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The impact of fat tailed returns on asset allocation

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Abstract. This paper analyzes the asset allocation problem of an investor who can invest in equity and cash when there is time variation in expected returns on the equity. The solution methodology is multistage stochastic asset allocation problem with decision rules. The uncertainty is modeled using economic scenarios with Gaussian and stable Paretian non-Gaussian innovations. The optimal allocations under these alternative hypothesis are compared. Our computational results suggest that asset allocation may be up to 20% different depending on the utility function and the risk aversion level of the investor. Certainty equivalent return can be increased up to .13% and utility can be improved up to .72% by switching to the stable Paretian model.

Key words: portfolio optimization, stable distribution, scenario generation

1 Introduction

Strategic investment planning is allocation of a portfolio across broad asset classes such as bonds, stocks, cash and real estate considering the legal and policy constraints facing an institution or individual. Empirical evidence by Culp et al. (1997) suggests that asset allocation is the most important factor in determining investment performance.

Most of the early models in this field are either myopic or represent deterministic formulations of multiperiod problems. An investor that has isoelastic utility function chooses the same investment proportions independent of the investment horizon if the market is frictionless and the returns are independent over time (Merton, 1969 and Samuelson, 1969). However, myopic models cannot capture long-term investment goals in the presence of transaction costs. There is considerable evidence of predictability in asset returns (Hodrick, 1992 and Kandel and Staumbaugh, 1996) and the myopic models do not take this empirical finding into account. These models tend to produce high portfolio turnovers and opportunistic asset trades.

There has been a growing interest in the development of multiperiod stochastic models for asset and liability management (ALM). Kusy and Ziemba (1986) developed a multiperiod stochastic linear programming model for Vancouver City Savings Credit Union. Another successful application of multistage stochastic programming is the Russell-Yasuda Kasai model by Carino et al. (1994). The investment strategy suggested by the model resulted in extra income of $79 million during the first two years of its application (1991 and 1992). Boender (1997) reports the success of a hybrid simulation/optimization scenario model for ALM of pension funds in the Netherlands.

There are various approaches to modeling the predictability of asset returns. Wilkie (1986, 1995) suggests using ARMA model for each variable of interest in a cascade structure rather than a multivariate model. Mulvey (1996) describes an economic projection model that uses stochastic differential equations in a similar cascade framework. Hodrick (1992) uses VAR to model time variation in asset returns. Boender et al. (1998) extend VAR model to a Vector Error Correction Model (VECM) which additionally takes economic regime changes and long term equilibria into account.

Most of these models assume that the variables or the innovations of these variables follow normal distribution or the continuous time counterpart, Brownian motion. In response to the empirical evidence about the heavy tail, high peak and possible skewness in financial data, Fama (1965) and Mandelbrot (1963, 1967) propose stable Paretian distribution as an alternative model. Among the alternative non-Gaussian distributions in the literature, stable distribution has unique characteristics that make it an ideal candidate. The stable laws are the only possible limit distributions for properly normalized and centered sums of independent identically distributed random variables (Embrechts et al., 1997 and Rachev and Mittnik, 2000). If a financial variable can be regarded as the result of many microscopic effects, then it can be described by a stable law. Stable distributions are leptokurtotic. When compared to normal distribution, they typically have fatter tails and higher peak around the center. They allow modeling various levels of skewness. Due to these flexibilities, stable model fits the empirical distribution of the financial data better (see Mittnik et al. 2000). Gaussian distribution is a special case of stable distribution. In fact, it is the only distribution in the stable family with a finite second moment. Although autoregressive conditional heteroskedastic models driven by normally distributed innovations imply unconditional distributions that possess heavier tails, there is still considerable kurtosis unexplained by this model. Mittnik et al. (2000) present empirical evidence favoring stable hypothesis over the normal assumption as a model for unconditional, homoskedastic conditional, and heteroskedastic conditional distributions of several asset return series.

