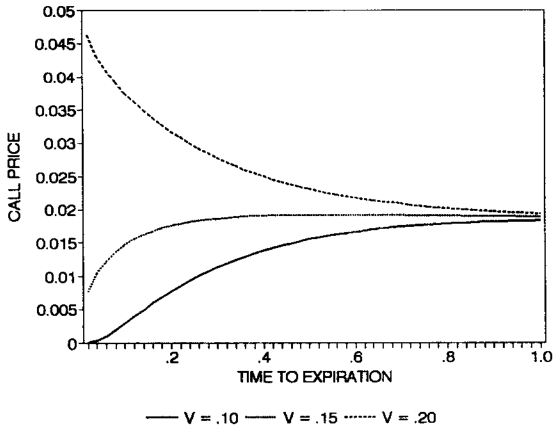

Fig. 4. The price of a volatility call graphed as a function of its time to expiration measured in years. The parameters used are $\alpha = 0.60$, $\beta = 4.00$, $\sigma^2 = 0.133$, $r = 0.05$, and $K = 0.15$.

these types of calls. The 'moneyness' of a call is a function not only of the difference between the current value of V and K, but also of the difference between the long-run mean of V and K.

In the Black–Scholes model, call options are increasing functions of the riskless rate r. The intuition for this is that an increase in r increases the upward drift of the risk-neutral process for the underlying asset. In contrast, volatility calls are decreasing functions of the riskless rate, through the discount factor $D(T)$. This is because an increase in r has no effect on the drift of the process for V. Consequently, the only effect of an increase in r is to reduce the discount factor $D(T)$, which in turn, reduces the value of the volatility call.

Finally, volatility call prices also depend on the values of the α, β, and σ^2 parameters. To examine the sensitivity to changes in these parameters, we compute the elasticity of the price of an at-the-money volatility call with strike price $K = 0.25$ and $T = 0.25$. A one percent increase in the value of α increases the call price by 2.7 percent; a one percent increase in β decreases the call price by 2.8 percent; and a one percent increase in σ^2 increase the call price by 4.0 percent.

Brenner and Galai (1989) and Whaley (1993) also present models for volatility option prices. Although innovative, these models do not consider the effects of mean reversion on the option prices. Hence, the properties of volatility options implied by these models are similar to those implied by the Cox et al. (1979) or Black (1976) models.

4.3. Properties of volatility puts

As for volatility calls, the price of a volatility put converges to its payoff value $\max(0, K - V_T)$. Differentiation shows that the deltas of volatility puts are negative. In addition, the value of a put option can again be less than its intrinsic value. As in the case of volatility calls, the delta of a put is a decreasing function of T. As T increases, the price function for the put flattens out and the delta of the option converges to zero.

The relations between put deltas and T and put deltas and V mirror the patterns for volatility calls. These results again imply that longer-maturity volatility puts will be of limited use to hedgers since put deltas approach zero as T increases. The second derivative of volatility put prices with respect to V is again positive.

The relation between put prices and T is similar to that for calls. As $T \to \infty$, the put price converges to zero. The rationale for this follows from the boundedness of the payoff function and from the fact that the discount factor for the expected payoff decreases to zero as $T \to \infty$. This feature implies that the theta of a volatility put can be either positive or negative. For some values of V, the in-the-money puts are decreasing functions of T, while the opposite is true for the out-of-the-money puts.

The remaining comparative statics parallel those of volatility calls. An increase in the riskless interest rate has the effect of reducing the discount factor and decreasing the value of the put. In this respect, the prices of volatility puts are similar to those implied by the Black–Scholes formula. An increase in the strike price tends to make the put option further in the money for all values of V and T. Hence, the put is an increasing function of its strike price.

5. Volatility futures options

In addition to options on volatility, it is important to consider the valuation of volatility futures options. Let $CF(V,T,t+T,K)$ and $PF(V,T,t+T,K)$ denote the respective prices for a call and put option with maturity T on a futures contract expiring at time $t+T$. These options can be priced directly given the results in the previous sections. In particular, recall that the payoff function for a T-maturity call option on a futures contract expiring at time $t+T$ can be written as

$$\max(0,F(V_T,t)-K). \tag{16}$$

Substituting in for $F(V_T,t)$ from the futures price expression in (8) gives

$$\max(0,\exp(-\beta t)\,V_T+(\alpha/\beta)(1-\exp(-\beta t))-K). \tag{17}$$

Rearranging terms and factoring out $\exp(-\beta t)$ gives

$$\exp(-\beta t)\max(0,V_T+(\alpha/\beta)(\exp(\beta t)-1)-K\exp(\beta t)). \tag{18}$$

However, this is equal to $\exp(-\beta t)$ times the payoff for a simple volatility call option where the strike price differs from K. Thus, the current value of a volatility futures call option is

$$CF(V,T,t+T,K) = \exp(-\beta t)\, C(V, K\exp(\beta t) - (\alpha/\beta)(\exp(\beta t) - 1), T). \tag{19}$$

Because of the functional form of the volatility futures price, the value of a volatility futures call equals a scale factor $\exp(-\beta t)$ times the value of a simple volatility call where the strike price of the option is transformed. Because of this, the properties of these options are basically similar to those described in the previous section for volatility options. Note, however, that because $\exp(-\beta t) < 1$, volatility futures calls may be even less sensitive to changes in V than simple volatility calls.

One important difference between volatility futures calls and simple volatility calls arises because of the transformation of the strike price in (19). Recall that even though volatility can take on any positive value, volatility futures prices are bounded below. This implies that if the strike price of the volatility futures call is small enough, then the call option is guaranteed to expire in the money. Thus, when

$$K < (\alpha/\beta)(1 - \exp(-\beta t)), \tag{20}$$

the call option expires in the money and the valuation expression in (19) becomes

$$D(T)\exp(-\beta t)\, F(V,T) + D(T)(\alpha/\beta)(1 - \exp(-\beta T)) - KD(T). \tag{21}$$

This representation of the value of a deep-in-the-money futures call makes it clear that the delta of this option is again less than one since the V term in (21) is multiplied by an exponential term with value less than one.

Similar arguments can be applied to derive the value of a volatility futures put. The value of this option can be shown to be

$$PF(V,T,t+T,K) = \exp(-\beta t)\, P(V, K\exp(\beta t) - (\alpha/\beta)(\exp(\beta t) - 1), T). \tag{22}$$

This time, the lower bound on the value of the futures price implies that if K satisfies the inequality in (20), the futures put option is guaranteed to be out of the money at expiration. In this case, the value of the put option is zero even though it has a positive strike price. This again demonstrates how different the properties of volatility derivatives are from those of more conventional derivative securities.

6. Conclusion

Volatility derivatives have the potential to be one of the most important new financial innovations of this decade. In light of this, this paper develops simple closed-form valuation models for futures and options contracts on stock index volatility and examines their implications for the pricing and hedging behavior of these securities. We show that the properties of volatility derivatives can be very different from those for futures and options on traded assets.

A major implication of our analysis is that longer-term volatility futures and options may be less effective in hedging and managing volatility risk than commonly believed. The underlying reason for this is that mean reversion dampens the effect of current shocks in volatility on the option or futures price. These results have many important implications for the design of volatility derivative products.

7. For further reading

For further reading, see Abramowitz and Stegun (1970) and Cox and Ross (1976).

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