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Prepayment, Default, and the Valuation of Mortgage Pass-through Securities*

This article investigates the interaction of prepayment and default decisions in the valuation of mortgage pass-through securities. Even though a mortgage pass-through security is typically guaranteed by a financial intermediary, default decisions affect the timing of its cash flows and therefore its value. We also investigate the equilibrium valuation of the default insurance provided by the financial intermediary. The equilibrium insurance fee varies with prevailing interest rates, with prepayment possibilities, and, most significantly, with the volatility and value of the underlying mortgaged house. To the extent that a fixed insurance fee is charged, our analysis suggests that default insurance is not properly priced.

Approximately three trillion dollars worth of mortgage debt is currently outstanding in the United States. Concurrently, approximately one trillion dollars worth of mortgage pass-through securities have been issued.

A mortgage pass-through security is a security whose cash flows depend on an underlying pool of residential mortgages. In particular, the mortgage cash flows, both principal and interest, are passed through to the investor via an intermediary who retains a portion of the interest cash flows as compensation for services rendered including guaranteeing these pass-through payments.

Given their popularity, there has been significant interest in the valuation of mortgage pass-through securities. What makes the valuation of these default-free securities particularly interesting is the apparent suboptimal prepayment behavior of borrowers. That is, certain borrowers

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prepay their mortgage loan when prevailing mortgage rates exceed their loan's interest rate. Alternatively, other borrowers do not prepay their mortgage loan when their loan's interest rate exceeds prevailing mortgage rates.

Previous research on the valuation of mortgage pass-through securities includes Dunn and McConnell (1981), who, in a one-factor default-free security valuation framework, assume that there exists an autonomous probability that the borrower will prepay when refinancing rates exceed the mortgage's coupon rate. Brennan and Schwartz (1985), in a two-factor default-free security valuation framework, assume that there exists an autonomous probability that the borrower will prepay when refinancing rates are either below or above the mortgage's coupon rate. Schwartz and Torous (1989) extend the above analyses by integrating an empirical prepayment function into a two-factor default-free security valuation framework. Prepayment probabilities vary with the mortgage's age, the fraction of the pool outstanding, seasonality, and lagged refinancing rates.

The purpose of this article is to investigate the interaction of prepayment and default decisions as they affect the valuation of mortgage pass-through securities. The possibility of default allows us to value the insurance provided to the pass-through security holder. A key insight we provide is that prepayment and default decisions are taken at the level of the borrower (mortgagor) and that different economic circumstances underlie these decisions. What appears to be suboptimal prepayment behavior may indeed be optimal once the possibility of default is admitted.

The relevance of default possibilities to mortgage valuation has been recognized in previous research. Titman and Torous (1989) examine the effects of optimal default behavior in the absence of transaction costs on the pricing of bullet commercial mortgages where, in contrast to the residential mortgages we consider, prepayment is effectively prohibited. Kau, Keenan, Muller, and Epperson (1986) investigate the effects of both optimal default and prepayment decisions in the absence of transaction costs on the pricing of residential mortgages. Our analysis relaxes their assumption of optimal default and prepayment behavior in the absence of transaction costs, allowing for a potentially more accurate pricing analysis of residential mortgages, insurance, and mortgage pass-through securities.

Recently, much attention has been focused on the distress and insolvency of various financial institutions. A major concern in this discussion has been the pricing of default insurance. While we consider only the equilibrium pricing of default insurance in the mortgage pass-through securities market, our methodology and conclusions have implications for all financial markets.

The plan of this article is as follows. In Section I we detail the risky mortgage, insurance, and the mortgage pass-through security. We characterize the mortgagor's prepayment and default decisions in Section II. Assuming transactions costs, the mortgagor's conditional probability of prepayment is given by a prepayment function, while the mortgagor's conditional probability of default is given by a default function. We put forward our valuation framework in Section III, and in Section IV we illustrate the valuation of the risky mortgage, insurance, and the mortgage pass-through security for realistic parameter values and exogenously specified coupon rates. These security values vary with prevailing interest rates, prepayment possibilities, and mortgaged house values. In Section V we use our valuation procedures to determine equilibrium mortgage rates, while in Section VI we determine the fair fee to charge for the default insurance provided to the pass-through security holder. The fair insurance fee equates the costs and benefits of insurance and also varies with prevailing interest rates, prepayment possibilities, and mortgaged house values. Interest rate volatility and housing volatility significantly affect the fair insurance fee as well as equilibrium mortgage rates. Section VII provides our summary and conclusions.

I. The Mortgage Securities

We consider a fully amortizing mortgage having an original principal of $F(0)$ with a fixed continuously compounded coupon rate of c for an original term to maturity of T years. This implies a continuous payout (annuity) rate of C, where

$$C = cF(0)/[1 - \exp(-cT)],$$

with principal outstanding at time t, $F(t)$, given by

$$F(t) = F(0)\{1 - \exp[-c(T - t)]\}/[1 - \exp(-cT)].$$

The mortgagor possesses two options. First, the mortgagor has the option to prepay the loan. Second, the mortgagor has the option to default on the loan. In other words, the mortgage is subject to default risk. We assume the mortgage is not guaranteed by a government agency.[1] Later we detail the prepayment and default behavior of the

1. Government National Mortgage Association (GNMA) pass-through securities are backed by Federal Housing Administration (FHA)-insured or Veterans Administration (VA)-guaranteed mortgages. While FHA mortgages are fully guaranteed, VA mortgages are partially insured with a maximum guarantee of only \$27,500. Federal Home Loan Mortgage Corporation (FHLMC) and Federal National Mortgage Association (FNMA) pass-through securities are backed primarily by conventional mortgages that are either not insured (loan-to-value ratio of no more than 80%) or otherwise partially insured by a private insurer. Our later analysis considers the case where the mortgage is partially insured.

mortgagor and, in particular, the interaction of prepayment and default decisions.