In this paper we analyze the multistage asset allocation problem of an investor under the Gaussian and stable returns scenarios. We use stochastic programming with decision rules to solve the allocation problem. Our model captures uncertainty by a branching event tree. Each node of the tree represents a joint outcome of all the random variables at that decision stage. Each path through the event tree represents a 'scenario'. The major advantage of stochastic programming is that it permits a very rich description of the state of world at each node by easily accommodating a large number of random variables. This characteristic leads to several important practical applications of the model such as Carino et al. (1994), Zenios (1993), and Mulvey (1996). Stochastic programming with decision rules simplifies the computation of a solution by imposing the use of an allocation rule such as fix-mix at each decision stage. Some commercial applications of this approach include Boender et al. (1998) and Berger and Mulvey (1998).

The investor is assumed to invest in two assets-cash and an equity portfolio. While we do not model any liabilities, it is relatively straightforward to generalize the model to include them (see Boender et al., 1998). The variables that predict the return on the equity portfolio are dividend yield and dividend growth of the equity portfolio, Treasury bill rate, Treasury bond yield, and inflation. The investor updates his portfolio at every decision stage using the fix-mix allocation rule. Although fix-mix rule is the optimal strategy under certain conditions, it is still widely used by financial practitioners. Fixed mix strategy maintains a fixed exposure to equity market by requiring the purchase of stocks as they fall in value, and the sale of stocks as they rise in value (Perold and Sharpe, 1988).

We consider two alternative objective functions for the investor. We first analyze an investor that maximizes the classical power utility function of final wealth. Then we model an investor who trades off between the mean return and a risk measure, which is a power (< 2) of mean absolute deviation of the return. This objective function is an analog of mean-variance but it assigns less importance to extreme observations.

Our computational results suggest that the significance of the asset allocation and certainty equivalent return differences between Gaussian and stable returns models depend on the objective function of the investor. We find that if the investor has very high or low risk aversion, then the normal and stable scenarios result in similar asset allocations for all of the objective functions analyzed. However, when the risk aversion level is between the two cases, the two distributional assumptions may result in considerably different asset allocations depending on the objective function and the risk aversion level of the decision maker. The investor may reduce his equity allocation up to 20%, increase his certainty equivalent wealth up to .13% and improve his utility by .72% by switching to stable model. Since stable economic scenarios model extreme events more realistically, they suggest more conservative asset allocations. Ortobelli et al. (1999) report similar observations in their single period asset allocation model.

Section 2 introduces stable distribution. The asset allocation model is set up in Section 3 with the discussion of the scenario generation and asset allocation modules. The computational results are reported in Section 4. Section 5 concludes.

2 Stable distribution

There are several important reasons for modeling financial variables using stable distributions. The stable law is supported by a generalized central limit theorem (Embrechts et al., 1997 and Rachev and Mittnik, 2000). Stable distributions are leptokurtotic. Since they can accommodate the fat tails and asymmetry, they fit empirical distribution of the financial data better.

Any distribution in the domain of attraction of a specified stable distribution will have properties which are close to the ones of stable distribution.

Even if the observed data does not exactly follow the ideal distribution specified by the modeler, in principle, the resulting decision is not affected.

Each stable distribution has an index of stability which remains the same regardless of the sampling interval adopted. The index of stability can be regarded as an overall parameter that can be employed in inference and decision making. However, we should note that for some financial data empirical analysis shows that the index of stability increases as the sampling interval increases.

It is possible to check whether a distribution is in the domain of attraction of a stable distribution or not by examining the tails of the distribution. The tails dictate the properties of the distribution.

This section describes the properties of stable distribution and addresses the estimation issues.

