Gonzalo Cortazar
Pontificia Universidad Catolica de Chile

Eduardo S. Schwartz
University of California, Los Angeles

A Compound Option Model of Production and Intermediate Inventories*

This article extends the option approach to valuing real assets by modeling the firm as a two-stage process with bounded output rates, in which the output of the first stage may be held as work-in-process. In this setting, the real asset becomes a compound option, which, if exercised, gives the option to finish the work-in-process and sell the output as its final payoff. The existence of intermediate inventories may arise as an optimal investment strategy for exploiting possible future price increases. The framework allows us to analyze the effect of uncertainty on output rates and the effect of interest rates changes on inventory levels.

I. Introduction

The value of assets is one of the central issues of financial economics. Since the seminal work of Fisher (1930), discounting cash flows has been the preferred valuation method used by academicians and practitioners. Although powerful in its simplicity, the net present value (NPV) model has sometimes proved difficult to use. In particular, it requires estimating future cash flows and determining the relevant risk-adjusted discount rate, not easy tasks in markets with high levels of uncertainty. This problem is especially acute for valuing real asset investments. For example, in an investment in a copper mine, relevant cash flows critically depend on the future copper spot prices, as well as on the physical quantities sold. However, copper spot prices, like most commodity prices, exhibit high volatility, making them very difficult to predict. In addition, physical production levels are also difficult to estimate because the firm may optimally adjust them in

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response to the actual price realization, increasing production at booming prices, and decreasing (or stopping) production at low prices. A further complication arises when sequential production decisions must be made. In this case, we may be required to commit resources for a production process, only to acquire an option to complete the process and sell the goods in the future.

Recently, a new literature, under the name of real options, has been unfolding in which real assets are valued using techniques initially developed for financial options (e.g., Brennan and Schwartz 1985; McDonald and Siegel 1986; Majd and Pindyck 1987, 1989; Pindyck 1988, 1991; He and Pindyck 1992). This approach has several advantages over the NPV model, overcoming most of its drawbacks. For example, real option models do not require prediction of future prices or estimation of the risk adjustment on the interest rate. In addition, they adequately value the operation options available to the firm.

In the last decade, an array of real option models has been developed. Most of them agree in their prediction that increases in price volatility should induce a nonpositive effect on output rates, giving support to the commonly held belief in the benefits of price stabilization. For example, in a model with finite resources, to produce now implies forgoing the option to produce tomorrow, and, therefore, stability induces higher current output (Pindyck 1991). In another example,[1] Majd and Pindyck (1989, p. 342) examine the implications of the learning curve in a world of uncertainty and conclude that "for those industries in which the learning curve is an important determinant of cost, this has a curious implication: other things being equal, during periods of high volatility, firms ought to be producing less, but are worth more."

Previous studies, using other modeling approaches, have also noted the perverse effects of volatility (Van Wijnbergen 1985; Dornbusch 1987; Ingersoll and Ross 1992). However, the positive effect on output rates, induced by reductions in price volatility, has been surprisingly difficult to detect in empirical studies. For example, a number of econometric studies analyzed the effect of volatility of exchange rates on output rates and international trade, finding contradictory results.[2] This article provides one possible explanation for why uncertainty may lead some firms to increase output and others to decrease output.[3]

1. See also Pindyck (1991, p. 1141): "If a goal of macroeconomic policy is to stimulate investment, stability and credibility may be more important than tax incentives or interest rates."

2. Empirical studies by Abrahms (1980), Cushman (1983, 1986), Thursby and Thursby (1985), and Kenen and Rodrik (1986) conclude that reductions in volatility increase exports. In contrast, studies by Hooper and Kohlhagen (1978) and Gotur (1985) find a decrease in exports under similar conditions.

3. See Cortazar (1992) for another possible explanation for this difficulty in empirically confirming the benefits from stabilization: some firms actually benefit from volatility through the exercise of options in certain states of the world. These firms will see their

Another limitation of current real option models is their (lack of) treatment of inventories. Existing real option models assume that the firm technology can be represented by a one-stage production process and that the firm instantaneously sells its production. In this setting, there is no place for inventories. However, other branches of the economic literature have long recognized the importance of analyzing the effect of different variables on inventory levels. For example, understanding the determinants of inventory investment has become critical to analyzing business cycles, considering that during U.S. postwar recessions, the decline in total inventory investment has averaged 101% of the decline in gross national product (Blinder and Holtz-Eakin 1986). Among the inventory components according to their stage of processing, Ramey (1989) found that the most volatile is the aggregate of raw materials and work-in-process. Moreover, work-in-process represents more than a third of the total inventories held by U.S. manufacturing companies, amounting to almost 140 billion dollars in 1989 (*Annual Survey of Manufactures* 1989). This research specifically analyzes the behavior of work-in-process inventories.

The effect of interest rates on inventory investment is an issue that has generated a long debate in the literature, with some authors finding a negative effect (Rubin 1979; de Leeuw 1982; Akhtar 1983; Ramey 1989) and others failing to uncover any significant effect (Feldstein and Auerbach 1976; Lovell 1976; Blinder 1981; Maccini and Rossana 1981). This difficulty in documenting the negative effect of increases in interest rates on inventory levels is quite surprising, considering that standard economic models usually treat inventories only as a buffer stock between sales and production.[4] Thus, standard economic theory predicts that an increase in holding costs should unambiguously reduce inventory levels. If work-in-process is considered only a buffer stock, it makes sense to use its level as a measure of the inefficiency of manufacturing systems. This is consistent with the widespread use of just-in-time manufacturing systems, which promote the reduction or even the elimination of inventories as the major thrust of production management.

The objective of this article is to extend the real option approach by modeling the firm as a two-stage process with bounded output rates, in which the output of the first stage may be held as work-in-process. We are especially interested in analyzing the case where there is a bottleneck in the process; that is, the bound on the output rate of the

value diminished with stability and will tend to make foreign direct investments in countries that provide high-profit volatility by having revenues and costs expressed in different and volatile currencies.

4. Ramey (1989), in her econometric work, is one of the few who consider inventories as factors of production that provide the firm with a flow of services.

first stage is lower than that of the second stage.[5] We assume that the price of the output demanded by the final consumers is the driving stochastic process, while the intermediate output should be priced as a derivative asset. In this setting, the asset becomes a compound option, which, if exercised (first-stage process), has an option to finish the work-in-process (second-stage process) and sell the output as its payoff. Unlike in previous literature, in our model the existence of intermediate inventories may arise, not only because of inefficiencies in the production system, but also as an optimal investment strategy for exploiting possible future price increases.