Now consider a mortgage pass-through security backed by the risky mortgage. The risky mortgage's cash flows are passed through to the investor via an intermediary who retains a portion of the cash flows as compensation for its *guarantee* of the mortgage pass-through security.[2] That is, if the mortgagor defaults on the loan, then the intermediary immediately pays off the outstanding principal to the mortgage pass-through security holder.[3]

As a consequence of this guarantee, the pass-through security is default free. This does not imply, however, that the cash flows to the pass-through security, and therefore the value of this security, are not affected by the default behavior of the mortgagor. In particular, default by the mortgagor triggers immediate payment of the principal outstanding and, as such, terminates the pass-through security. From the pass-through security holder's perspective, default is identical to prepayment since both default and prepayment result in the payment of the principal outstanding. However, as our later analysis indicates, default occurs under different economic circumstances from prepayment.

Given the underlying characteristics of the risky mortgage, the pass-through security has an original principal of $F(0)$ and original term to maturity of T years. We assume that it has a continuously compounded fixed coupon rate of $p < c$, the difference representing the insurance rate. Thus, the continuous payout rate to the pass-through security, $P(t)$, is given by

$$P(t) = C - (c - p)F(t).$$

Notice that, while the mortgage's payout rate C is constant, the pass-through security's payout rate $P(t)$ is not since it increases slightly with increasing time as the principal outstanding, $F(t)$, decreases.

As mentioned earlier, in the event of default by the mortgagor, the intermediary pays off the pass-through security holder the principal

2. Services rendered by the intermediary typically include not only guaranteeing pass-through payments but also collecting mortgage payments in a timely fashion as well as making pass-through payments to the investor in a timely fashion. As such, a portion of the spread between the mortgage rate and the pass-through rate represents compensation for these other services. However, since our partial equilibrium framework does not explain this servicing fee, our analysis ignores these other services.

3. In practice, the pass-through security is also typically reinsured by a government agency. For example, a GNMA pass-through security's servicing fee of 50 basis points per month includes 6 basis points received by GNMA for its guarantee. The GNMA guarantees the timely payment of both interest and principal in the event of default by the mortgagor *and* the intermediary's inability to pass through interest and principal to the investor. We value the total insurance provided to the pass-through investor since we do not separately model the financial behavior of the intermediary.

outstanding. This insurance offered by the intermediary can be viewed as an American put option written by the intermediary on the underlying mortgaged house with exercise price equal to the principal outstanding. When the put option is exercised, instigated by the mortgagor's default decision, the insurance makes up the shortfall between the principal outstanding and the value of the mortgaged house.

In this article we value the three aforementioned claims: the risky mortgage, M, the insurance, I, and the mortgage pass-through security, G. To capture the mortgagor's prepayment and default decisions, we employ a two-state variable framework with the two state variables being the instantaneous riskless rate of interest, r, and the value of the mortgaged house, H. That is, we assume

$$M = M(r, H, t),$$
$$I = I(r, H, t),$$

and

$$G = G(r, H, t).$$

The dynamics of the instantaneous riskless rate are assumed to be given by the following square-root mean-reverting process due to Cox, Ingersoll, and Ross (1985):

$$dr = k(m - r)dt + \sigma_r \sqrt{r} dz_r,$$

where

k = the speed of adjustment coefficient,
m = long-term mean instantaneous riskless rate, and
$\sigma_r^2 r$ = instantaneous variance of changes in r.

Mortgaged house value dynamics are assumed to be given by

$$dH = (\mu - b)Hdt + \sigma_H H dz_H$$

with

$$(dz_r)(dz_H) = \rho dt,$$

where

μ = instantaneous expected housing rate of return,
b = housing payout rate,
σ_H^2 = instantaneous variance of housing returns, and
ρ = instantaneous correlation coefficient between the increments to the standardized Wiener processes dz_r and dz_H.

This specification implies log-normally distributed house prices if μ is constant.

II. Prepayment and Default Decisions

The mortgagor's prepayment and default decisions are critical not only to the valuation of the risky mortgage but also to the valuation of insurance and the mortgage pass-through security since their corresponding cash flows are affected by the mortgagor's prepayment and default decisions.

We assume that the mortgagor's conditional probability of prepaying (conditional on not having previously prepaid) is given by a prepayment function

$$\pi = \pi(r, H, t).$$

We interpret π as the rate per unit time that outstanding mortgages are prepaid. We motivate this prepayment function by appealing to transaction costs associated with prepayments that vary across mortgagors. In addition, mortgagors may differ in their utility gain from prepaying and transferring the underlying property.

We also assume that the mortgagor's conditional probability of default (conditional on not having previously defaulted) is given by a default function

$$\delta = \delta(r, H, t).$$

We interpret δ as the rate per unit time that outstanding mortgages are defaulted on. We motivate this default function by appealing to transaction costs, borne by mortgagors, associated with defaults that vary across mortgagors. As noted by Foster and Van Order (1984), many mortgagors with house values less than their mortgage values do not default.

A. *Interaction of Prepayment and Default*

We assume that the probability of default is positive ($\delta > 0$) only when both the value of the mortgaged house is less than the market value of the risky mortgage *and* the value of the mortgaged house is less than the principal outstanding. That is,

$$\begin{aligned}\delta &> 0 \quad \text{when } H(t) < M(r, H, t) \text{ and } H(t) < F(t),\\ \delta &= 0 \quad \text{otherwise.}\end{aligned}$$

In the absence of both default costs and prepayment possibilities, optimal default requires δ equals infinity when $H(t) < M(r, H, t)$, which implies the standard default condition that the mortgagor defaults when the value of the mortgaged house is less than the market value of the risky mortgage. We require the second condition ($H(t) < F(t)$) since it is possible that for low interest rates the value of the mortgaged house is less than the market value of the risky mortgage but greater than the principal outstanding:

$$F(t) < H(t) < M(r, H, t).$$

In this case prepayment dominates default.