2.1 Description of stable distributions

If the sums of linear functions of independent identically distributed (iid) random variables belong to the same family of distributions, the family is called stable. Formally, a random variable r has stable distribution if for any $a > 0$ and $b > 0$ there exists constants $c > 0$ and $d \in R$ such that

$$ar_1 + br_2 \stackrel{d}{=} cr + d, \tag{1}$$

where r_1 and r_2 are independent copies of r, and $\stackrel{d}{=}$ denotes equality in distribution. The distribution is described by the following parameters: $\alpha \in (0, 2]$ (index of stability), $\beta \in [-1, 1]$ (skewness parameter), $\mu \in R$ (location parameter), and $\sigma \in [0, \infty)$ (scale parameter). The variable is then represented as $r \sim S_{\alpha,\beta}(\mu, \sigma)$. Gaussian distribution is actually a special case of stable distribution when $\alpha = 2$, $\beta = 0$. The smaller the stability index is, the stronger the leptokurtic nature of the distribution becomes, i.e. with higher peak and fatter tails. If the skewness parameter is equal to zero, as in the case of Gaussian distribution, the distribution is symmetric. When $\beta > 0$ ($\beta < 0$), the distribution is skewed to the right (left). If $\beta = 0$ and $\mu = 0$, then the stable random variable is called symmetric α-stable ($S\alpha S$). The scale parameter generalizes the definition of standard deviation. The stable analog of variance is variation, v_α, which is given by σ^α.

Stable distributions generally do not have closed form expressions for density and distribution functions. They are more conveniently described by characteristic functions. The characteristic function of random variable r, $\Phi_r(\theta) = E[\exp(ir\theta)]$, is given by

$$\begin{aligned}\Phi_r(\theta) &= \exp\left\{-\sigma^\alpha|\theta|^\alpha\left(1 - i\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right) + i\mu\theta\right\}, \quad \text{if } \alpha \neq 1,\\ &= \exp\left\{-\sigma|\theta|\left(1 - i\beta\frac{2}{\pi}\mathrm{sign}(\theta)\ln\theta\right) + i\mu\theta\right\}, \quad \text{if } \alpha = 1.\end{aligned} \tag{2}$$

The p^{th} absolute moment of r, $E|X|^p = \int_0^\infty P(|X|^p > y)\,dy$, is finite if $0 < p < \alpha$, and infinite otherwise. Hence, when $\alpha < 1$ the first moment is infinite, and when $\alpha < 2$ the second moment is infinite. The only stable distribution that has finite first and second moments is the Gaussian distribution.

In models that use financial data, it is generally assumed that $\alpha \in (1, 2]$. There are several reasons for this:

1) When $\alpha > 1$, the first moment of the distribution is finite. It is convenient to be able to speak of expected returns.
2) Empirical studies support this parametrization.
3) Although the empirical distributions of the financial data sometimes depart from normality, the deviation is not "too much".

In scenario generation, one may need to use multivariate stable distributions. The extension to the multivariate case is nontrivial. Although most of the literature concentrates on the univariate case, recently some new results have become available. See for example Samorodnitsky and Taqqu (1994), Rachev and Mittnik (2000).

If R is a stable d-dimensional stable vector, then any linear combination of the components of R is also a stable random variable. However, the converse is true under certain conditions (Samorodnitsky and Taqqu, 1994). The characteristic function of R is given by:

$$\begin{aligned}\Phi_Y(\theta) &= \exp\left\{-\int_{S_d} |\theta^T s|\left(1 - i\,\mathrm{sign}(\theta^T s)\tan\frac{\pi\alpha}{2}\right)\Gamma(ds) + i\theta^T\mu\right\},\\ &\quad \text{if } \alpha \neq 1,\\ &= \exp\left\{-\int_{S_d} |\theta^T s|\left(1 + i\frac{2}{\pi}\,\mathrm{sign}(\theta^T s)\ln|\theta^T s|\right)\Gamma(ds) + i\theta^T\mu\right\},\\ &\quad \text{if } \alpha = 1,\end{aligned} \tag{3}$$

where Γ is the spectral measure which replaces the scale and skewness parameters that enter the description of the univariate stable distribution. It is a bounded nonnegative measure on the unit sphere S_d, and $s \in S_d$ is the integrand unit vector. The index of stability is again α, and μ is the vector of locations.

Stable distributions have infinite variances. The stable equivalent of covariance for $S\alpha S$ variables is covariation:

$$[R_1, R_2]_\alpha = \int_{S_d} s_1 s_2^{\langle\alpha-1\rangle}\Gamma(ds), \tag{4}$$

where (R_1, R_2) is a $S\alpha S$ vector ($\alpha \in (1, 2)$), and $x^{\langle\alpha-1\rangle} = |x|^k\,\mathrm{sign}(x)$. The matrix of covariations determines the dependence structure among the individual variables.

