Rational Pricing of Internet Companies

Eduardo S. Schwartz and Mark Moon

We apply real-options theory and capital-budgeting techniques to the problem of valuing an Internet company. We formulate the model in continuous time, form a discrete time approximation, estimate the model parameters, solve the model by simulation, and then perform sensitivity analysis. We report that, depending on the parameters chosen, the value of an Internet stock may be rational if growth rates in revenues are high enough. Even with a real chance that a company may go bankrupt, if the initial growth rates are sufficiently high and if this growth rate contains enough volatility over time, then valuations can reach a level that would otherwise appear dramatically high. In addition, the valuation is highly sensitive to initial conditions and exact specification of the parameters, which is consistent with observations that the returns of Internet stocks have been strikingly volatile.

Probably no recent investment topic elicits stronger feelings than Internet stocks. The skyrocketing valuations of these companies have made millionaires and billionaires out of many Internet entrepreneurs while the actual companies were generating significant, and often growing, losses. Interestingly, as the Internet has grown, so have the means by which individuals can trade over the Internet easily and with relatively low transaction costs.

The view among some traditional money managers is that Internet stocks have been bid upward irrationally by individual day traders sitting at home at their computers and buying any stock that begins with "e-" or ends with ".com." Such managers see the current frenzy as a spectacular example of a market bubble, the likes of which many will witness only once in a lifetime. These traditionalists fear significant negative consequences to the real economy after this bubble bursts. Others see the Internet as dramatically transforming the way in which business is transacted. These investors believe that some of the upstart Internet companies will rapidly grow to dominate and even make irrelevant their traditional bricks-and-mortar competitors.

We apply real-options theory and modern capital-budgeting techniques to the problem of valuing an Internet stock. We formulate the model in continuous time, form a discrete time approximation, estimate the model parameters, solve the model by simulation, and then perform sensitivity analysis.

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Continuous-Time Model

In developing the simple model to value Internet stocks, for simplicity, we initially describe the model in continuous time. Its implementation, however, will use the quarterly accounting data available from Internet companies and be in discrete time.

Consider an Internet company with instantaneous rate of revenues (or sales) at time t given by R_t. Assume that the dynamics of these revenues are given by the stochastic differential equation

$$\frac{dR_t}{R_t} = \mu_t dt + \sigma_t dz_1, \qquad (1)$$

where μ_t, the drift, is the expected rate of growth in revenues and is assumed to follow a mean-reverting process with a long-term average drift $\bar{\mu}$; σ is volatility in the rate of revenue growth; and z_1 is a random variable that reflects the draw from a normal distribution. That is, the initial very high growth rates of the Internet company are assumed to converge stochastically to the more reasonable and sustainable rate of growth for the industry to which the company belongs:

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + \eta_t dz_2, \qquad (2)$$

where η_0 is the initial volatility of expected rates of growth in revenues. The mean-reversion coefficient, κ, describes the rate at which the growth is expected to converge to its long-term average; so, $\ln(2)/\kappa$ can be interpreted as the "half-life" of the deviations, in that any deviation μ is expected to be halved in this time period.

The unanticipated changes in revenues are also assumed to converge (deterministically) to a more normal level, and the unanticipated changes in the drift are assumed to converge (also deterministically) to zero:

$$d\sigma_t = \kappa_1(\bar{\sigma} - \sigma_t)\,dt; \tag{3}$$

$$d\eta_t = -\kappa_2\eta_t\,dt. \tag{4}$$

The unanticipated changes in the growth rate of revenues and the unanticipated changes in its drift may be correlated:

$$dz_1 dz_2 = \rho dt. \tag{5}$$

The net after-tax rate of cash flow to the company, Y_t, is then given by

$$Y_t = (R_t - \text{Cost}_t)(1 - \tau_c), \tag{6}$$

where τ_c is the corporate tax rate.

The costs at time t have two components. The first is the cost of goods sold (COGS), which is assumed to be proportional to the revenues. The second is other expenses, which are assumed to have a fixed component, F, and a variable component proportional to the revenues:

$$\begin{aligned}\text{Cost}_t &= \text{COGS}_t + \text{Other expenses}_t \\ &= \alpha R_t + (F + \beta R_t) \\ &= (\alpha + \beta)R_t + F,\end{aligned} \tag{7}$$

where α is COGS as a percentage of revenues and β is the variable component of other expenses.

More-complicated cost structures can be easily accommodated in the model. For example, the cost function could be stochastic, reflecting the uncertainty about future potential competitors, market share, or technological developments.[1] The corporate tax rate in Equation 6 is only paid if there is no loss carry-forward (i.e., if the loss carry-forward is positive, the tax rate is zero).

For simplicity in this framework, we have neglected the depreciation tax shields in the computation of the after-tax cash flow. These shields could be easily incorporated, however, in the analysis.

The dynamics of the loss carry-forward, L_t, are given by

$$dL_t = -Y_t\,dt \qquad \text{if } L_t > 0 \tag{8a}$$

or

$$dL_t = \max(-Y_t\,dt, 0) \qquad \text{if } L_t = 0. \tag{8b}$$

Finally, the company is assumed to have an amount of cash available, X_t, that evolves according to

$$dX_t = Y_t\,dt. \tag{9}$$

The company is assumed to go bankrupt when the amount of its available cash reaches zero. That is, bankruptcy in the model is defined as the first time X_t hits zero. This bankruptcy condition is clearly a simplification of reality. It does not take into account the possibility of additional financing in the future. In particular, the company could run out of cash but have good enough prospects to be able to raise cash, sell all its equity, or merge with another company. Later, we discuss a more realistic alternative bankruptcy condition that addresses some of these issues.

If future financing is planned, the cash raised could be added to the cash balances available at the time of issue. The possible future financing could even be state dependent; that is, it could be a function of the revenues and the expected rate of growth in revenues at the time of issue. To keep things simple, we assume that there will be no additional financing in the future.

To avoid having to define a dividend policy in the model, we assume that the cash flow generated by the company's operations remains in the company, earns the risk-free rate of interest, and will be available for distribution to the shareholders at an arbitrary long-term horizon, T, by which time the company will have reverted to a "normal" company. This assumption may induce an underestimation of the probability of bankruptcy, but because this type of company is unlikely to start paying dividends until the cash flows are reliably positive, this underestimation will probably be small. Then, the interest earned on the cash available has to be added to the revenues in Equation 6.