We also assume that, when there is a positive probability of default, the probability of prepayment is zero. In other words, we assume that the mortgage would never be prepaid when default dominates prepayment.[4] That is,

$$\begin{aligned} &\pi = 0 \quad \text{when } H(t) < M(r, H, t) \text{ and } H(t) < F(t) \qquad [\delta > 0], \\ &\pi > 0 \quad \text{otherwise} \qquad [\delta = 0]. \end{aligned}$$

The interaction of prepayment and default decisions can help explain a number of the stylized facts associated with prepayments experienced by the pass-through security holder. For example, some high coupon rate mortgages are not prepaid in the presence of low refinancing rates. If the value of the mortgaged house is relatively low, the mortgagor will choose not to prepay even given relatively low refinancing rates since in this case default dominates prepayment. As default rates are generally lower than prepayment rates, this leads to relatively lower termination rates on premium mortgage pass-through securities. An implication of this argument is that premium mortgage pass-through securities with low collateral values should have higher prices than premium mortgage pass-through securities with high collateral values.

Alternatively, some low coupon rate mortgages are prepaid in the presence of high refinancing rates. If the value of the mortgaged house is low enough, the mortgagor may default and the mortgage pass-through security holder experiences a prepayment. While default clearly does not explain all prepayments of low coupon rate mortgages in the presence of high refinancing rates, it may be important in certain geographic regions.

B. Default and Prepayment Functions

To implement our valuation procedures, we must explicitly specify potentially estimable default and prepayment functions.

For illustrative purposes, we assume that the default function is given by the following hazard function:

$$\delta(r, H, t) = \begin{cases} \{[M(r, H, t) - H(t)]/H(t)\} \\ \times \exp(\eta\{[M(r, H, t) - H(t)]/H(t)\}) \\ \qquad \text{for } H(t) < M(r, H, t) \text{ and } H(t) < F(t), \\ 0 \quad \text{otherwise.} \end{cases}$$

4. It is also possible to model the interaction of prepayment and default assuming that there is some positive probability of prepayment when $\delta > 0$. In this case the mortgagor may prepay even when the house value is less than both the market value of the mortgage and the principal outstanding by depleting other sources of wealth.

Notice that, as the mortgagor's equity position becomes more negative, then δ approaches infinity. The parameter η determines the speed of default. In particular, the larger η, the closer the implied default behavior is to optimal behavior in the absence of default costs. Apart from the fact that the mortgagor's equity position determines the probability of default, the motivation for this particular default function lies in the fact that, given default data, a hazard function can be estimated in a straightforward fashion.[5]

For illustrative purposes, we assume that the prepayment function is given by the following proportional hazards model originally specified by Green and Shoven (1986):

$$\pi(r, H, t) = \begin{cases} \pi_0(t)\exp\{\beta[M(r, H, t) - F(t)]/H(t)\} & \text{for } \delta = 0, \\ 0 & \text{for } \delta > 0. \end{cases}$$

The baseline hazard function is given by $\pi_0(t)$ and measures the effect of mortgage age or seasoning on prepayment behavior. Given prepayment data, $\pi_0(t)$ can be estimated or, alternatively, exogenously specified.

Notice that, as the market value of the mortgage increases relative to the principal outstanding for a given mortgaged house value, then the probability of prepayment increases, ceteris paribus. We can interpret β as measuring the speed of prepayment. The larger β, the closer the implied prepayment behavior is to optimal prepayment behavior in the absence of prepayment costs: the more likely prepayment is when $M(r, H, t) > F(t)$, and the less likely prepayment is when $M(r, H, t) < F(t)$.

Also note that, for a given difference between the market value of a mortgage and its corresponding principal outstanding, the greater the value of the mortgaged house, the further the mortgagor's prepayment behavior is from optimal prepayment behavior in the absence of transaction costs, ceteris paribus. In other words, for a given savings due to refinancing, the mortgagor is less likely to prepay the more valuable the mortgaged house; conversely, for a given loss due to refinancing, the mortgagor is less likely not to prepay the more valuable the mortgaged house.[6]

5. Letting $z = (M - H)/H$, the hazard function can be written as $\delta(z; \eta) = z\exp(\eta z)$. Assuming independence across mortgages, the likelihood function of η is

$$L(\eta) = \prod_{i=1}^{n} (z_i e^{\eta z_i})^{\gamma_i} \exp(-z_i t_i e^{\eta z_i}),$$

where t_i is an observed survival time for mortgage i with corresponding covariate z_i and γ_i indicates default ($\gamma_i = 1$) or censoring ($\gamma_i = 0$). This specification is sufficiently flexible to admit other covariates that may influence the mortgagor's default decision. This would permit a more accurate description of actual default behavior necessary if the model is to be used for trading or hedging.

6. It should be pointed out that there are other possible ways in which house values could affect prepayment rates distinct from that specified and tested by Green and

III. Valuation Equation

In the absence of arbitrage opportunities, our previous assumptions together with the assumption that prepayments and default decisions are purely nonsystematic imply that the value of a claim, V—risky mortgage, insurance, or mortgage pass-through security—must satisfy the following second-order partial differential equation:

$$1/2\,\sigma_r^2 r\frac{\partial^2 V}{\partial r^2} + 1/2\,\sigma_H^2 H^2\frac{\partial^2 V}{\partial H^2} + \sigma_H\sigma_r\rho H\sqrt{r}\,\frac{\partial^2 V}{\partial r\partial H} + [k(m - r) + \lambda r]\frac{\partial V}{\partial r}$$
$$+ (r - b)H\frac{\partial V}{\partial H} - rV + \xi_i(r, H, t) = \frac{-\partial V}{\partial t}, \qquad i = M, I, G,$$

where λ equals the market price of interest rate risk.