Subordinated Gaussian distribution can be used to model the dependence between stable variables (Mercury, 1999). The main idea of this approach is to capture the dependence structure using the Gaussian distribution and to model the heavy tails with the stable subordinator. Subordinated Gaussian is defined as follows: Let $X \sim N(0, 2\sigma^2)$, and $A \sim S_{\alpha/2,0}(1; c)$, X and A being independent. Then, one can generate $Z = A^{1/2}X \sim S_{\alpha,0}(0, \sigma^*)$, where $c = \left(\frac{\sigma^{*2}}{\sigma^2}\right)[\cos(\pi\alpha/4)]^{2/\alpha}$.

The 'truncated' covariance matrix can be used to capture the dependence by leaving out the very extreme events. Let Σ be the truncated covariance matrix. It can be estimated by exponential smoothing (Riskmetrics, 1996) as follows:

$$c^2_{j,t+1|t} = (1-\lambda)\sum_{i=1}^{\infty} \lambda^i R^2_{j|t-i} \tag{5}$$

is the diagonal element of the truncated covariance matrix, and

$$c^2_{jk,t+1|t} = (1-\lambda)\sum_{i=1}^{\infty} \lambda^i R_{j|t-i} R_{k|t-i}, \tag{6}$$

where λ is the smoothing parameter, measures the truncated covariance between j and k. Hence, $\Sigma = \{c_{ik}\}$ and $c_{jj} = 2\sigma_j^2$. Suppose the truncation points are x and $-x$, then $R_{j|t-i}$ is defined as the following:

$$R_{j|t-i} = \left\{ \begin{array}{c} r_t^j, \text{ if } r_t^j \text{ is in } (-x, x) \\ -x, \text{ if } r_t^j < -x \\ x, \text{ if } r_t^j > x \end{array} \right\}. \tag{7}$$

2.2 *Financial modeling and estimation*

Financial modeling frequently involves information on past market movements. Examples include technical analysis to derive investment decisions, or researchers assessing the efficiency of financial markets. In such cases, it is not the unconditional return distribution which is of interest, but the conditional distribution, which is conditioned on information contained in past return data, or a more general information set. The class of autoregressive moving average (ARMA) models is a natural candidate for conditioning on the past of a return series. These models have the property that the conditional distribution is homoskedastic. In view of the fact that financial markets frequently exhibit volatility clusters, the homoskedasticity assumption may be too restrictive. As a consequence, conditional heteroskedastic models, such as Engle's (1982) autoregressive conditional heteroskedastic (ARCH) models and the generalization (GARCH) of Bollerslev (1986), possibly in combination with an ARMA model, referred to as an ARMA-GARCH model, are now common in empirical finance. It turns out that ARCH-type models driven by normally distributed innovations imply unconditional distributions which themselves possess heavier tails. Thus, in this respect, ARCH models and stable distributions can be viewed as competing hypotheses.

Mittnik et al. (1997) present empirical evidence favoring stable hypothesis over the normal assumption as a model for unconditional, homoskedastic conditional, and heteroskedastic conditional distributions of several asset return series.

2.2.1 Maximum likelihood estimation

We will describe an approximate conditional maximum-likelihood (ML) estimation procedure suggested by Mittnik et al. (1996). The unconditional ML estimate of $\theta = (\alpha, \beta, \mu, \sigma)$ is obtained by maximizing the logarithm of the likelihood function

$$L(\theta) = \prod_{t=1}^{T} S_{\alpha,\beta}\left(\frac{r_t - \mu}{\sigma}\right)\sigma^{-1}. \tag{8}$$