The objective of the model is to determine the value of the Internet company at the current time (assumed to be time zero), V_0. According to standard theory, this value is obtained by discounting the expected net cash flow to the company under the risk-neutral measure (the equivalent martingale measure), E_Q, at the risk-free rate, which for simplicity is assumed to be constant:[2]

$$V_0 = E_Q\left(X_t e^{-rT}\right), \tag{10}$$

where e^{-rT} is the continuously compounded discount factor.

An implicit assumption in Equation 10 is that the company is liquidated at the horizon T and all cash flows are distributed. In most cases, a terminal value for the company that is related to the net cash flow at the horizon (given by Equation 6) might be more appropriate. For example, the value of the company at the horizon could be assumed to be a multiple (e.g., 10 times) of earnings before interest, taxes, depreciation, and amortization (EBITDA), which would make the value of the company less sensitive to the horizon chosen.

The model has two sources of uncertainty. The first is uncertainty about the changes in revenues, and the second is uncertainty about the expected rate of growth in revenues. Under some simplifying assumptions (see, for example, Brennan and Schwartz 1982), the risk-adjusted processes for the state variables can be obtained from the true processes, as in

$$\frac{dR_t}{R_t} = (\mu_t - \lambda_1 \sigma_t)\,dt + \sigma_t\,dz_1^*, \tag{11}$$

$$d\mu_t = [\kappa(\bar{\mu} - \mu_t) - \lambda_2 \eta_t]\,dt + \eta_t\,dz_2^*, \tag{12}$$

and

$$dz_1^* dz_2^* = \rho\,dt, \tag{13}$$

where the market prices of factor risks, λ_1 and λ_2, are constant and the asterisk indicates that the process is risk adjusted.

The expectation in Equation 10 is taken with respect to these risk-adjusted processes. Note that because the cash flow in Equation 10 is discounted at the risk-free rate and is also assumed to earn the risk-free rate if retained in the company, if the probability of bankruptcy is negligible, then the timing of the cash flow does not affect V_0.

Implicit in the model is that the value of the company at any point in time is a function of the value of the state variables (revenues, expected growth in revenues, loss carry-forward, and cash balances) and time. That is, the value of the company can be written as

$$V \equiv V(R, \mu, L, X, t). \tag{14}$$

Applying Ito's lemma to this expression, we can obtain the dynamics of the value of the company as

$$\begin{aligned} dV &= V_R dR + V_\mu d\mu + V_L dL + V_X dX + V_t dt \\ &\quad + \frac{1}{2}V_{RR} dR^2 + \frac{1}{2}V_{\mu\mu} d\mu^2 + V_{R\mu} dR\,d\mu. \end{aligned} \tag{15}$$

The volatility of the company's value can be derived directly from

$$\begin{aligned} \sigma_V^2 &= \frac{1}{dt}\operatorname{var}\left(\frac{dV}{V}\right) \\ &= \left(\frac{V_R}{V}\sigma R\right)^2 + \left(\frac{V_\mu}{V}\eta\right)^2 + 2\frac{V_R V_\mu}{V^2} R\sigma\eta\rho. \end{aligned} \tag{16}$$

The model can then be used to determine not only the value of the company but also its volatility.

Discrete Version of the Model

The model developed in the previous section is path dependent. The cash available at any time, which determines when bankruptcy is triggered, depends on the whole history of past cash flows. Similarly, the loss carry-forward, which determines when the company has to pay corporate taxes, is also path dependent. In a more general model that also included depreciation tax shields, which would affect the after-tax cash flow, path dependencies would become even more complex.

These path dependencies can easily be taken into account by using Monte Carlo simulation to solve for the value of the Internet company. To implement the simulation, the discrete version of the risk-adjusted process, Equations 11–13, is used:[3]

$$R_{t+\Delta t} = R_t e^{\{[\mu_t - \lambda_1 \sigma_t - (\sigma_t^2/2)]\Delta t + \sigma_t \sqrt{\Delta t}\varepsilon_1\}} \tag{17}$$

and

$$\begin{aligned} \mu_{t+\Delta t} &= e^{-\kappa\Delta t}\mu_t + \left(1 - e^{-\kappa\Delta t}\right)\left(\bar{\mu} - \frac{\lambda_2 \eta_t}{\kappa}\right) \\ &\quad + \sqrt{\frac{1 - e^{-2\kappa\Delta t}}{2\kappa}}\,\eta_t \sqrt{\Delta t}\varepsilon_2, \end{aligned} \tag{18}$$

where

$$\sigma_t = \sigma_0 e^{-\kappa_1 t} + \bar{\sigma}\left(1 - e^{-\kappa_1 t}\right) \tag{19}$$

and

$$\eta_t = \eta_0 e^{-\kappa_2 t}. \tag{20}$$

Equations 19 and 20 were obtained by integrating Equations 3 and 4, with initial values σ_0 and η_0; ε_1 and ε_2 are standard normal variates with correlation ρ.

The net after-tax cash flow is still given by Equation 6, where both revenues and costs are measured over the period Δt. The discrete versions of the dynamics of the loss carry-forward and the amount of cash available are immediate from, respectively, Equations 8 and 9.

Estimation of the Parameters

Even the simple model, described in the previous section, requires more than 20 parameters for its implementation. Some of these parameters are easily observable; others can be estimated from the quarterly data available for most Internet companies. The determination of some parameters, however, requires the use of judgment, which can come only from a thorough knowledge of the specific situation.

The estimation of the parameters of the model is probably the most critical in the analysis—and the one that requires the most expertise about the particular Internet company being valued and its industry. We describe the parameters of the model in **Exhibit 1** and give some suggestions about how to estimate them. For the actual implementation of the approach, detailed study would be required. Because these companies have limited past histories from which to estimate the parameters, the analyst must use judgment and knowledge of the company's industry and characteristics to infer the parameters.

Keep in mind also that, at this stage, the whole company is being valued. To obtain the value of the stock, we will investigate the details of the capital structure and the options that most of these companies grant generously to their employees. We explore this issue in the next section.