What differentiates one claim from another is their corresponding payout rates:

$$\xi_M(r, H, t) = C + \pi(r, H, t)[F(t) - M(r, H, t)] + \delta(r, H, t)[H(t) - M(r, H, t)],$$

$$\xi_I(r, H, t) = [F(t) - H(t) - I(r, H, t)]\delta(r, H, t) - \pi(r, H, t)I(r, H, t),$$

and

$$\xi_G(r, H, t) = C - F(t)(c - p) + [\pi(r, H, t) + \delta(r, H, t)][F(t) - G(r, H, t)].$$

The payout rate to the risky mortgage $\xi_M(r, H, t)$ reflects the fact that with probability $\pi(r, H, t)$ the mortgage is prepaid and the principal outstanding is received, while with probability $\delta(r, H, t)$ the mortgage is defaulted on and the house value is received. The insurance payout rate $\xi_I(r, H, t)$ corresponds to the fact that with probability $\delta(r, H, t)$ the mortgage is defaulted on and the shortfall between the principal outstanding and the house value will be paid[7] and with probability

Shoven. For example, it could be argued that prepayments are more likely to occur the *higher* house values are relative to the principal outstanding as mortgagors take advantage of increased equity to increase their borrowings or improve their housing quality. Also, the mortgagor's prepayment decision can be influenced by other covariates. The proportional hazards framework can easily incorporate these covariates, permitting a more accurate description of actual prepayment behavior (Schwartz and Torous 1989).

7. As noted earlier, our analysis can accommodate partially insured mortgages. If the bottom f% (<100%) of the principal is guaranteed, then the insurance payout rate becomes

$$\xi_I(r, H, t) = \{\max[fF(t) - H(t), 0] - I(r, H, t)\}\delta(r, H, t) - \pi(r, H, t)I(r, H, t).$$

Alternatively, if the top f% (<100%) of the principal is guaranteed, then

$$\xi_I(r, H, t) = \{\max[F(t) - H(t), 0] - \max[(1 - f)F(t) - H(t), 0] - I(r, H, t)\} \times \delta(r, H, t) - \pi(r, H, t)I(r, H, t).$$

$\pi(r, H, t)$ the mortgage is prepaid and insurance becomes worthless. The payout rate to the mortgage pass-through security $\xi_G(r, H, t)$ reflects the fact that in the event of either default or prepayment the holder receives the principal outstanding.

Since the underlying risky mortgage is fully amortizing, the value of the claims must satisfy the terminal condition

$$V(r, H, T) = 0,$$

where T denotes the maturity date of the mortgage. In addition, each of the claims' values must satisfy standard boundary conditions.[8]

To value these claims, we must specify the market price of interest rate risk, λ. Given the assumed dynamics of the instantaneous riskless rate, we use the corresponding Cox, Ingersoll, and Ross (1985) default-free long-term yield, r_L, to determine that value of λ consistent with a specified r_L. That is,

$$\lambda = k[1 - (m/r_L)] + \sigma_r^2 r_L/2km.$$

It must be emphasized that the three claims, the risky mortgage, insurance, and the pass-through security, must be valued simultaneously by solving a system of partial differential equations[9] since π and δ, which appear in each of the partial differential equations, both depend on $M(r, H, t)$. Intuitively, we cannot value insurance or the pass-through security in isolation since their values depend on the mortgagor's prepayment and default decisions.

IV. Valuation Results

In this section, we illustrate the valuation of the risky mortgage, insurance, and the mortgage pass-through security for realistic underlying parameter values and exogenously specified coupon rates.

We consider a 30-year fully amortizing mortgage with a fixed continuously compounded coupon rate of $c = 10.5\%$ and a $p = 10\%$ fixed continuously compounded coupon rate pass-through security backed by this risky mortgage. This mortgage pass-through security is guaranteed so that, in the event of default by the mortgagor, the investor receives the principal outstanding.

8. For $r = 0$ we impose corresponding natural boundaries, while for sufficiently large values of r and H we impose $V_{rr} = 0$ and $V_{HH} = 0$, respectively.

9. The partial differential equations are solved using a finite difference method (the alternating difference implicit method of Peaceman and Rachford [1955]). Since M, I, and G are homogeneous of degree 1 in the principal outstanding together with the fact that our assumed default and prepayment functions are not path dependent, we are able to implement this precise solution method. Path-dependent default and prepayment functions would require us to approximate partial differential equation solutions using less precise Monte Carlo simulation techniques (see Schwartz and Torous [1989] for further details).

Based on the analysis of Bunce, MacRae, and Szymanoski (1988), we characterize the dynamics of the instantaneous riskless rate of interest by assuming that the speed-of-adjustment coefficient is $k = 0.10$, the long-term mean instantaneous riskless rate is $m = 0.065$, while the volatility of instantaneous riskless rates of interest is specified by setting $\sigma_r = 0.075$.

To characterize the dynamics of the value of the underlying mortgaged house, we assume a housing payout rate of $b = 0.065$ and a housing return volatility of $\sigma_H = 0.100$. This specification is based on the analysis of Cunningham and Hendershott (1984).[10] To complete the specification of the underlying state variable processes, we assume that unanticipated increments to the instantaneous riskless rate of interest are uncorrelated with unanticipated increments to housing returns, $\rho = 0$.[11]

For illustrative purposes, the default function is assumed to be consistent with an annualized probability of 0.50 that a mortgagor with -20% equity in a house will default. The implies $\eta = 4.58$. A more accurate specification of this speed-of-default parameter awaits the estimation of the hazards model given actual default data on individual homes.