One needs to use conditional ML to estimate ARMA and ARMA-GARCH models. The ML estimation is conditional, in the sense that, when estimating, for example, an ARMA(p, q) model, one conditions on the first p realizations of the sample, $r_p, r_{p-1}, \dots, r_1$, and, when $\alpha > 1$ holds, sets innovations $\varepsilon_p, \varepsilon_{p-1}, \dots, \varepsilon_{p-q+1}$ to their unconditional mean $\mathbf{E}(\varepsilon_t) = 0$. The estimation of all stable models is approximate in the sense that the stable density function, $S_{\alpha,\beta}(\mu, \sigma)$, is approximated via fast Fourier transformation (FFT) of the stable characteristic function,

$$\int_{-\infty}^{\infty} e^{itx}\, dH(x) = \begin{cases} \exp\left\{-\sigma^{\alpha}|t|^{\alpha}\left[1 - i\beta\,\mathrm{sign}(t)\tan\frac{\pi\alpha}{2}\right] + i\mu t\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma|t|\left[1 + i\beta\frac{2}{\pi}\mathrm{sign}(t)\ln|t|\right] + i\mu t\right\}, & \text{if } \alpha = 1, \end{cases} \tag{9}$$

where H is the distribution function corresponding to $S_{\alpha,\beta}(\mu, \sigma)$.

This ML estimation method essentially follows that of DuMouchel (1973), but differs in that the stable density is approximated numerically by an FFT of the characteristic function rather than some series expansion. As DuMouchel shows, the resulting estimates are consistent and asymptotically normal with the asymptotic covariance matrix of $T^{1/2}(\hat{\theta} - \theta_0)$ being given by the inverse of the Fisher information matrix. The standard errors of the estimates are obtained by evaluating the Fisher information matrix at the ML point estimates. For details on stable ML estimation see Mittnik et al. (1996), Mittnik and Rachev (1993), and Paulauskas and Rachev (1999).

2.3 Comparison of estimation methods

When the residuals of the ARMA model have Gaussian distribution, Least Squares (LS) estimation is equivalent to condition ML estimation. Furthermore, Whittle estimator is asymptotically equivalent to LS and conditional ML estimation methods. However, when the innovations have stable distribution, the properties of conventional estimation methods may change due to the infinite variance property. In the stable case, ML estimates are still consistent and asymptotically normal (DuMouchel, 1987); LS and Whittle estimates are consistent but they are not asymptotically normal. The LS and Whittle estimates have infinite variance limits with a convergence rate that is faster than that of the Gaussian case (Mikosch et al., 1995). When $\alpha < 2$, Mikosch (1998) suggests using the classical confidence bands and test regions based on L^2 in a conservative sense.

Calder and Davis (1998) compare LS, Least Absolute Deviation (LAD), and ML methods for the estimation of ARMA model with stable innovations. Their simulations reveal that the difference between the estimates of the three methods is insignificant when the index of stability of the residuals is 1.75. However, when $\alpha = 1$ or $\alpha = .75$, they report that the LAD and ML estimation procedures are superior to LS estimation.

ML estimation has desirable properties in both the Gaussian and stable setting, but it is computationally very demanding. Since the variables of interest in this paper have indices of stability greater than 1.5, nonlinear LS estimation method has been utilized in this study. Our parameter estimates are consistent, but they are not asymptotically normal. However, due to the high index of stability, the parameter estimates are comparable to those that would be achieved if ML estimation were to be used.

3 Model setup

3.1 Multistage asset allocation model

The asset allocation problem for an investor that maximizes isoelastic utility function or an analog of mean-variance objective function at the end of the investment horizon is formulated as follows:

$$\max E[u(\acute{R}^i_{s,T})]$$
s.t.

$$\acute{R}^i_{s,T} = \prod_{t=1}^{T}(1 + R^i_{s,t}) - 1,$$

$$R^i_{s,t} = \sum_{j=1}^{J} w^i_j r_{jst},$$

$w^i_j \geq 0$, where w^i_j is the proportion of funds of portfolio i invested in asset j, $\acute{R}^i_{s,T}$ is compound return of allocation i in time period of 1 through T under scenario $s \in \{1, 2, \ldots, S\}$,

$R^i_{s,t}$ is the return of the portfolio i under scenario $s \in \{1, 2, \ldots, S\}$ in time period $t \in \{1, 2, \ldots, T\}$, and

r_{jst} is the percentage return of asset $j \in \{1, 2, \ldots, J\}$ under scenario s in time period t.