Exhibit 1. Key Parameters of the Model

Parameter	Notation	Proposed Estimation Procedure
Initial revenue	R_0	Observable from current income statement
Initial loss carry-forward	L_0	Observable from current balance sheet
Initial cash balance available	X_0	Observable from current balance sheet
Initial expected rate of growth in revenues	μ_0	From past income statements and projections of future growth
Initial volatility of revenues	σ_0	Standard deviation of percentage change in revenues over the recent past
Initial volatility of expected rates of growth in revenues	η_0	Inferred from market volatility of stock price
Correlation between percentage change in revenue and change in expected rate of growth	ρ	Estimated from past company or cross-sectional data
Long-term rate of growth in revenues	$\bar{\mu}$	Rate of growth in revenues for a stable company in the same industry as the company being valued
Long-term volatility of the rate of growth in revenues	$\bar{\sigma}$	Volatility of percentage changes in revenues for a stable company in the same industry as the company being valued
Company's corporate tax rate	τ_c	Observable from tax code
Risk-free interest rate	r	One year U.S. T-bill rate
Speed of adjustment for the rate of growth process	κ	Estimated from assumptions about the half-life of the process to $\bar{\mu}$
Speed of adjustment for the volatility of revenue process	κ_1	Estimated from assumptions about the half-life of the process to $\bar{\sigma}$
Speed of adjustment for the volatility of the rate of growth process	κ_2	Estimated from assumptions about the half-life of the process to zero
COGS as a percentage of revenues	α	Analysts' future projections
Fixed component of other expenses	F	Analysts' future projections
Variable component of other expenses	β	Analysts' future projections
Market price of risk for the revenue factor	λ_1	Obtained from the product of the correlation between percentage changes in revenues and return on aggregate wealth multiplied by the standard deviation of aggregate wealth
Market price of risk for the expected rate of growth in revenues factor	λ_2	Obtained from the product of the correlation between changes in growth rates in revenues and return on aggregate wealth multiplied by the standard deviation of aggregate wealth
Horizon for the estimation	T	An arbitrary long-term horizon at which the company is deemed to become a "normal" company
Time increment for the discrete version of the model	Δt	Chosen according to data availability, which is usually quarterly

Simulation Results

We illustrate the methodology for valuing Internet companies by applying it to one of the best-known companies in the sector—Amazon.com. The basic data are given in **Table 1** and include quarterly sales, COGS, and other expenses for the last 15 quarters. In addition to these data, we used balance sheet data to estimate the loss carry-forward and the amount of cash available. We performed the evaluation with the information available as of December 31, 1999, which included financial statements from the third quarter (Q3) of 1999, and supplementary analyst projections as of that quarter.

Sales grew dramatically at the beginning of the sample period, as **Figure 1** shows, but then began to slow. **Figure 2** shows that the growth rate during the sample period started out very high and then declined. **Figure 3** and **Figure 4** show, respectively, the relationship between COGS and sales and between selling, general, and administrative expenses (SG&A) and sales. The relationship between COGS and sales seems to have been stable; the relationship between SG&A and sales was more erratic. Part of the reason is the extraordinary expense of building infrastructure, some of which did not reflect actual cash outlays.

Figure 5 shows the stock price from May 1997 to December 1999. Clearly, the stock price grew dramatically up to December 1998, after which it exhibited great volatility without an apparent trend.

We present the parameters we used in our basic valuation of Amazon in **Exhibit 2**. Some of these parameters came from the financial statements or were otherwise directly observable, so they need no further explanation. Others were estimated from past data and/or future projections and will be discussed further. We performed sensitivity analyses to determine the sensitivity of Amazon's value to changes in such estimated parameters.

For the initial expected rate of growth in revenues, we took the average growth rate over the last two quarters, and for the rate of growth over the next four quarters, we used analyst expectations from I/B/E/S International. The standard deviation of past percentage changes in revenue was used as the initial volatility of revenues. The initial volatility of the expected rate of growth in revenues was inferred from the observed stock price volatility. We assumed that the changes in revenues and changes in expected growth rates were uncorrelated.[4] For the long-term rate of growth in revenues for the industry, we chose 1.5 percent per quarter (6 percent per year), and for the long-term volatility of revenues, we chose 5 percent per quarter (10 percent per year). To obtain the three speed-of-adjustment or mean-reversion coefficients, we assumed that the half-life of the deviations was approximately 10 quarters.

Table 1. Quarterly Sales and Costs for Amazon, March 1996–September 1999 (millions)

Date	Sales	COGS	Gross Profit	Selling, General, and Administrative Expenses	Operating Profit before Taxes (EBITDA)
1996					
March	$ 0.875	$ 0.678	$ 0.197	$ 0.516	–$0.319
June	2.230	1.725	0.505	1.253	–0.748
September	4.173	3.172	1.001	3.383	–2.382
December	8.468	6.426	2.042	4.286	–2.244
1997					
March	16.005	12.484	3.521	6.623	–3.102
June	27.855	22.641	5.214	13.067	–7.853
September	37.887	30.717	7.170	17.486	–10.316
December	66.040	53.127	12.913	24.237	–11.324
1998					
March	87.361	66.222	21.139	29.283	–8.144
June	116.044	89.793	26.251	44.651	–18.400
September	153.698	118.823	34.875	76.381	–41.506
December	252.893	199.476	53.417	95.486	–42.069
1999					
March	293.643	223.629	70.014	95.386	–25.372
June	314.377	246.846	67.531	190.005	–122.474
September	355.800	285.300	70.500	260.945	–190.445