The main contributions of this article are to provide analytical expressions for valuing a firm that has a production bottleneck and for determining its optimal output rate and capacity. These issues have already been analyzed in the literature for the one-stage firm. For example, Brennan and Schwartz (1985) solve for the value and optimal output rate of the firm, while Pindyck (1988) analyzes the capacity choice problem. We extend their analyses for a firm with a two-stage process, providing new insights into the effect of price volatility and of interest rate levels on firm value and on optimal production and inventory levels.

In addition to being relevant for the two-stage technology firm, most of the conclusions derived from our model can be extended to analyses at the industry level. As the input-output economic analysis of major economies reveals (Leontief 1966), a low fraction of the output of any industry is demanded by final consumers, while most of the output becomes input to the same or other industries. For example, in a detailed analysis of the structure of the U.S. economy in 1977, the Bureau of Economic Analysis of the U.S. Department of Commerce classified industry output into 85 commodities and studied its final use. A detailed analysis shows that for more than half of these commodities, final demand represents less than a third of total output, while the remaining two-thirds serve as industry inputs.

There are two major empirical implications of our model. The first one is that increases in price volatility may induce a higher first-stage output rate, a lower second-stage output rate, and, therefore, an increase in work-in-process inventory.[6] The intuition for this result is that, when the bound for the first-stage output rate is lower than that of the second stage, increases in the price volatility of final output raise the option value of work-in-process inventory, inducing a higher

5. Our model does not require the existence of a bottleneck, but most of the interesting conclusions are obtained for this case. We will later argue why this bottleneck assumption is reasonable in many cases.

6. The model predicts that in general the firm will start first-stage production at prices that are lower than the marginal cost of production and refrain from completing the output (second-stage process) unless prices are well above marginal cost. This effect is similar to the *hysteresis* effect described by Dixit (1989, 1992).

first-stage output rate and inventory. By contrast, all previous real option research has modeled the firm as a one-stage production process, predicting that increases in uncertainty have an unambiguous nonpositive effect on output levels. Our model could provide one explanation for why it has been so difficult to detect aggregate output increases with more stable economic conditions. The model predicts that some industries should optimally increase output, while others should decrease it.

A second empirical implication of our model is that increases in interest rates raise first-stage output rate and work-in-process and reduce second-stage output rate. In addition to the standard treatment of inventories as a buffer between different stages of production, we unveil a second purpose of inventories: a profitable investment that will allow the firm to make use of future short-term price increases in conditions of limited production capacity. Again, the intuition for this result is that the option value of work-in-process inventories increases with interest rates, inducing a higher first-stage output rate. This effect could explain the difficulty in empirically detecting the negative effect of interest rates on inventory investment, when the analysis is at the aggregate level.

This article is organized in the following way. In the next section we present the valuation framework for a firm with a two-stage process with bounded output rates. In Section III we develop the model of production and intermediate inventories and discuss the main implications of the analysis. In Section IV we summarize the key new insights that can be obtained from the model. Finally, Section V presents some final remarks.

II. The Valuation Framework

To be able to analyze intermediate inventories at the firm and industry levels, we extend the real options model by assuming that the technology of the firm can be described as two sequential processes; thus, the valuation problem has two decision variables: the output rate of each process. To focus our discussion, assume that we are valuing a copper mine in which the first-stage process represents the extraction of the mineral, while the second-stage process includes all activities related to the processing of the mineral and the distribution and sale of the final output. We assume that, for all practical purposes, the ore reserves of the mine are infinite. Our model can be viewed as an extension to a simplified version of the infinite resource model described in Brennan and Schwartz (1985),[7] in which a one-stage process was assumed.

7. We do not consider in our analysis a cost of opening and closing the mine, as opposed to Brennan and Schwartz (1985).

In this section, we introduce the framework and present the major assumptions of the model. There are two main concerns we want to address. The first is how to value a firm that has two sequential processes, each with a different output capacity. A second related issue is how to value an investment project that expands the capacity of one stage only. By answering these questions, we can determine the optimal capacity of each production stage. Capacity choice in various economic settings has been studied by many authors. Pindyck (1988) is probably the closest in spirit to our analysis, but his is restricted to a one-stage production process.

We now present the notation used in the model:

S = spot price of a unit of final output (second stage);

$\overline{q}_p$ = the maximum first-stage output rate;

q_p = actual first-stage output rate, with $q_p \in (0, \overline{q}_p)$;

$\overline{q}_m$ = the maximum second-stage output rate;

q_m = actual second-stage output rate, with $q_m \in (0, \overline{q}_m)$;

$A_p(q_p)$ = average first-stage unit cost, if first-stage output rate is q_p;

$A_m(q_m)$ = average second-stage unit cost, if second-stage output rate is q_m;

I = work-in-process inventory (output after first-stage process but before second-stage process);

$dI = (q_p - q_m)dt$ = change in work-in-process inventory;

cS = convenience yield on holding one unit of finished output;[8]

r = risk-free rate of interest, assumed constant;

ϕ = operating policy of the firm that specifies q_p and q_m for any value of (S, I); and

$H(S, I; \phi)$ = current value of the firm if the spot price is S, the firm holds work-in-process inventory, I, and the firm follows the operating policy ϕ.

We assume that the spot price for a unit of final (second-stage) output is determined competitively and follows a Brownian motion. Let

$$\frac{dS}{S} = \mu dt + \sigma dz,$$

8. Defined as a flow of services accruing to the holder of one unit of the final output, but unavailable to the holder of a futures contract.

where μ is the instantaneous trend, σ is the known instantaneous standard deviation, t represents time, and dz is the increment to a standard Gauss-Wiener process.

Another critical assumption is the existence of a sufficiently complete market to allow the firm to hedge the final output price risk. This assumption would be met, for example, if a market for futures contracts on the final output of the firm exists.[9]

The rate of cash flows, or *dividends*, that accrue to the owner of the firm are

$$\{q_m[S - A_m(q_m)] - q_p A_p(q_p)\}.$$

Then, standard arguments imply that the optimal operating policy, as well as the value of the firm under this policy, may be obtained by solving

$$\max_{q_p, q_m} \left[\frac{1}{2} H_{SS} S^2 \sigma^2 + (q_p - q_m) H_I + \{q_m[S - A_m(q_m)] - q_p A_p(q_p)\} + (r - c) S H_S - rH \right] = 0,$$

subject to

$$\begin{aligned} q_p &\leq \overline{q}_p \\ q_m &\leq \overline{q}_m \\ q_m &\leq \overline{q}_p \quad \text{if } I = 0. \\ q_p, q_m &\geq 0 \end{aligned}$$

This model does not have an analytical solution. However, it is straightforward to solve using numerical methods. The main contribution of this general formulation is that it provides a practical way for valuing real assets, when the firm can be modeled as having two sequential processes, with bounded output rate. Moreover, this valuation model is specially useful for valuing investments in capacity expansion. Typically, technological and economic reasons induce different bounds on the output rate of each stage of the process. However, most expansion projects add capacity only to some of the processing stages, leaving others unchanged. Thus, to correctly value an expansion project, we should compare the value of the firm, computed using the initial bound (e.g., q_p), to the one using the bound that would result after the investment project is completed. This procedure for obtaining the value of an expansion project takes into account the probability

9. In our copper mine example, the existence of a market for futures on copper satisfies this assumption.

that future decreases in output price may induce the firm not to make use of the additional capacity generated by the project.