The restrictions on the model are that there are no short sales and the asset allocation is updated every month according to fixed mix decision rule. In general, fixed mix strategy requires the purchase of stocks as they fall in value, and the sale of stocks as they rise in value. Fixed mix strategy does not have much downside protection, and tends to do very well in flat but oscillating markets. However, it tends to do relatively poorly in bullish markets (Perold and Sharpe, 1988).

We use two alternative objective functions: the first one is power utility' function and the second one is an analog of mean-variance analysis. The power utility function, which has constant relative risk aversion, is calculated as follows:

$$U(W^i) = \frac{1}{S}\sum_{s=1}^{S}\frac{1}{(1-\gamma)}(W_s^i)^{(1-\gamma)}, \quad \gamma > -1, \tag{10}$$

where γ is the coefficient of relative risk aversion, and W_s^i is the final wealth. $U(W^i)$ is finite if $(1-\gamma) < 2$. Assuming that the initial wealth is 1, we compute the final wealth as follows:

$$W_s^i = 1 \cdot (1 + \acute{R}_{s,T}^i). \tag{11}$$

A constant relative risk aversion investor chooses the same investment proportions independent of the investment horizon if the market is frictionless and returns are independent over time. Fix mix is the optimal portfolio choice in this setting. However, if the returns are predictable, which is the conjecture of this paper, then the portfolio choice depends on the investment horizon. Although the fix mix strategy is no longer optimal in this economic environment, the investor is assumed to follow this decision strategy for computational simplicity.

The second objective function trades off between mean final return and a measure of risk:

$$U(\acute{R}_T^i) = E[\acute{R}_T^i] - c \cdot MD(\acute{R}_T^i), \tag{12}$$

where c is the coefficient of risk aversion.

The mean compound portfolio return of fixed mix rule $i \in \{1, 2, \ldots, I\}$ at the final date is:

$$E[\acute{R}_T^i] = \frac{1}{S}\sum_{s=1}^{S}\acute{R}_{s,T}^i. \tag{13}$$

We consider the following risk measure which gives less importance to outliers than variance does:

$$MD(\acute{R}_T^i) = \frac{1}{S}\sum_{s=1}^{S}|\acute{R}_{s,T}^i - E[\acute{R}_T^i]|^r, \quad \textit{where } 1 < r < 2. \tag{14}$$

Notice that when $r = 2$, the above risk measure becomes the variance. Since variance is not defined for non-Gaussian stable variables, we use those values of $r < 2$ for which $MD(\acute{R}_T^i)$ is finite, such as $r = 1.5$.

The scenario generation module generates asset return scenarios, r_{jst}, for each time period. At each stage, n new offspring scenarios are generated from the parent scenarios. If the horizon of interest is T periods, then we produce n^T alternative asset return scenarios for the final date. Optimal asset allocation is calculated for this scenario tree. The scenario tree is repeated 100 times and the sample average of optimal allocations is reported as the optimal asset allocation.

3.2 *Time series analysis*

The portfolio we analyze is composed of Treasury bill and S&P 500. The monthly return on Treasury bill is assumed to be constant at 6% annualized