Figure 1. Amazon Quarterly Sales, Q1 1996–Q3 1999

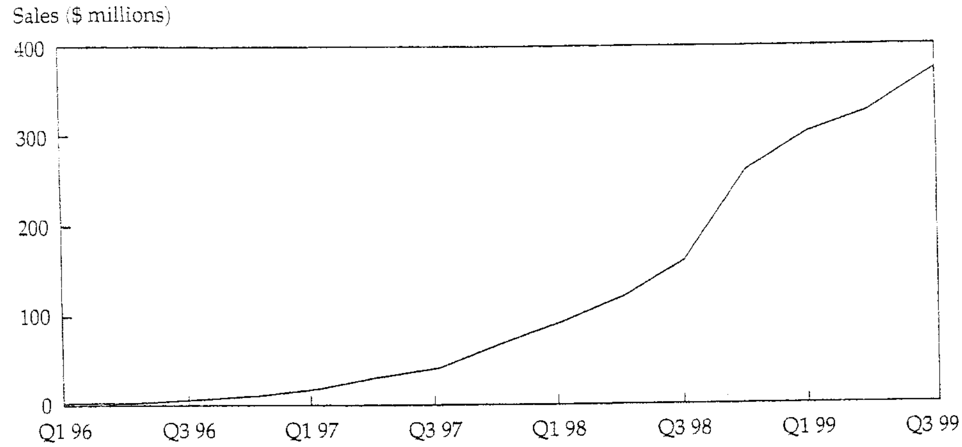
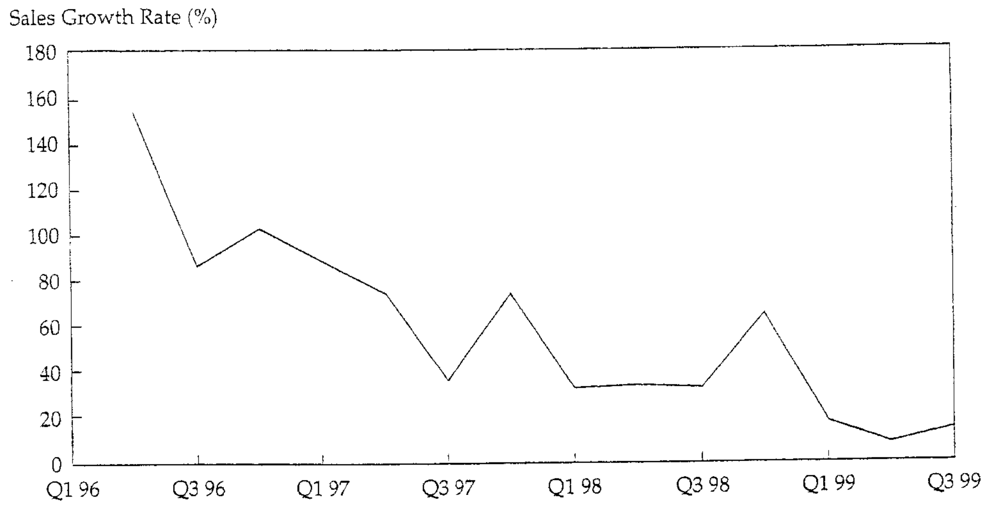


Figure 2. Amazon Quarterly Sales Growth Rate, Q1 1996–Q3 1999



For the critical cost parameters, which we have simplified in this illustration, we assumed that COGS would be 75 percent of the revenues, very much in accordance with the available data. We used a higher fixed component of other expenses ($75 million per quarter) and a lower variable component as a proportion of revenues (19 percent) than the historical past to reflect some recent extraordinary expenses. Had we used the cost parameters estimated from the historical data, the model would have projected that Amazon would never make any profits because the historical profit margins were negative.

To estimate the two market prices of risk, we used as the standard deviation for aggregate wealth 5 percent per quarter (or 10 percent per year). We assumed a correlation of 0.2 between the percentage changes in revenue and the return on aggregate wealth, but we assumed that the changes in growth rates were uncorrelated with aggregate wealth. Finally, we took 25 years as the horizon of the estimation and, because all the data we had were provided quarterly, one quarter as the time increment. For a terminal value at the 25-year horizon, we assumed the value of Amazon would be equal to 10 times pretax operating profit (EBITDA), which is an approach practitioners frequently use.

For all the valuations, we used 100,000 simulations. For the base valuation, which used the parameters of Exhibit 2, the total value of Amazon was $5,457 million. We obtained this value despite the company going bankrupt in 27.9 percent of the simulations. **Table 2** reports the proportion of bankruptcies per year for the base case. Note that the bankruptcies start only in Year 5, when cash has been exhausted, and that no bankruptcies show up after Year 18. The majority of the bankruptcies projected by the model occur in Year 6, and the number decreases slowly up to Year 18.