The above framework could also be used to analyze the effects of volatility and interest rates on the optimal operating policy for the firm. In this general formulation of the problem, however, a rather complex interrelation may exist between the optimal output rates of the production stages and the state variables S and I, which could be analyzed through numerical simulations. We choose, however, to impose some further restrictions on the model that allows us to obtain analytical solutions and therefore enable us to perform formal comparative static analysis.

III. The Model of Production and Intermediate Inventories

A. *Simplifying Assumptions*

In the previous section, we presented a framework for valuing real assets as compound options. In this section, we add two simplifying assumptions on the technology of the firm, which allow us to obtain analytic solutions, and which provide new insights on the effect of volatility and interest rates on firm value, and on optimal capacity, output, and work-in-process inventory.

First, we assume both stages have a constant-returns-to-scale technology; that is, the average costs, $A_p(q_p)$ and $A_m(q_m)$, are constant. Second, we assume the second-stage process has no upper bound on the output rate ($\bar{q}_m \rightarrow \infty$). This implies that the first stage becomes the bottleneck process and that all work-in-process can be processed instantaneously, should the price of the output become such that it is optimal for the firm to do so. For example, in a copper mine, the bottleneck process could be the extraction of the mineral and not the processing of the mineral and distribution and selling activities. The two new assumptions may be reasonable in some cases, and the conclusions obtained from the model may be relevant for many real-world situations.

B. *Optimal Operating Policy*

Under the new assumptions of constant-returns-to-scale technology and no upper bound on the second-stage output rate, it will always be optimal for the firm to have either zero or the maximum output rate at each of the stages. This can be seen by first analyzing the second-stage process. Given that marginal benefits (S) and marginal costs (A_m) are constant, whenever it is optimal to process and sell one unit, it will be optimal to process and sell them all.[10] A similar situation arises for the

10. Recall that in this type of model it is never optimal to hold inventories of the final output.

first-stage process in which, because the second-stage output rate is unbounded, the benefit of adding one unit of work-in-process and the marginal cost of doing so (A_p) are also constant, inducing a bang-bang solution.

Therefore, the optimal operating policy becomes fully defined by just describing the critical values of the state variables that induce a change in output rate. Furthermore, with no upper bound on the second-stage output rate, the only relevant critical variable will be the spot price.[11] Thus, we define two critical spot prices, $\hat{S}_p$ and $\hat{S}_m$, over which first-stage output and second-stage output are resumed at their maximum rate. For spot prices of the final output below $\hat{S}_p$ and $\hat{S}_m$, the optimal output rate for the correspondent stage is zero.

Given that the firm has the ability to instantaneously process and sell all work-in-process at the market price, should it decide to do so, the critical second-stage price will not depend on the level of these intermediate inventories.[12] Therefore, for the purpose of determining the critical spot price $\hat{S}_m$, we can solve the modified problem of a firm that has an inventory I which can be processed and sold at any moment at a unit profit of $S - A_m$. Future first-stage production will not affect the critical second-stage price and therefore will not be considered.

Let $H^m(S, I)$ be the value of the firm in this modified problem, when the spot price is S and inventory is I. Notice that $H^m(S, I)$ corresponds exactly to the value of I perpetual American calls with an exercise price of A_m, written on a stock of price S and constant dividend yield c. Let $P(S)$ be the value of one of these American calls. Then

$$H^m(S, I) = I \times P(S). \tag{1}$$

Following Ingersoll (1987), the critical price $\hat{S}_m$ at which the call should be optimally exercised is given by

$$\hat{S}_m = A_m\left(\frac{d_1}{d_1 - 1}\right),$$

and the value of each unit of inventory, for a firm that follows the optimal exercise strategy, is

$$P(S) = \begin{cases} (\hat{S}_m - A_m)\left(\dfrac{S}{\hat{S}_m}\right)^{d_1} & \text{if } S < \hat{S}_m \\ (S - A_m) & \text{if } S \geq \hat{S}_m, \end{cases} \tag{2}$$

11. The inventory level does not affect marginal benefits or costs in the processing stages.

12. That is, if it is optimal to process and sell one unit of inventory, it is optimal to process and sell the entire inventory.

where

$$d_1 = \alpha_1 + \alpha_2,$$
$$\alpha_1 = \frac{1}{2} - \frac{(r - c)}{\sigma^2},$$

and

$$\alpha_2 = \sqrt{\left(\alpha_1^2 + \frac{2r}{\sigma^2}\right)}.$$

The results on the critical spot price corresponding to the second-stage process and its sensitivity to changes in volatility and interest rates can be summarized in the following proposition.

PROPOSITION 1. Assume a two-stage firm as above. Then

i) The second-stage critical spot price, $\hat{S}_m$, can be computed as

$$\hat{S}_m = A_m\left(\frac{d_1}{d_1 - 1}\right),$$

with

$$d_1 = \alpha_1 + \alpha_2,$$
$$\alpha_1 = \frac{1}{2} - \frac{(r - c)}{\sigma_2},$$

and

$$\alpha_2 = \sqrt{\left(\alpha_1^2 + \frac{2r}{\sigma^2}\right)}.$$

ii) The second-stage critical spot price, $\hat{S}_m$, is always greater than the second-stage marginal cost, $A_m(\hat{S}_m > A_m)$. Moreover, a one-dollar increase in the second-stage marginal cost, A_m, raises the second-stage critical spot price, $\hat{S}_m$, by more than one dollar ($\partial\hat{S}_m/\partial A_m > 1$).

iii) An increase in the volatility of the spot price, σ^2, or in the risk-free interest rate, r, raises the second-stage critical spot price, $\hat{S}_m(\partial\hat{S}_m/\partial\sigma^2 > 0, \partial\hat{S}_m/\partial r > 0)$.