We characterize the prepayment function as follows. The baseline hazard function is given by the Public Securities Association's (PSA) standard prepayment model. That is, the annualized baseline probability of prepayment is zero at the mortgage's origination, increases by 0.002 per month for the first 30 months of the mortgage's life, and then remains constant at an annualized rate of 0.06 from the thirtieth month until maturity. Based on the empirical analysis of Green and Shoven (1986), we choose two values of the speed-of-prepayment parameter: $\beta = 4.37$ and $\beta = 13.07$. The value of $\beta = 4.37$ was estimated by Green and Shoven on the basis of a sample of mortgages where the due-on-sale clause[12] was enforceable, implying that mortgages were less likely to be assumed, increasing the likelihood of prepayment in this sample. By contrast, the value of $\beta = 13.07$ was based on a sample of mortgages where the due-on-sale clause was not enforceable, decreasing the likelihood of prepayment in this sample since mortgages were more likely to be assumed.

We summarize our valuation results in tables 1 and 2. In table 1 we present risky mortgage, insurance, and pass-through security values assuming a speed-of-prepayment parameter of 4.37; while in table 2

10. In a more general information, the housing payout rate can be made dependent on the prevailing level of interest rates.

11. We choose this particular value of ρ for illustrative purposes. The valuation equation allows for nonzero values of ρ.

12. The due-on-sale clause allows the lender to demand the mortgage's principal outstanding at the time of sale.

TABLE 1 **Risky Mortgage, Insurance, and Mortgage Pass-through Security Values for Varying Short Rates r Given a Long Rate r_L = 10% and a Smaller Speed-of-Prepayment Parameter β = 4.37**

	H											
	M				I				G			
r	100	120	160	200	100	120	160	200	100	120	160	200
.04	111.5	120.3	125.0	126.7	7.29	2.97	.61	.09	116.4	120.3	122.4	123.4
.05	109.2	116.1	119.7	121.1	5.90	2.21	.42	.05	112.5	115.3	116.9	117.8
.06	106.4	111.7	114.5	115.6	4.66	1.62	.29	.02	108.4	110.3	111.6	112.3
.07	103.2	107.2	109.3	110.2	3.58	1.16	.19	.01	104.0	105.3	106.4	107.1
.08	99.6	102.6	104.3	105.1	2.67	.82	.13	.00	99.4	100.4	101.3	102.0
.09	95.8	98.0	99.4	100.3	1.93	.57	.09	.00	94.9	95.6	96.5	97.3
.10	91.8	93.6	94.8	95.6	1.37	.39	.06	.00	90.4	91.0	92.0	92.7
.11	87.8	89.2	90.4	91.3	.95	.27	.04	.00	86.0	86.6	87.6	88.5
.12	83.9	85.0	86.2	87.1	.65	.18	.02	.00	81.8	82.4	83.5	84.4
.13	80.0	81.0	82.2	83.2	.43	.12	.01	.00	77.7	78.4	79.5	80.6
.14	76.2	77.1	78.3	79.4	.29	.08	.01	.00	73.9	74.6	75.8	77.0
.15	72.5	73.4	74.7	75.9	.19	.05	.01	.00	70.2	70.9	72.3	73.5
.16	69.0	69.8	71.2	72.5	.12	.03	.00	.00	66.6	67.4	68.9	70.3
.17	65.5	66.3	67.8	69.3	.07	.02	.00	.00	63.2	65.0	65.6	67.1

NOTE.—This table tabulates c = 10.5% risky mortgage values, p = 10% mortgage pass-through security values, as well as corresponding insurance values for short rates varying between r = 4% and r = 17% given a long rate of r_L = 10% and a smaller speed-of-prepayment parameter β = 4.37.

TABLE 2 **Risky Mortgage, Insurance, and Mortgage Pass-through Security Values for Varying Short Rates r Given a Long Rate r_L = 10% and a Larger Speed-of-Prepayment Parameter β = 13.07**

	H											
	M				I				G			
r	100	120	160	200	100	120	160	200	100	120	160	200
.04	110.0	116.3	119.3	120.9	6.92	2.36	.33	.03	114.6	116.1	116.9	118.1
.05	107.7	112.8	115.2	116.5	5.60	1.79	.24	.01	110.8	111.9	112.7	113.6
.06	104.9	108.9	110.9	111.9	4.41	1.34	.17	.00	106.7	107.4	108.1	109.0
.07	101.6	104.7	106.3	107.2	3.38	.97	.12	.00	102.2	102.7	103.4	104.2
.08	97.9	100.2	101.6	102.5	2.51	.70	.08	.00	97.5	97.9	98.7	99.5
.09	93.9	95.7	96.9	97.8	1.82	.49	.05	.00	92.8	93.2	94.0	94.8
.10	89.8	91.2	92.3	93.2	1.29	.34	.04	.00	88.2	88.6	89.4	90.3
.11	85.7	86.8	87.9	88.7	.89	.23	.02	.00	83.6	84.1	85.0	85.9
.12	81.6	82.5	83.5	84.4	.61	.15	.02	.00	79.3	79.7	80.7	81.6
.13	77.6	78.3	79.4	80.3	.41	.10	.01	.00	75.1	75.6	76.6	77.6
.14	73.7	74.3	75.4	76.4	.27	.07	.01	.00	71.2	71.7	72.7	73.7
.15	69.9	70.5	71.5	72.6	.18	.04	.00	.00	67.4	67.9	68.9	70.0
.16	66.2	66.8	67.9	68.9	.11	.03	.00	.00	63.7	64.3	65.3	66.4
.17	62.7	63.2	64.3	65.4	.07	.02	.00	.00	60.2	60.8	61.9	63.0

NOTE.—This table tabulates c = 10.5% risky mortgage values, p = 10% mortgage pass-through security values, as well as corresponding insurance values for short rates varying between r = 4% and r = 17% and a larger speed-of-prepayment parameter β = 13.07.

we present these values assuming a speed-of-prepayment parameter of 13.07. In both tables we fix the long rate at $r_L = 10\%$ and vary the short rate from $r = 4\%$ to $r = 17\%$. In each case we report resultant values of the risky mortgage, insurance, and the pass-through security for mortgaged house values of $H = 100$, 120, 160, and 200, assuming a prevailing principal of 100.