Figure 3. Amazon COGS versus Sales

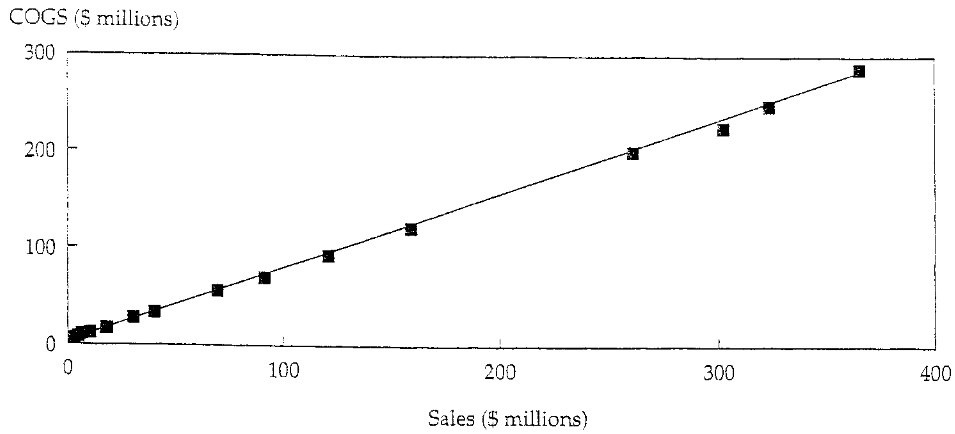


Figure 4. Amazon SG&A versus Sales

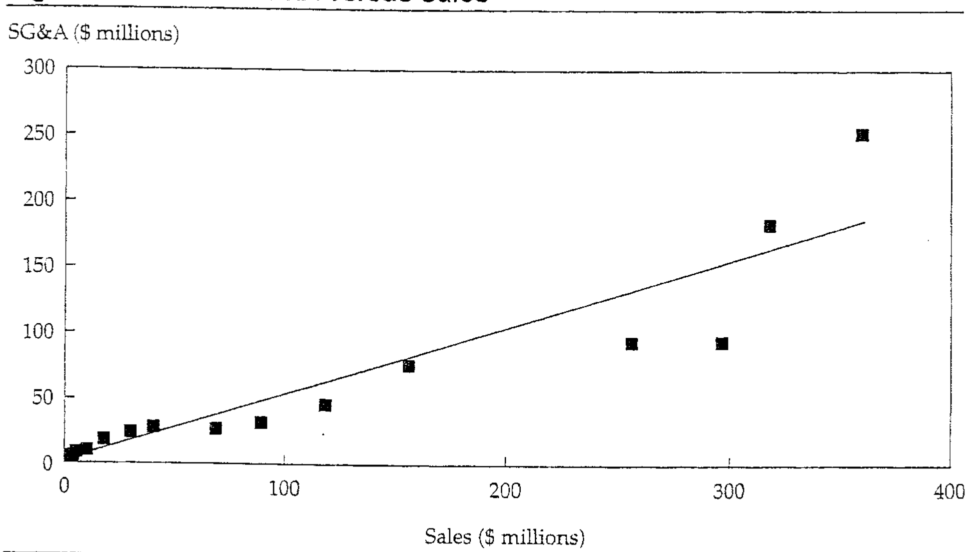
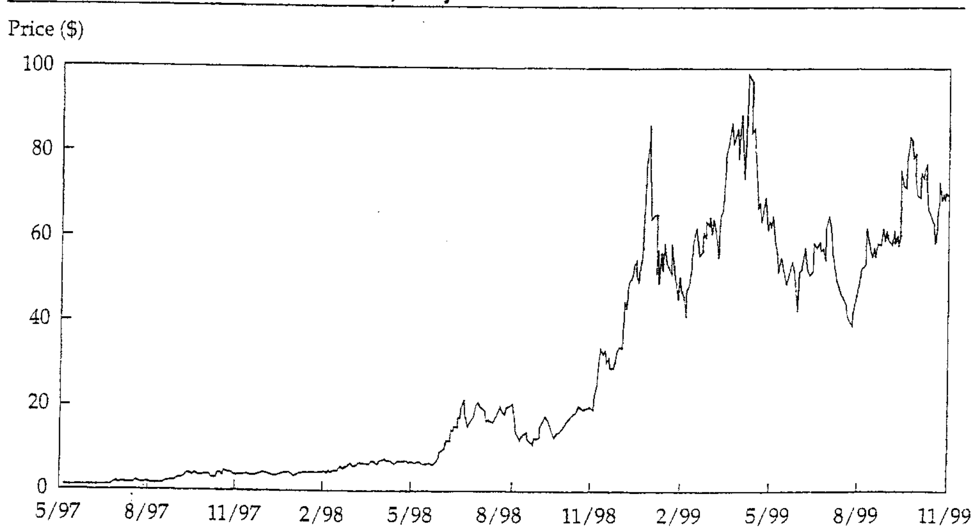


Figure 5. Amazon Share Price, May 1997–November 1999



Note: Adjusted for stock splits.

Exhibit 2. Parameters Used in the Base Valuation of Amazon

Parameter	Notation	Proposed Estimation Procedure
Initial revenue	R_0	\$356 million / quarter
Initial loss carry-forward	L_0	\$559 million
Initial cash balance available	X_0	\$906 million
Initial expected rate of growth in revenues	μ_0	0.11 / quarter
Initial volatility of revenues	σ_0	0.10 / quarter
Initial volatility of expected rates of growth in revenues	η_0	0.03 / quarter
Correlation between percentage change in revenue and change in expected rate of growth	ρ	0.0
Long-term rate of growth in revenues	$\bar{\mu}$	0.015 / quarter
Long-term volatility of the rate of growth in revenues	$\bar{\sigma}$	0.05 / quarter
Company's corporate tax rate	τ_c	0.35
Risk-free interest rate	r	0.05 / year
Speed of adjustment for the rate of growth process	κ	0.07 / quarter
Speed of adjustment for the volatility of revenue process	κ_1	0.07 / quarter
Speed of adjustment for the volatility of the rate of growth process	κ_2	0.07 / quarter
COGS as a part of revenues	α	0.75
Fixed component of other expenses	F	\$75 million / quarter
Variable component of other expenses	β	0.19
Market price of risk for the revenue factor	λ_1	0.01 / quarter
Market price of risk for the expected rate of growth in revenues factor	λ_2	0.0 / quarter
Horizon for the estimation	T	25 years
Time increment for the discrete version of the model	Δt	1 quarter

Table 3 reports the sensitivity of the total value of Amazon to the most critical parameters. We obtained the numbers by using a perturbation (usually a 10 percent higher value) for the indicated parameter while leaving all the other parameters the same as the base valuation. The table shows that two sets of parameters have a significant effect on the value of the firm. First, and most obviously, is the variable component of the cost function, which is proportional to the revenues. Equation 7 indicates that an increase in either α or β has the same effect on the cost function and, therefore, also on the value of the company. In the base example, the sum of these two variable costs is 94 percent of sales, leaving a profit margin of only 6 percent of sales. If any of these variable costs are increased by 1 percent, as in Table 3, the profit margin decreases to 5 percent of sales and the value of Amazon decreases from \$5.5 billion to about \$4.3 billion (a 22 percent decrease, in line with the decrease in profit margin). This discussion emphasizes the importance of correctly assessing the variable components of the cost function.[5]

The second, and not so obvious, set of parameters that have a significant effect on the value of the firm are the parameters for the stochastic process of changes in the growth rate in revenues (Equation 2)—in particular, those parameters that affect the future distribution of rates of growth in revenues. An increase in the initial volatility of this rate of growth, η_0, from 30 percent to 33 percent per quarter (a 10 percent increase) increases the value of Amazon from \$5.5 billion to about \$6.3 billion. Similarly, but in the opposite direction, an increase in the mean-reversion coefficient, κ, from 70 percent to 77 percent decreases the value of Amazon from \$5.5 billion to about \$4.3 billion. The deterministic mean-reversion coefficient for the volatility of this process, κ_2, also has a significant effect but not as large as η_0 and κ. These three parameters affect the

Table 2. Probability of Bankruptcy per Year for Base Valuation

Year	Bankruptcy
1	0.0%
2	0.0
3	0.0
4	0.0
5	3.9
6	9.0
7	6.2
8	3.5
9	2.0
10	1.1
11	0.7
12	0.5
13	0.3
14	0.2
15	0.2
16	0.1
17	0.1
18	0.1
19	0.0
20	0.0
21	0.0
22	0.0
23	0.0
24	0.0
25	0.0
Total	27.9%