To obtain the first-stage critical spot price, $\hat{S}_p$, we can consider the modified problem of a firm with only a one-stage process and marginal cost A_p, that produces output that is sold to another firm that performs the activities corresponding to the previous second-stage process. The price at which the first-stage output is sold to the second-stage firm is assumed to be equal to the value that this unit represents to the second-stage firm, given in equation (2).

Let $H^p(S)$ be the value of the modified one-stage firm, just described. Standard arguments imply that the value of this firm and the optimal first-stage production rate, q_p, are given by

$$\max_{q_p}\left[\frac{1}{2}H^p_{SS}S^2\sigma^2 + q_p(P(S) - A_p) + (r - c)SH^p_S - rH^p\right] = 0, \quad (3)$$

subject to

$$q_p \le \bar{q}_p$$

and

$$q_p \ge 0.$$

Given our constant-returns-to-scale technology, a bang-bang solution is obtained. Whenever $(P(S) - A_p) > 0$, it will be optimal to produce $q_p = \bar{q}_p$, otherwise $q_p = 0$. By definition, the first-stage critical spot price, $\hat{S}_p$, is such that it makes $(P(\hat{S}_p) - A_p)$ equal to zero. We must analyze two cases, whether the first-stage critical spot price, $\hat{S}_p$, is greater or smaller than the second-stage critical spot price, $\hat{S}_m$.

1. If $\hat{S}_p \ge \hat{S}_m$, then from equation (2) we know that $P(\hat{S}_p) = \hat{S}_p - A_m$, so

$$\hat{S}_p = A_m + A_p.$$

Therefore, substituting in $\hat{S}_p \ge \hat{S}_m$, we have

$$A_m + A_p \ge A_m\left(\frac{d_1}{d_1 - 1}\right),$$

which is equivalent to

$$A_p \ge A_m\left(\frac{1}{d_1 - 1}\right).$$

2. If $\hat{S}_p < \hat{S}_m$, using equation (2) we obtain

$$\hat{S}_p = A_p^{1/d_1}A_m^{(d_1-1)/d_1}\left[\frac{d_1}{(d_1 - 1)^{(d_1-1)/d_1}}\right].$$

The results on the critical spot price corresponding to the first-stage process, $\hat{S}_p$, can be summarized in the following proposition.

PROPOSITION 2. Assume a two-stage firm as above. Then

i) The first-stage critical spot price, $\hat{S}_p$, can be computed as

$$\hat{S}_p = \begin{cases} A_p^{1/d_1}A_m^{(d_1-1)/d_1}\left[\dfrac{d_1}{(d_1 - 1)^{(d_1-1)/d_1}}\right] & \text{if } A_p < A_m\left(\dfrac{1}{d_1 - 1}\right) \\ A_p + A_m & \text{if } A_p \ge A_m\left(\dfrac{1}{d_1 - 1}\right). \end{cases}$$

ii) $\hat{S}_p < \hat{S}_m$ if and only if $A_p < A_m[1/(d_1 - 1)]$. If $\hat{S}_p < \hat{S}_m$, then the first-stage critical spot price, $\hat{S}_p$, is less than the combined marginal cost of both stages, $A_p + A_m(\hat{S}_p < A_p + A_m)$. Moreover, a one-dollar increase in the first-stage cost, A_p, raises the first-stage critical spot price, $\hat{S}_p$, by more than one dollar ($\partial\hat{S}_p/\partial A_p > 1$), while a one-dollar increase in the second-stage cost, A_m, raises the first-stage critical spot price, $\hat{S}_p$, by less than one dollar ($\partial\hat{S}_p/\partial A_m < 1$).

iii) If $\hat{S}_p \geq \hat{S}_m$, then $\hat{S}_p$ is equal to the combined marginal cost of both stages, $A_p + A_m(\hat{S}_p = A_p + A_m)$. Therefore, a one-dollar increase in either the first or second-stage cost, A_p or A_m, raises the first-stage critical spot price, $\hat{S}_p$, by one dollar ($\partial\hat{S}_p/\partial A_p = \partial\hat{S}_p/\partial A_m = 1$).

iv) If $\hat{S}_p < \hat{S}_m$, then an increase in the volatility of the spot price, σ^2, or in the risk-free interest rate, r, lowers the first-stage critical spot price, $\hat{S}_p$. Otherwise, neither volatility nor interest rates have any effect on $\hat{S}_p$.

C. *Value and Optimal Capacity of the Firm*

It is easy to see that H^p, the value of the modified one-stage-process firm described in the previous section, is actually equal to the value of our original two-stage-process firm, when no work-in-process is in stock ($I = 0$). The rationale is that, given that the first stage sells its output to the second stage for its economic value, the second stage becomes a zero NPV activity. In other words, we are transferring all the economic surplus to the first stage for computational reasons.[13] The solution to equation (3), H^p, becomes the value of all future output of the firm. Thus, to obtain the total value of the original two-stage firm, we should add to H^p the value of current work-in-process inventories. So

$$H(S, I) = H^p(S) + I \times P(S), \tag{4}$$

in which $P(S)$ is computed using equation (2).

To derive the analytical expression for $H^p(S)$, we rewrite equation (3) as

$$\frac{1}{2}H^p_{SS}S^2\sigma^2 + (r - c)SH^p_S - rH^p = -q_p(P(S) - A_p). \tag{5}$$

The optimal value for the output rate, q_p, and the value of $P(S)$ are dependent on the spot price. Assuming that the cost structure of the firm is such that $A_p < A_m[1/(d_1 - 1)]$, there are three different regions, depending on the current spot price.[14] Let

13. Should both stages represent different firms, we are assuming that the second-stage firm participates in a much more competitive environment, while the first-stage extracts all economic surplus.

14. If the cost structure of the firm does not satisfy the above inequality, there will be two regions only, but all basic results are the same.

$W(S)$ = the value of the firm when it is closed ($q_p = 0$). This is the optimal operating policy for $0 \le S < \hat{S}_p$.
$Z(S)$ = the value of the firm when it is open ($q_p = \overline{q}_p$) and it is optimum to hold inventories. This is the optimal operating policy for $\hat{S}_p \le S < \hat{S}_m$.
$V(S)$ = the value of the firm when it is open ($q_p = \overline{q}_p$) and it is optimum to process and sell all work-in-process inventories. This is the optimal operating policy for $\hat{S}_m \le S$.