As expected, equilibrium claim values are sensitive to prevailing interest rates. As interest rates increase, then risky mortgage, insurance, and pass-through security values decrease.[13] This follows since these are interest-sensitive claims. Notice that for low mortgaged house values ($H = 100$) and low interest rates, the value of the mortgage pass-through security exceeds the value of the risky mortgage, even though the latter has a higher coupon rate. Intuitively, for low mortgaged house values and low interest rates, default is likely, in which case the pass-through security holder is paid in full while the risky mortgage holder is not.

The value of the underlying mortgaged house has a significant effect on equilibrium claim values. In other words, we can more accurately value the risky mortgage, insurance, or the pass-through security by more accurately valuing the underlying mortgaged house. Notice that the value of the risky mortgage decreases while the value of insurance increases with decreasing house values as the probability of default increases. However, the value of the pass-through security does not necessarily decrease with decreasing house values and increasing probability of default. For high interest rates and very low mortgaged house values, the pass-through security holder desires default as default triggers payment of the principal outstanding by the financial intermediary. For example, in unreported sensitivity analyses, we have that, for $r = 9\%$, $r_L = 10\%$, and $\beta = 4.37$, then $G = 92.00$ for $H = 90$, $G = 93.84$ for $H = 70$, and $G = 98.08$ for $H = 50$.

Finally, the assumed prepayment behavior of the mortgagor has a significant effect on equilibrium claim values. In particular, the larger the speed-of-prepayment parameter β, the lower the values of the risky mortgage, insurance, and the pass-through security, ceteris paribus. With regard to the risky mortgage and the pass-through security, the larger the speed-of-prepayment parameter, the closer are values to those assuming optimal prepayment in the absence of prepayment costs, resulting in lower risky mortgage and pass-through security values. When the risky mortgage is selling at a premium relative to the principal outstanding, the larger the speed-of-prepayment parameter, the faster are prepayments, which is detrimental to premium security holders. Conversely, when the risky mortgage is selling at a discount

13. This result and all our other conclusions also hold if we fix the short rate and vary the long rate.

relative to the principal outstanding, the larger the speed-of-prepayment parameter, the slower are prepayments, which is detrimental to discount security holders. With regard to insurance, the larger the speed-of-prepayment parameter, the larger prepayments on average and hence the lower the probability of default, implying lower insurance values.

V. Equilibrium Mortgage Rates

In the previous section, we examined risky mortgage, insurance, and mortgage pass-through security values under varying interest rate and loan-to-value ratio environments, while holding fixed exogenously specified coupon rates on the mortgage and the mortgage pass-through security. We now use our valuation procedures to determine equilibrium mortgage rates. We also investigate insurance and mortgage pass-through security values assuming a pass-through rate that differs from corresponding equilibrium mortgage rates by a fixed servicing rate. In the next section, we determine fair insurance fees and, as such, equilibrium pass-through rates.

The equilibrium mortgage rate is that coupon rate, c^*, which results in the mortgage being priced at par at origination. This coupon rate depends on the current interest rate environment, including prevailing interest rates, r and r_L, and the assumed parameters of the riskless instantaneous interest rate process, as well as the value of the mortgaged house relative to the mortgage's principal, and the assumed parameters of the mortgaged house value process. In addition, mortgagors' assumed default and prepayment behavior affect prevailing equilibrium mortgage rates.

In table 3, we tabulate equilibrium 30-year mortgage rates for varying house values given an original principal of 100, assuming $r = 9\%$ and $r_L = 10\%$. The dynamics of the instantaneous riskless rate and the mortgaged house value are as previously specified. As before, the default function is consistent with an annualized probability of 0.50 that a mortgagor with -20% equity in a house will default, and we again report results for speed-of-prepayment parameters of $\beta = 4.37$ and $\beta = 13.07$.

Notice that equilibrium mortgage rates decrease as the value of the mortgaged house increases (loan-to-value ratio decreases). Higher house values decrease the probability of default. For example, assuming $\beta = 4.37$, a loan-to-value ratio of 91% (house value of 110 at origination) requires an equilibrium mortgage rate of approximately 50 basis points more than a loan-to-value ratio of 53% (house value of 190 at origination). Also, the larger the speed-of-prepayment parameter β, the higher equilibrium mortgage rates, ceteris paribus. That is, as the assumed prepayment behavior approaches optimal prepayment be-

TABLE 3 Equilibrium Mortgage Rates and Corresponding Insurance and Mortgage Pass-through Security Values Assuming a Servicing Fee of 15 Basis Points Given a Short Rate of 9% and a Long Rate of 10%