Table 3. Sensitivity of Amazon's Value to Changed Parameters

Parameter	Value of Perturbed Parameter	Total Amazon Value (millions)	Standard Deviation	Probability of Bankruptcy
Base case		$5,457	34%	27.9%
μ_0	0.121/quarter	6,558	39	22.8
σ_0	0.11/quarter	5,446	34	28.7
η_0	0.033/quarter	6,256	44	29.6
ρ	0.01	5,483	34	28.0
$\bar{\mu}$	0.0165/quarter	6,064	14	26.9
$\bar{\sigma}$	0.055/quarter	5,437	34	28.5
κ	0.077/quarter	4,282	24	29.9
κ_1	0.077/quarter	5,461	33	27.8
κ_2	0.077/quarter	5,134	30	27.2
α	0.76	4,349	28	37.1
F	$82.5 million/quarter	5,253	34	35.6
β	0.20	4,349	28	37.1
λ_1	0.011/quarter	5,429	33	28.1
λ_2	0.001/quarter	5,423	33	28.1
T	26 years	5,620	35	28.2

distribution of future rates of growth in revenues. Increases in the initial volatility of the growth rate in revenues will increase the variance of this distribution, and increases in the mean-reversion coefficient or the mean-reversion coefficient for the volatility of this process will reduce this variance.

The variance of the distribution of future growth rates is important in the valuation because it determines the *option value* of the Internet firm. High variance of future rates of growth implies a higher probability of both very high rates of growth and of very low (or even negative) rates of growth. For individual paths of the growth rate over time, higher growth rates lead to larger cash flows, which imply a more valuable company. In contrast, if growth rates are sufficiently low, the company may go bankrupt. In the event that the company goes bankrupt, however, it will be worth zero if growth rates are just low enough for the company to go bankrupt or even if growth rates are far lower than that critical level. Limited liability for the shareholders of the company implies a nonlinearity in the valuation function, which results in a more valuable company, given a more variable distribution of future growth rates. **Figure 6** shows the probability density of rates of growth in revenues 5 years and 10 years into the future for the parameters of the base valuation. Because the variance of this distribution is important to the valuation, parameters should be jointly chosen to give what is believed to be a reasonable distribution of future rates of growth (and of future revenues) for an Internet company.

The variance of the rate of growth in revenues has an effect not only on the option value of the company but also on the mean of revenue distribution. Higher volatility implies higher mean revenues because of Jensen's inequality and the inference of Equation 17.

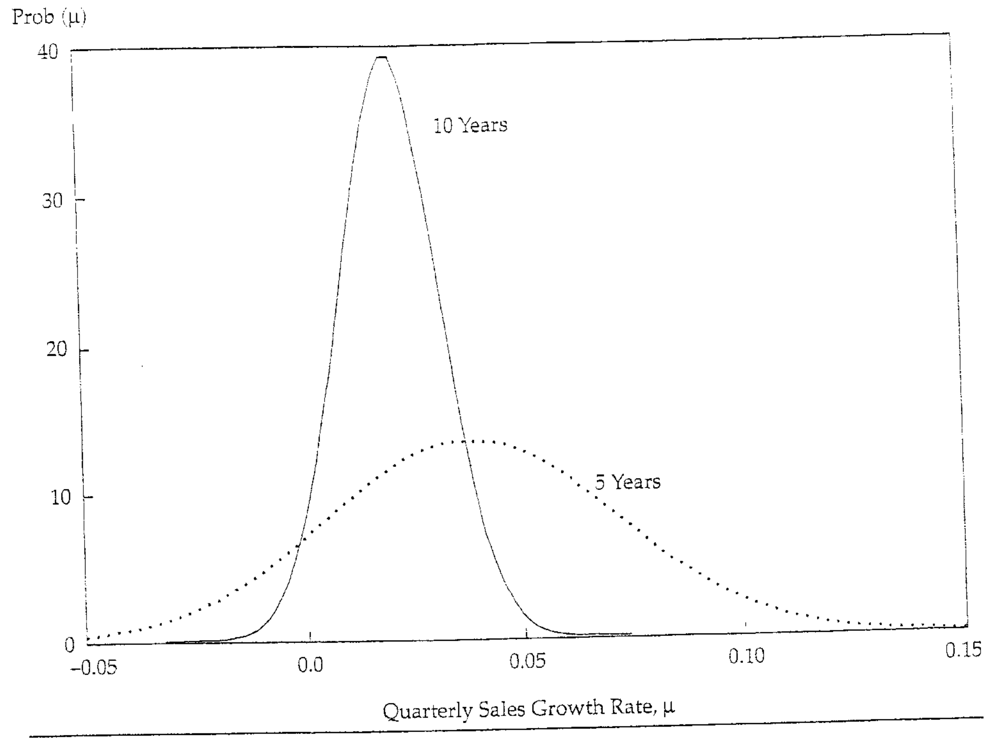
Table 4 shows the quarterly distribution of revenues 1, 3, 5, 7, and 10 years in the future. The means for one and three years are approximately consistent with analysts' forecasts. Note that the mean quarterly revenues grow substantially over time, reaching $3.8 billion in 10 years.

Determining Share Value

To obtain the share price of an Internet company, we need to examine the capital structure of the company in more detail than we did in determining the value. We need to know how many shares are outstanding and how many shares are likely to be issued to employees who hold stock options and holders of convertible bonds. We also need to know how much of the cash flow will be available to the shareholders after coupon and principal payments to the bondholders.

To simplify the analysis, we assume that options will be exercised and convertible bonds will be converted into shares whenever the company survives. That is, in the no-default paths of the simulations, we adjust the number of shares to reflect the exercise of options and convertibles. To obtain the cash flow available to shareholders from the cash flow available to all securityholders (which determines the total value of the company), we subtract the principal and after-tax coupon payments

Figure 6. Amazon Sales: Growth Rate Probability Density



on the debt and add the payments by optionholders at the exercise of the options. Because we are assuming that the company pays no dividends, the exercise of the options and convertibles occurs at their maturity. If all optionholders exercise their options optimally, this procedure overvalues the stock by undervaluing the options and convertibles (because there may be some countries of the world where a company survives but exercising the options or converting the convertibles is not optimal).