The value of the firm under the optimal operating policy can then be obtained by solving the following system of equations:

$$\frac{1}{2} W_{SS} S^2 \sigma^2 + (r - c) S W_S - rW = 0, \tag{6}$$

$$\frac{1}{2} Z_{SS} S^2 \sigma^2 + (r - c) S Z_S - rZ = -\overline{q}_p \left(\frac{(\hat{S}_m - A_m)}{(\hat{S}_m)^{d_1}} S^{d_1} - A_p \right), \tag{7}$$

and

$$\frac{1}{2} V_{SS} S^2 \sigma^2 + (r - c) S V_S - rV = -\overline{q}_p (S - A_p - A_m), \tag{8}$$

with the following boundary conditions:

$$W(0) = 0, \tag{9}$$

$$W(\hat{S}_p) = Z(\hat{S}_p), \tag{10}$$

$$W_S(\hat{S}_p) = Z_S(\hat{S}_p), \tag{11}$$

$$Z(\hat{S}_m) = V(\hat{S}_m), \tag{12}$$

$$Z_S(\hat{S}_m) = V_S(\hat{S}_m), \tag{13}$$

and

$$\lim_{S \to \infty} \frac{V(S)}{S} < \infty. \tag{14}$$

The analytical solution to this system of equations is given in the Appendix.

The main results of this section can be summarized in the following proposition.

PROPOSITION 3. Assume a two-stage firm as above. Assume, also, that currently the firm has no work-in-process inventory ($I = 0$). Then

i) The value of the two-stage process firm increases with the spot price level and volatility, S and σ^2; with the output rate bound for the first stage, $\overline{q}_p$; with the risk-free interest rate, r; and with the proportion of second-stage marginal cost over total costs $(A_m/(A_p + A_m))$. However, increases in the convenience yield, c, reduce the value of the firm.

ii) The value of the two-stage-process firm is linear in the maximum output rate for the first stage, $\overline{q}_p$. Thus, the optimal capacity of the firm can be determined by equating the marginal cost of adding one unit to $\overline{q}_p$ to the current plant value per unit of capacity. Moreover, given part i of this proposition, increases in S, σ^2, r, or $(A_m/(A_p + A_m))$, as well as decreases in c, induce a higher optimal capacity, $\overline{q}_p$.

iii) If $\hat{S}_p < \hat{S}_m$, then the option to stock work-in-process inventories has a positive value, which increases with the output rate bound for the first stage, $\overline{q}_p$, and with the proportion of second-stage marginal cost over total costs $(A_m/(A_p + A_m))$.

iv) If $\hat{S}_p \geq \hat{S}_m$, then the option to stock work-in-process inventories has no value.

D. *Discussion and Implications*

We now discuss the implications of propositions 1, 2, and 3, when $\hat{S}_p < \hat{S}_m$, which leads to the most interesting results. Proposition 1 analyzes the characteristics of the critical spot price of the second-stage process, $\hat{S}_m$. Figure 1 illustrates how an increase in the volatility of the spot price, σ^2, raises the critical second-stage spot price,[15] thus reducing second-stage output. This result is in contrast with standard one-stage models that predict that optimal output rate is not affected by volatility if the firm has infinite resources.[16] Also notice that $\hat{S}_m$ is always higher than the marginal second-stage cost, A_m.

Figure 2 shows how an increase in the risk-free rate, r, also raises the critical second-stage spot price. This result highlights the special function that inventories perform in our model. Standard economic models have treated inventories as a cost-reducing buffer stock. Under this perspective, it follows that increases in the holding costs should induce selling accumulated inventories. Our model, in contrast, considers inventories as an investment asset that increases its value with the discount rate. This approach leads to an increase in inventory levels, when interest rates rise.

Note that if A_m is zero, work-in-process effectively becomes final inventory,[17] and proposition 1 predicts that the critical second-stage spot price is zero, leading to the well-known result that it is never optimal to invest in final inventories, when there is a convenience yield that does not accrue to the inventory holder.[18] In addition, it can easily

15. For presentation purposes, we normalize $\hat{S}_m$ by A_m.
16. Recall that we assumed that the resources of the firm have infinite life.
17. For example, no additional cost is added prior to sales.
18. For example, never hold final inventories for speculative reasons.

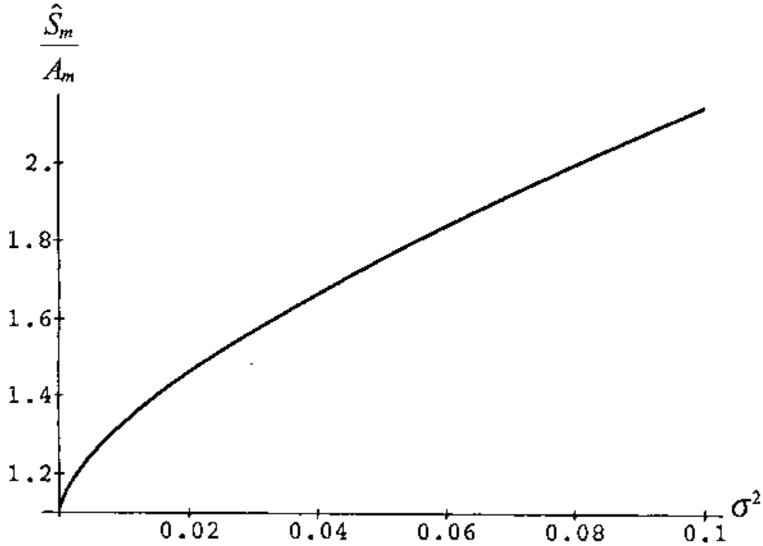

FIG. 1.—Effect of volatility on second-stage critical spot price. This figure is constructed using the following parameter values: $r = 0.10$, $c = 0.09$.

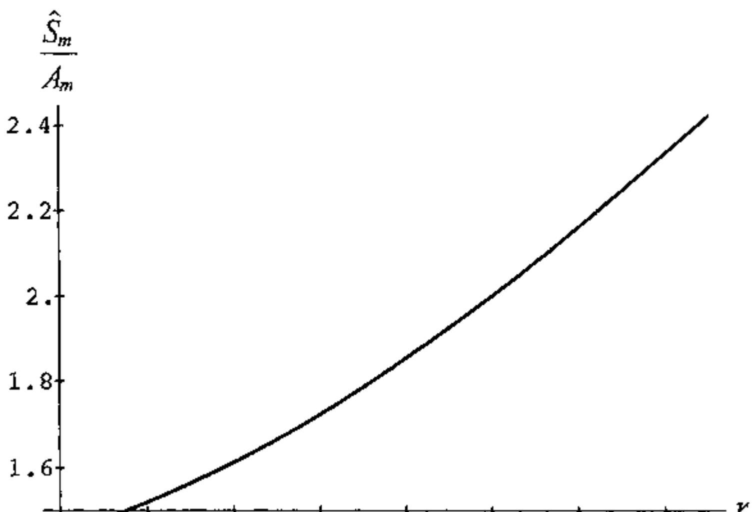

FIG. 2.—Effect of interest rates on second-stage critical spot price. This figure is constructed using the following parameter values: $\sigma^2 = 0.08$, $c = 0.09$.

be shown that $\lim_{\sigma^2 \to 0} d_1 = r/(r - c)$, in which case $\hat{S}_m = A_m(r/c)$, or, in other words, in a certainty world, the expected cost of waiting ($\hat{S}_m c$) equals the benefit in doing so ($A_m r$), while in a risky economy there is an added benefit in waiting, the option to wait, which increases the critical spot price.