	β = 4.37			β = 13.07		
H	c^* (in %)	I	G	c^* (in %)	I	G
100	11.35	2.17	101.35	11.73	2.11	101.30
110	11.01	1.07	100.20	11.43	.98	100.14
120	10.85	.59	99.70	11.29	.50	99.65
130	10.76	.35	99.45	11.21	.27	99.41
140	10.69	.21	99.31	11.15	.15	99.29
150	10.64	.13	99.24	11.11	.09	99.23
160	10.60	.09	99.19	11.06	.05	99.18
170	10.56	.05	99.16	11.01	.03	99.16
180	10.52	.03	99.12	10.97	.02	99.12
190	10.49	.01	99.11	10.93	.00	99.11
200	10.46	.00	99.09	10.89	.00	99.09

NOTE.—This table tabulates equilibrium risky mortgage rates, c^*, as well as insurance and mortgage pass-through security values for varying house values assuming $p = c^* - .15\%$ given $r = 9\%$ and $r_L = 10\%$, for both smaller, $\beta = 4.37$, and larger, $\beta = 13.07$, speed-of-prepayment parameters.

havior without transaction costs, the option to call the loan becomes more valuable, requiring greater compensation for the mortgagee. For example, by increasing the speed-of-prepayment parameter from $\beta = 4.37$ to $\beta = 13.07$, equilibrium mortgage rates increase by approximately 40 basis points.

For illustrative purposes, table 3 also reports insurance and mortgage pass-through security values assuming a pass-through rate of 15 basis points less than the corresponding equilibrium mortgage rate, $p = c^* - .15\%$. As previously documented, it is still the case that the larger the speed-of-prepayment parameter β, the lower the value of insurance. Mortgage pass-through security values now decrease with increasing mortgaged house values (lower loan-to-value ratios). This follows since the assumed coupon rate of $p = c^* - .15\%$ decreases with increasing house values for the guaranteed mortgage pass-through security. Notice also that for relatively low mortgaged house values (relatively high loan-to-value ratios), which implies high equilibrium mortgage rates and, assuming a fixed servicing rate, high pass-through rates, the mortgage pass-through security is priced at a premium. By contrast, the mortgage pass-through security is priced at a discount for relatively high house values (relatively low loan-to-value ratios). Finally, the speed-of-prepayment parameter has a minimal effect on mortgage pass-through security values since differing equilibrium mortgage rates compensate for differing prepayment risk and the servicing rate is assumed constant. The small differences reported reflect differences in the timing of the cash flows due to the uneven nature of payments across the life of the mortgage pass-through security and the interaction between default and prepayment.

VI. Default Insurance

Our valuation procedures can also be used to determine the fair fee to charge for the default insurance provided to the mortgage pass-through security holder. A fair insurance fee is that fee that results in the pass-through security selling for par at origination. Intuitively, the pass-through security differs from the underlying risky mortgage in that it is guaranteed but at the cost of a reduction in coupon rate. When the value of the guarantee equals the cost of insurance, the insurance is fairly priced and the pass-through security sells for par at origination, assuming that the underlying risky mortgage also sells for par at origination. The fair insurance fee varies with prevailing interest rates, with prepayment possibilities, and, most significantly, with the value of the underlying mortgaged house as the probability of default correspondingly varies. As a result, insurance is not properly priced by charging a *fixed* insurance fee.

From table 3, the assumed insurance fee of 15 basis points is too small for those cases in which the mortgage pass-through security sells at a premium and too large for those cases in which the mortgage pass-through security sells at a discount. That is, when selling at a premium, the benefits of insurance outweigh costs, and when selling at a discount, the costs of insurance outweigh the benefits. For example, for the assumptions underlying table 3, an insurance fee of 15 basis points is too small for $H = 110$, or loan-to-value ratio of 90.9%, and too large for $H = 150$, or loan-to-value ratio of 66.7%.

For a given house value and interest rate environment, we determine the fair fee to charge for default insurance by iteratively solving for that pass-through rate, p^*, such that for a given equilibrium mortgage coupon rate, c^*, the mortgage pass-through security sells for par. The fair rate to charge for default insurance is given by $c^* - p^*$.

Assuming that at the origination of the risky mortgage the short rate is $r = 9\%$ and the long rate is $r_L = 10\%$, table 4 documents the fair rate to charge for default insurance given a variety of mortgaged house values, as well as for both the smaller, $\beta = 4.37$, and larger, $\beta = 13.07$, speed-of-prepayment parameters. Notice that the fair insurance fee is extremely sensitive to the assumed mortgaged house value. Fair insurance fees are significantly larger for loan-to-value ratios greater than 83.3% or, equivalently, for mortgaged house values less than 120 since the probability of default is significant. By contrast, fair insurance fees are minimal for loan-to-value ratios less than 66.7% or, equivalently, for mortgaged house values greater than 150 since the probability of default is negligible. Notice also from table 4 that even though the equilibrium pass-through rate is higher the larger the speed-of-prepayment parameter β, the fair insurance fee is only

TABLE 4 **Fair Insurance Fees (in %)**

H	Fair Insurance Rate for $\beta = 4.37$	Fair Insurance Rate for $\beta = 13.07$
100	.40	.39
110	.19	.17
120	.10	.09
130	.06	.05
140	.03	.03
150	.02	.02
160	.01	.01
170	.01	.00
180	.00	.00
190	.00	.00
200	.00	.00

NOTE.—This table tabulates fair insurance fees, $c^* - p^*$, for varying house values assuming $r = 9\%$ and $r_L = 10\%$, for both smaller, $\beta = 4.37$, and larger, $\beta = 13.07$, speed-of-prepayment parameters.

slightly lower, reflecting the fact that the probability of default is only slightly lower given the larger value of β.[14]

Comparing tables 3 and 4, we see that, even though equilibrium mortgage rates and fair insurance fees are extremely sensitive to underlying mortgaged house values, equilibrium pass-through rates are much less sensitive. This follows from the fact that in the case of default the insurer will make up any shortfall to the pass-through investor.