Table 4. Amazon Quarterly Revenue Distributions (millions)

	Years Forward				
Percentile	1	3	5	7	10
5	$370	$ 398	$ 379	$ 366	$ 374
10	399	476	495	511	547
15	421	538	597	641	715
20	438	593	692	766	879
25	453	643	782	893	1,051
30	468	693	873	1,024	1,234
35	482	743	967	1,161	1,427
40	495	794	1,066	1,311	1,648
45	508	846	1,172	1,472	1,887
50	522	899	1,286	1,651	2,158
55	535	956	1,411	1,850	2,468
60	550	1,019	1,550	2,078	2,827
65	565	1,088	1,709	2,346	3,265
70	581	1,166	1,893	2,661	3,775
75	600	1,257	2,114	3,053	4,431
80	621	1,365	2,388	3,559	5,300
85	646	1,503	2,770	4,254	6,510
90	681	1,700	3,337	5,332	8,521
95	735	2,030	4,363	7,444	12,448
Mean	$533	$1,017	$1,692	$2,507	$ 3,810

In addition, employees frequently exercise stock options before maturity, if the options are exercisable, to allow for the sale of the underlying stock for diversification purposes. Also, even if the options are in the money, not all of them will be exercised because many employees will leave the company before they are vested in their stock options. If the number of shares to be issued at exercise and conversion is small relative to the total number of shares outstanding, the impact of these shares on share value is likely to be small. In the next section, we discuss an extension that takes into account the optimal exercise of these options.

At the valuation date in this illustration, Amazon had 339 million shares outstanding. In addition to equity, the capital structure consisted of a convertible bond, a discount note, and employee stock options. The convertible debt issue has a face value of $1,250 million with a coupon rate of 4.75 percent; it matures in February of 2009 and is convertible into 16 million shares. The senior discount note has a face value of $265 million and matures in May of 2008. The employee stock options outstanding as of December 31, 1999, were obtained from the company's 10-K form and have been adjusted for a subsequent stock split. In total, there were 76 million options outstanding, of which 60 million (more than 78 percent) had average exercise prices below $7.50. Because the stock price on the valuation date was approximately $75 a share, these options are likely to be exercised if the company survives.

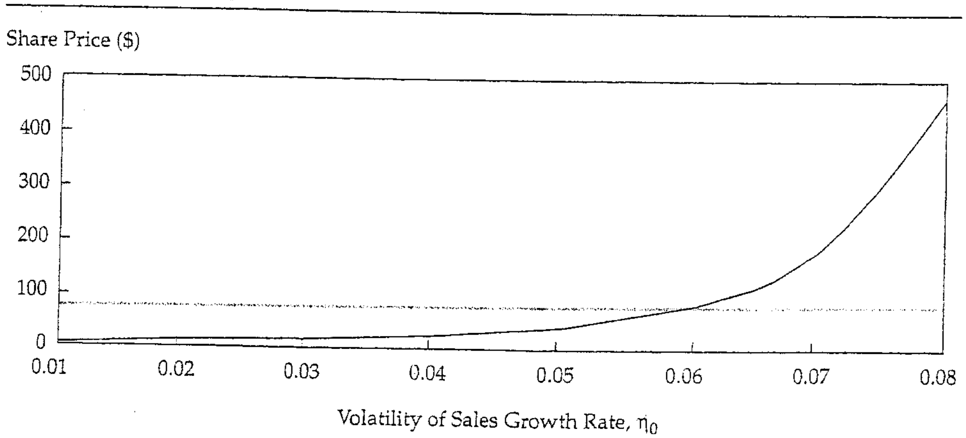
We modified the simulation program to take into account the shares issued at the exercise of the options and conversion of the convertible and to compute the part of the cash flow belonging to the shareholders. The stock value obtained for the base valuation was $12.42. This value is strikingly lower than the market price of $76.125 at the close of 1999.

This analysis implicitly assumed that the total cash flows available to all securityholders are independent of the capital structure. Recall that bankruptcy occurs in the model when the cash balances are driven to zero. Therefore, when a debt matures and is paid, for example, an equal amount of debt is issued to keep the cash balances the same. Alternative financing assumptions can easily be incorporated into the analysis if the analyst judges them to be more reasonable.

The volatility of the company was obtained from Equation 16, and the volatility of the equity was obtained from an identical equation in which we substituted the equity value for the company value. The partial derivatives of company (and equity) value with respect to the level of revenues and to the expected rate of growth in revenues were obtained by simulation.[6] With the parameters used in the base valuation, we obtained a volatility for the equity of 106 percent a year. This volatility is consistent with observed historical volatility of Amazon equity in the preceding year. (Recall that we chose the volatility of the expected growth rate in revenues to give this result.)

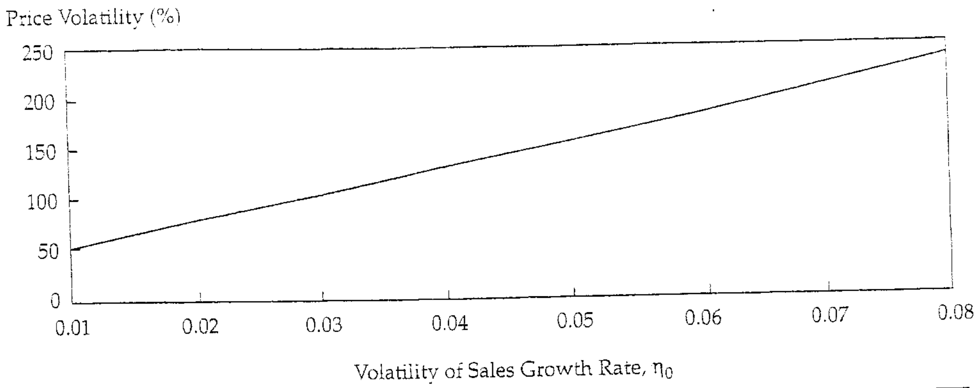
Because the volatility of the expected growth rate of revenues, η_0, is the most critical parameter in the valuation model, we show its effect on the stock price and its volatility in, respectively, **Figure 7** and **Figure 8**. As a comparison of the two figures shows, the stock price increases dramatically with increasing η_0 whereas the volatility of the stock price increases linearly. Furthermore, to obtain a model stock price consistent with the market price, a value of 0.06 for η_0 would be required. Such a value would also produce a model volatility of 182 percent, however, which is almost double the market volatility. In addition, the revenue distribution implied by this parameter appears to be unrealistic.