Proposition 2 focuses primarily on the behavior of the first-stage critical spot price, $\hat{S}_p$. Figures 3 and 4 illustrate how increases in the spot price volatility and in the risk-free interest rate decrease the first-stage critical spot price, $\hat{S}_p$, effectively increasing the first-stage output rate.[19] This result contrasts with previous real option models that emphasized how increases in these two factors had either a negative or no effect at all on optimal output rate, depending if the firm had finite or infinite resources. Also notice that the critical first-stage spot price, $\hat{S}_p$, is smaller than the combined marginal cost for both stages.

Proposition 3 analyzes the value and optimal capacity of a two-stage-process firm. Figure 5 plots the normalized value of the firm, as a function of the spot price, S. The plot is divided into three regions, depending on which is the optimal operating policy, given current spot prices. The first region corresponds to $W(S)$ (no output from any of the two processes), the second to $Z(S)$ (positive first-stage output, but no second-stage output), and the third to $V(S)$ (positive output from both processing stages). Figure 6 plots the value of the firm as both r and σ^2 vary. It can be seen that increases in either of these two parameters raise the value of the firm.

Finally, Figure 7 plots the percentage increase in firm value for a two-stage firm, as compared to a one-stage firm. This represents the value increase due to the option to invest in work-in-process inventories. It can be seen that for high spot prices, this option has less relative value. Also, the value of this option increases as the ratio of the second-stage marginal cost to total marginal cost ($A_m/(A_p + A_m)$) increases.

IV. Key Insights of the Model

Our two-stage-process model of the firm may provide one explanation for the difficulty in obtaining empirical evidence on the aggregate economic effects of increases in the spot price volatility and interest rates, as discussed in the introduction. The model predicts that for some processing stages of the firm, the effect on output rate of an increase in these variables may be the reverse of other processing stages and industrial sectors, depending on their characteristics.

To be able to relate the findings of our model with the empirical studies in the literature, we need to extend the insights obtained at the

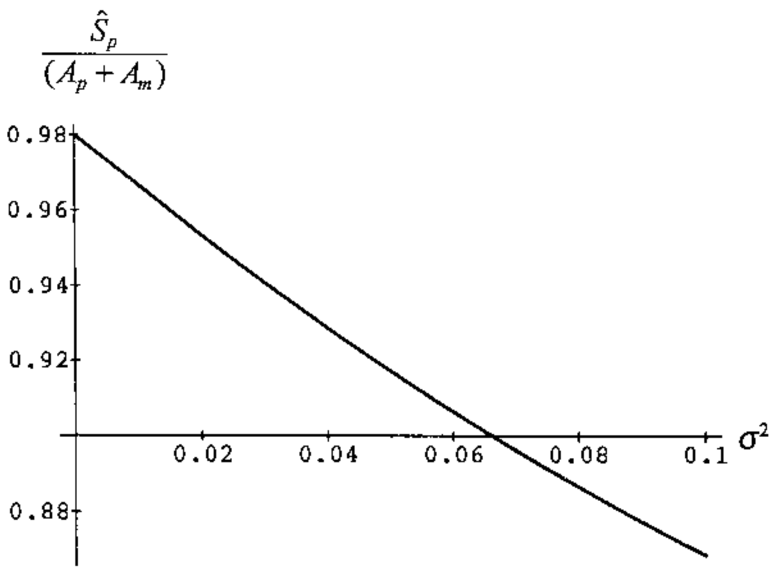

FIG. 3.—Effect of volatility on first-stage critical spot price. This figure is constructed using the following parameter values: $r = 0.10$, $c = 0.04$, $A_p = 50$, $A_m = 50$.

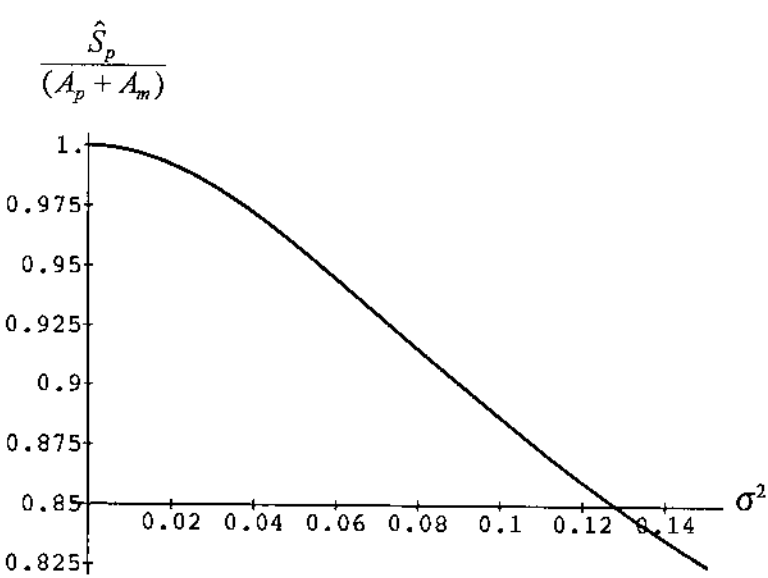

FIG. 4.—Effect of interest rates on first-stage critical spot price. This figure is constructed using the following parameter values: $\sigma = 0.08$, $c = 0.04$, $A_p = 50$, $A_m = 50$.