The sensitivity of the fair insurance fee to changes in housing volatility, σ_H, and interest rate volatiity, σ_r, is examined in table 5. Since the fair insurance fee requires that we solve for the corresponding equilibrium mortgage rate, we also investigate how equilibrium mortgage rates vary with changes in these volatilities. In particular, housing volatility is increased to $\sigma_H = .20$ and decreased to $\sigma_H = .05$ from the previously assumed $\sigma_H = .10$, while interest rate volatility is increased to $\sigma_r = .10$ and decreased to $\sigma_r = .05$ from the previously assumed $\sigma_r = .075$.

Notice that the fair insurance fee is extremely sensitive to changes

14. In addition to the insurance fee, a portion of the total servicing rate also represents compensation for rendering such services as collecting mortgage payments in a timely fashion. Assuming that the fee for services rendered is exogenously specified, the appropriate total servicing rate to charge is the sum of this fee and the previously determined equilibrium insurance fee. If the market for pass-through securities is competitive, these transaction costs will be borne by the mortgagor in the form of a higher mortgage coupon rate. That is, the equilibrium mortgage rate calculated under perfect market assumptions must be adjusted to take into account these transaction costs. The appropriate mortgage rate will then be that rate that prices the pass-through security at par, given the servicing rate that reflects both insurance and transaction costs.

TABLE 5 Equilibrium Mortgage Rates and Fair Insurance Fees for Varying Housing Volatility σ_H and Varying Interest Rate Volatility σ_r (in %)

	$\sigma_H = .10$, $\sigma_r = .075$		$\sigma_H = .20$, $\sigma_r = .075$		$\sigma_H = .05$, $\sigma_r = .075$		$\sigma_H = .10$, $\sigma_r = .10$		$\sigma_H = .10$, $\sigma_r = .05$	
H	c^*	$c^* - p^*$	c^*	$c^* - p^*$	c^*	$c^* - p^*$	c^*	$c^* - p^*$	c^*	$c^* - p^*$
$\beta = 4.37$:										
110	11.01	.19	11.84	1.01	10.72	.04	12.64	.24	10.09	.14
120	10.85	.10	11.48	.70	10.63	.02	12.40	.15	9.98	.07
150	10.64	.02	10.87	.25	10.43	.01	12.02	.05	9.86	.01
$\beta = 13.07$:										
110	11.43	.17	12.21	1.01	11.14	.04	13.34	.20	10.29	.12
120	11.29	.09	11.86	.70	11.05	.02	13.18	.11	10.19	.05
150	11.11	.02	11.30	.24	10.78	.01	12.94	.03	10.07	.01

NOTE.—This table tabulates equilibrium mortgage rates, c^*, and fair insurance fees, $c^* - p^*$, for varying housing volatility σ_H, varying interest rate volatility σ_r, and varying house values given r = 9% and r_L = 10%, for both smaller, $\beta = 4.37$, and larger $\beta = 13.07$, speed-of-prepayment parameters.

in housing volatility. For example, assuming $\sigma_r = .075$ and the smaller speed-of-prepayment parameter $\beta = 4.37$, increasing housing volatility from $\sigma_H = .05$ to $\sigma_H = .20$ increases the fair insurance fee from 4 to 101 basis points for $H = 110$, from 2 to 70 basis points for $H = 120$, and from 1 to 25 basis points for $H = 150$. Increasing housing volatility increases the probability of default and, as such, the fair insurance fee. Equilibrium mortgage rates also increase with increasing housing volatility reflecting the greater probability of default. However, equilibrium pass-through rates are far less sensitive to changes in housing volatility, suggesting that the pass-through investor should be less concerned with the details of underlying house value dynamics.

In addition, equilibrium mortgage rates vary significantly with changes in interest rate volatility. For example, assuming $\sigma_H = .10$ and the smaller speed-of-prepayment parameter $\beta = 4.37$, increasing interest rate volatility from $\sigma_r = .05$ to $\sigma_r = .10$ increases the equilibrium mortgage rate from 10.09% to 12.64% for $H = 110$, from 9.98% to 12.40% for $H = 120$, and from 9.86% to 12.02% for $H = 150$. Since mortgages are interest-sensitive claims, it is not surprising that equilibrium mortgage rates vary with interest rate volatility. Fair insurance fees also increase with increases in interest rate volatility since the probability of default increases in a volatile interest rate environment.

VII. Summary and Conclusions

Mortgage pass-through securities are backed by mortgages potentially subject to default risk as well as prepayment risk. This implies that mortgagors' prepayment *and* default decisions affect the valuation of mortgage pass-through securities. While prepayment and default decisions both result in the pass-through security holder receiving the mortgage's principal outstanding, different economic circumstances underlie these decisions. This article has carefully characterized mortgagors' prepayment and default decisions and integrated them into a mortgage pass-through security valuation framework. To value a pass-through security, we simultaneously value the underlying risky mortgage since the risky mortgage's value is required to characterize the mortgagor's prepayment and default decisions. Mortgage pass-through security values are sensitive to the value of the underlying mortgaged house. In fact, if the value of the mortgaged house is sufficiently low, the value of the mortgage pass-through security may exceed the value of the underlying risky mortgage, even though the latter has a higher coupon rate, since the pass-through security is guaranteed. To value more precisely a mortgage pass-through security requires that we value more precisely the underlying mortgaged house.

The possibility of default allows us to value the insurance provided to the pass-through security holder. We also determine the fair fee to charge for this default insurance. This fair fee varies with prevailing interest rates, with prepayment possibilities, and, most significantly, with the volatility and the value of the underlying mortgaged house. To the extent that a fixed insurance fee is charged, our analysis suggests that default insurance is not properly priced.

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