Figure 7. Share Price versus Volatility of Expected Sales Growth Rate for Amazon



Note: Shaded line is the closing price on December 31, 1999.

Figure 8. Share Price Volatility versus Volatility of Expected Sales Growth Rate for Amazon



This analysis suggests that, given the profitability assumed in the base valuation (through the cost function), either Amazon equity is overpriced or the volatility of its sales growth is too low. Substantially higher profitability would be needed to obtain model prices and volatilities that are consistent with those observed in the market. The profit margin before taxes would have to increase from 6 percent to 30 percent to attain this result.

Extensions

For the model described here, we made some simplifying assumptions about the optimal exercise of American-type options. We assumed that bankruptcy depends only on the level of the cash balances and that when this level goes to zero, the value of the company also goes to zero. The value of the company depends not only on the level of cash balances, however, but also on all the other state variables of the problem: the level of revenues, the expected rate of growth in revenues, their volatilities, and the amount of loss carry-forward. The cash balances could very well go to zero, but at the same time, the prospects of the company could be good enough that the company could raise additional cash or merge with another company.

In determining the value of the common equity, we also assumed that the options would be exercised and the convertibles converted whenever the company survived, whereas the optimal exercise of these options depends on the value of the firm at the decision date. For example, at the maturity of the convertible debt, the face value of the debt could be larger than its conversion value, in which case, the bondholders would optimally not convert but receive the face value instead.

Longstaff and Schwartz (1998) developed a least-squares Monte Carlo approach to value American-type options by simulation, which can be easily adapted to deal with the issues just noted. In the case of American options, the issue is to compare the value of immediate exercise with the conditional expected value (under the risk-neutral measure) of continuation. The conditional expected value of continuation for each path at each point in time is obtained from the fitted value of the linear regression of the discounted value (at the risk-free rate) of the cash flow obtained from the simulation following the optimal policy in the future on a set of basis functions of the state variables. Because this procedure is a recursive procedure starting from the maturity of the option, the outcome is the optimal stopping time for each path in the simulation. Knowing the optimal stopping time for each path, an analyst can easily value the American option.

The objective here is to determine the conditional expected value of the company (under the risk-neutral measure) at each point in time. We would start from the horizon T, where the value of the company is equal to the maximum of the cash balances or zero. Note that now we would not stop a particular path when the value of the cash balances are zero, because we want to optimally determine the stopping time (bankruptcy). Next, we would move back to time $T - \Delta t$. To determine the conditional expected value of the company at this point, we would regress the discounted (at the risk-free rate) cash flow (firm value) in period T on a set of basis functions of the state variables (revenues, rate of growth in revenues, volatilities, cash balances, and loss carry-forward) at time $T - \Delta t$.[7] The fitted value of this regression is the conditional expected value of the company. If this value is less than or equal to zero, the company is bankrupt and the value of the company is zero. We would proceed recursively in the same manner up to the present time. This procedure would produce the optimal stopping time for each path, from which we could calculate the current value of the company.

To determine the optimal exercise of the options and convertibles, we would follow a similar procedure. Knowing the value of the company at each possible exercise date would allow us to determine whether the exercise value of the options is larger than the exercise price or whether the conversion value is larger than the face value of the convertible bonds. We could keep accurate track of the number of shares outstanding and of the part of total cash flows belonging to the shareholders, and therefore, we could estimate a more accurate share value than with a simpler approach.

Conclusions

The valuation of Internet companies is a subject of much discussion in the financial press and among financial economists. We developed a simple model to value these companies that is based fundamentally on assumptions about the expected growth rate of revenues and on expectations about the cost structure of the company. Because these expectations are likely to change continuously as new information becomes available, the model generates company values and stock prices that are highly volatile. The model gives a systematic way to think about the drivers of value of Internet companies, however, and directs attention to the parameters that are most important in the valuation.

To implement the model, we had to make many assumptions about possible future financing, about future cash distributions to shareholders and bondholders, about the horizon of the estimation, and so on. Alternative assumptions are possible and easily incorporated in the analysis. Potential users of a model such as the one presented here would need a deep knowledge of the company and its industry in order to make more-reasonable assumptions.

We conclude that, depending on the parameters chosen and *given high enough growth rates of revenues*, the value of an Internet stock may be rational. Even when the chance that a company may go bankrupt is real, if the initial growth rates are sufficiently high and if there is enough volatility in this growth over time, valuations can be what would otherwise appear to be unbelievably high. In addition, we found the valuation has great sensitivity to initial conditions and exact specification of the parameters. This finding is consistent with observations that the returns of Internet stocks have been strikingly volatile.

We also examined the value of an exit option for Internet companies. In 1996, Berger, Ofek, and Swary empirically investigated whether investors price the option to abandon a company at its exit value. They concluded that firm value does increase in exit value after controlling for other variables. Even though the exit value is assumed to be zero in our model, the abandonment option can be valuable.

One of the most challenging issues in this analysis is the estimation of the parameters to use in the model. To illustrate the application of the model, we used data from only one company and made some judgment calls for the parameters for which we had no data. A more thorough analysis would use cross-sectional data from a sample of Internet companies to estimate the parameters. The cross-sectional data could also be used to test the model.

An issue that we did not address (but plan to pursue in future research) is seasonality, which characterizes the revenues of companies in certain industries. If seasonality is not taken into account when parameters are being estimated, the effect will be to overestimate the volatility of the growth rate in revenues. When seasonality is significant, it should be accounted for in the estimation process by using seasonally adjusted revenues.

We thank Michael Brennan and Russ Fuller for helpful suggestions and Marc Kelly for editorial comments. This article was written while Professor Schwartz was a visiting scholar at the University of British Columbia in the summer of 1999.

Notes

1. In future research, we plan to introduce uncertainty into the cost function.
2. Stochastic interest rates could easily be incorporated in our framework.
3. For a discrete version similar to Equation 18, see Schwartz and Smith (1997).
4. Contemporaneous values of these variables to compute the correlation are hard to obtain.
5. Note, however, that these components play such a role for any method of analysis.
6. The initial value of the revenues (the rate of growth in revenues) was perturbed to obtain new values of the company and equity from which these derivatives were computed.
7. See Longstaff and Schwartz for details on selecting the basis functions.

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