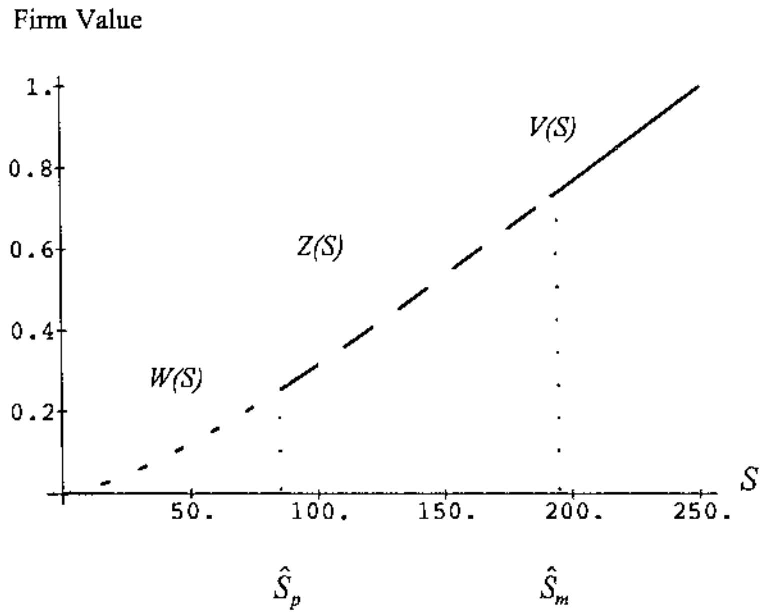

Fig. 5.—Value of the firm versus the spot price, S. Value is presented as a fraction of the value of the firm if the spot price reaches 250. This figure is constructed using the following parameter values: $\sigma = 0.08$, $r = 0.10$, $c = 0.04$, $A_p = 50$, $A_m = 50$, $I = 0$.

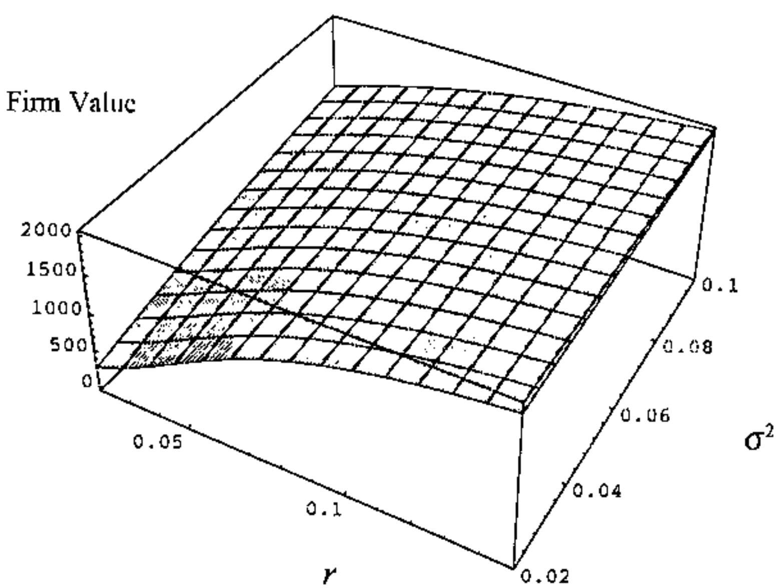

Fig. 6.—Value of the firm versus interest rates and volatility. This figure is constructed using the following parameter values: $S = 100$, $c = 0.04$, $A_p = 25$, $A_m = 75$, and $I = 0$.

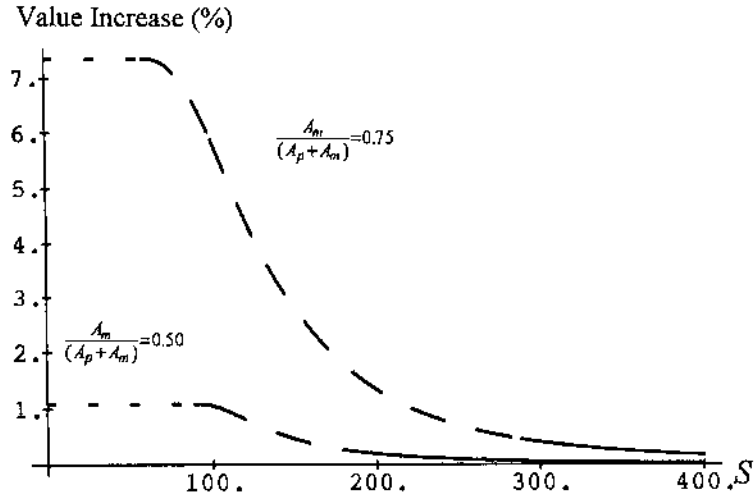

FIG. 7.—Percentage value increase of a two-stage firm. This figure is constructed using the following parameter values: $\sigma = 0.08$, $r = 0.10$, $c = 0.04$, and $I = 0$, $(A_m(A_p + A_m)) = 0.75$ and 0.50.

firm level to the industry level. Assume that the first-stage process of our firm corresponds to those industries that produce output to be used by other industries, which themselves correspond to the second-stage process in our model. Then the output of the first group of industries (not immediately used by the second group of industries) can also be considered work-in-process inventory for our purposes.

Consider first the effect of changes in interest rates on inventories. Our model predicts that increases in interest rates would increase work-in-process inventories, at least for those industries that have capacity constraints, as opposed to standard economic theory that predicts that increases in holding costs would reduce all kinds of inventories. Our approach then suggests that empirical studies looking at the effect of interest rates on inventories should distinguish between work-in-process and final inventories since the reasons for holding them might be different. Our model could be empirically tested by running regressions of changes in inventories on changes in interest rates (and perhaps other variables). The model would predict that the coefficient on the interest varible has a different sign for work-in-process (positive) and final inventory (negative). A weaker prediction, but still consistent with the model, is that the interest rate coefficient for the work-in-process regression is significantly larger than the one for the final inventory regression. The model then provides a possible explanation for the difficulty in documenting any significant effect of interest rates on aggregate inventories.

Similar considerations apply to the effect of increases in volatility on output rates. Our model predicts that some firms (in the first-stage industries) should increase output, whereas other firms (in the second-stage industries) should decrease output, when price volatility (or exchange rate volatility in the case of international trade) increases. Also, increases in volatility should increase work-in-process inventory. If appropriate data were available, these hypotheses could also be tested empirically.

V. Conclusions

In the last decade, an increasing number of papers have suggested that when there is high uncertainty it is more effective to adopt valuation techniques previously derived for financial options than to use the NPV approach. This article follows this trend by presenting a continuous-time real option model for valuing a firm that has fixed unit costs and the ability to modify its production level, depending on the output price realization.

Our model extends current one-stage models by assuming a two-stage technology, in which the output from the first-stage serves as input for the second stage. This new assumption provides the firm with a richer set of alternatives for dealing with uncertainty, which includes the control of the output rate in each processing stage and the use of work-in-process inventories. In addition to being a more realistic model of the firm, our approach offers predictions on the behavior of work-in-process under uncertainty that are not available using one-stage technology models.

The main results of the model are the following. First, it is optimal for firms to operate first-stage processes even at prices substantially lower than marginal costs and to require prices substantially higher than marginal costs to keep second-stage processes open.[20] Second, increases in the risk-free interest rate or in the price volatility raise the value of the firm and induce more first-stage output, less second-stage output, and an increase in work-in-process inventories. This optimal response by the firm is more complex than the one previously predicted by one-stage technology models and could account for the difficulty in empirically detecting those economywide output effects, as reported in the literature. Third, the value of the firm is linear in the capacity of the bottleneck process, providing a relatively simply, yet accurate, way of solving the capacity choice problem under uncertainty. Finally, the option to stock work-in-process has a nonnegative value that is higher the larger is the proportion of second-stage marginal cost over total costs.

20. Marginal costs include all remaining costs prior to the sale. Therefore, the marginal cost relevant for the first-stage process is $A_p + A_m$, while for the second stage it is A_m.

Appendix

Solution of Valuation Equations

The general solution to equation (6) is

$$W(S) = c_1 S^{d_1} + c_2 S^{d_2},$$

$$d_1 = \alpha_1 + \alpha_2,$$

$$d_2 = \alpha_1 - \alpha_2,$$

$$\alpha_1 = \frac{1}{2} - \frac{(r - c)}{\sigma^2},$$

and

$$\alpha_2 = \sqrt{\left(\alpha_1^2 + \frac{2r}{\sigma^2}\right)}.$$

Given that d_2 is negative, condition (9) requires that $c_2 = 0$. So

$$W(S) = c_1 S^{d_1}.$$

Similarly, the general solution to equation (7) is

$$Z(S) = c_5 S^{d_1} + c_6 S^{d_2} + \frac{-\bar{q}_p[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}] S^{d_1} \ln S}{\sigma^2 \alpha_2} - \frac{\bar{q}_p A_m}{r}.$$

Finally, the general solution to equation (8) is

$$V(S) = c_3 S^{d_1} + c_4 S^{d_2} + \frac{\bar{q}_p S}{c} - \frac{\bar{q}_p (A_m + A_p)}{r}.$$

Given that d_1 is greater than one, condition (14) requires that $c_3 = 0$. So

$$V(S) = c_4 S^{d_2} + \frac{\bar{q}_p S}{c} - \frac{\bar{q}_p (A_m + A_p)}{r}.$$

Let $k = [(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]$. To determine the coefficients c_1, c_4, c_5, and c_6; we use boundary conditions (10), (11), (12), and (13):

$$c_1 \hat{S}_p^{d_1} = c_5 \hat{S}_p^{d_1} + c_6 \hat{S}_p^{d_2} + \frac{-\bar{q}_p k \hat{S}_p^{d_1} \ln \hat{S}_p}{\sigma^2 \alpha_2} - \frac{\bar{q}_p A_p}{r},$$

$$c_1 d_1 \hat{S}_p^{d_1 - 1} = \hat{S}_p^{d_1 - 1}\left(c_5 d_1 - \frac{\bar{q}_p k}{\sigma^2 \alpha_2}\right) + \hat{S}_p^{d_2 - 1} c_6 d_2 + \hat{S}_p^{d_1 - 1} \ln \hat{S}_p \left(\frac{-\bar{q}_p k d_1}{\sigma^2 \alpha_2}\right),$$

$$c_4 \hat{S}_m^{d_2} + \frac{\bar{q}_p \hat{S}_p}{c} - \frac{\bar{q}_p (A_m + A_p)}{r} = c_5 \hat{S}_m^{d_1} + c_6 \hat{S}_p^{d_2} + \frac{-\bar{q}_p k \hat{S}_p^{d_1} \ln \hat{S}_p}{\sigma^2 \alpha_2} - \frac{\bar{q}_p A_p}{r},$$

and

$$c_4 d_2 \hat{S}_p^{d_2-1} + \frac{\bar{q}_p}{c} = \hat{S}_p^{d_1-1}\left(c_5 d_1 - \frac{\bar{q}_p k}{\sigma^2 \alpha_2}\right) + \hat{S}_p^{d_2-1} c_6 d_2 + \hat{S}_p^{d_1-1} \ln \hat{S}_p \left(\frac{-\bar{q}_p k d_1}{\sigma^2 \alpha_2}\right).$$

From the above equations, we obtain the following values for the constants,

$$c_1 = \frac{\bar{q}_p A_p d_2}{r\hat{S}_p^{d_1}(d_1 - d_2)} + \frac{\bar{q}_p[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]\ln(\hat{S}_m/\hat{S}_p)}{\alpha_2\sigma^2} - \frac{\bar{q}_p[r\hat{S}_m(-1 + d_2) - d_2 A_m c]}{cr\hat{S}_m^{d_1}(d_1 - d_2)},$$

$$c_4 = \frac{A_p d_1 \bar{q}_p}{r\hat{S}_p^{d_2}(d_1 - d_2)} + \frac{[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]\bar{q}_p(\hat{S}_m^{d_1-d_2} - \hat{S}_p^{d_1-d_2})}{\alpha_2\sigma^2(d_1 - d_2)} + \frac{\bar{q}_p[r\hat{S}_m(1 - d_1) + d_1 A_m c]}{cr\hat{S}_m^{d_2}(d_1 - d_2)},$$

$$c_5 = \frac{[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]\bar{q}_p \ln(\hat{S}_m)}{\alpha_2\sigma^2} + \frac{[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]\bar{q}_p}{\alpha_2\sigma^2(d_1 - d_2)} + \frac{\bar{q}_p[A_m d_2 c - r\hat{S}_m(-1 + d_2)]}{cr\hat{S}_m^{d_1}(d_1 - d_2)},$$

and

$$c_6 = \frac{\bar{q}_p(-[(\hat{S}_m - A_m)/(\hat{S}_m)^{d_1}]r\hat{S}_p^{d_1} + \alpha_2 A_p d_1 \sigma^2)}{\alpha_2 r\hat{S}_p^{d_2}\sigma^2(d_1 - d_2)},$$

which completely determine the value of a zero-inventory, two-stage firm, $H^p(S)$, for any spot price, S. To obtain the value of the firm that currently has intermediate inventories, $H(S, I)$, we must add to H^p the current value of the work-in-process inventories, according to equation (4).

A simple validation of the above expression can be done for a firm with $\hat{S}_p = \hat{S}_m$. For this case, it will never be optimal to hold inventories because when the spot price reaches the critical first-stage level, $\hat{S}_p$, it is also reaching the critical second-stage price. It can easily be shown that for this case our two-stage-firm solution is reduced to the more standard one-stage-firm solution reported in the literature (see, e.g., Brennan and Schwartz 1985).

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