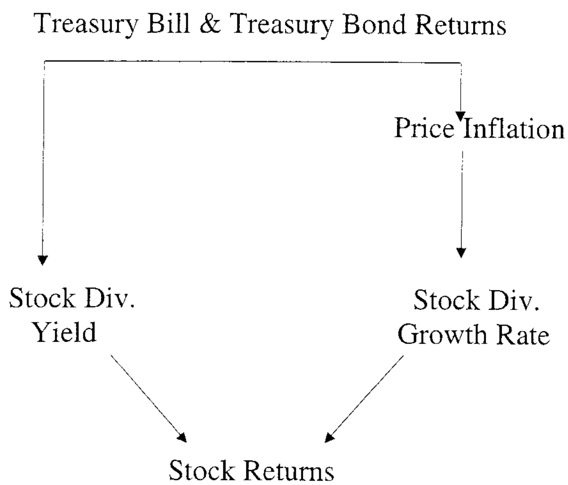

Fig. 1. Cascade Structure Model for Scenario Generation

rate of return. The main challenge is predicting the return scenarios for S&P 500. The financial variables that are used to generate the return scenarios for S&P 500 are modeled in a cascade structure similar to Mulvey (1996) (see Figure 1). However, the analysis is done in discrete time as in Wilkie (1995). Monthly data from 2/1965 through 12/1999 is used for the estimation of the time series models.3-month Treasury bill rate and 10-year Treasury bond rate are modeled first as measures of short term and long term interest rates. The price inflation depends on the Treasury bond rate and the previous values of inflation. Following Wilkie's and Mulvey's approaches, stock returns are analyzed in two components: dividend growth and dividend yield growth.

The relationship of economic variables does not denote a one way casual relationship, but rather indicates the sequencing of the modules. The economic variables are modeled using Box-Jenkins methodology. The standard Gaussian Box-Jenkins techniques carry over to the stable setting with some possible changes. There are two criteria that we used in the model selection:

1) Autocorrelation function (ACF) and Partial Autocorrelation function (PACF) were used to determine the order of the autoregressive and moving average terms and to detect the significance of the serial correlation of the residuals. Adler et al. (1998) compare stable, Cauchy and Gaussian limits in the construction of confidence interval for ACF and PACF. Their simulations show that when $\alpha \geq 1.7$, Gaussian and Cauchy limits are better than stable limits. However, for $\alpha < 1.7$, while Gaussian limits still perform in the acceptable range, Cauchy and stable limits are better than the Gaussian limits. Gaussian limits were used in our analysis since the indices of stability of all the residuals except for one are greater than 1.7[1].

2) Akaike Information Criteria was used to trade of between extra explanatory power and parsimonious parameter selection. It is valid is both the Gaussian and the stable setting (Adler, et al. 1998).

[1] The index of stability for the residuals of Treasury bill equation is 1.56. We will use the Cauchy limits for this variable in a later version of this paper.

We do not model the time varying volatility of the economic variables. Fitting ARMA-GARCH models may reduce the kurtosis in the residuals. However, Balke and Fomby (1994) show that even after estimating GARCH models, significant excess kurtosis and/or skewness still remains. Mittnik et al. (2000) present empirical evidence favoring stable hypothesis over the normal assumption as a model for ARMA-GARCH residuals. We postpone modeling the time varying volatility in the generation of economic scenarios to a further paper.

3.2.1 Level 1: Short term and long term interest rates

3-month Treasury bill rate is used as a proxy for short term risk free interest rate and 10-year Treasury bond yield is used as a proxy for the long term interest rate. Dickey-Fuller and Phillips Perron tests for unit root suggest that both Treasury bill and Treasury bond are first order unit root processes. There is no agreement in economic theory on whether short term and long term interest rate have a long-term equilibrium relationship or not. We analyze the data to decide on this issue on empirical grounds for our data set. One option is to ignore the nonstationarity and simply estimate in levels. However, the classical asymptotic theory for test statistics is nonstandard. Another option is to difference the apparently nonstationary variables before estimating the regression equations. If the true processes are regressions in first differences, then this approach is the right one. However, the series may have been in fact stationary, or a linear combination of the series might be stationary. In such circumstances, the analysis is misspecified. One needs to test for possible cointegration among series. The disadvantage of this approach is that alternative tests for unit roots and cointegration can produce conflicting results. One practical solution suggested by Hamilton (1994) is to employ parts of all three approaches. If the regression for the data in levels form yields similar inferences to those in the stationary first difference form, then one can be satisfied that the results were not governed by the assumptions made about unit roots; confidence in the specification increases.

When short term and long term interest rates are allowed to have linear rends, and the cointegration equation is allowed to have intercept and trend, Johansen Cointegration test (1991, 1995) suggests that there is no cointegration relationship between the two series. However, if one imposes the restriction that the individual series have no trend and the cointegration equation has no trend, but possibly an intercept then Johansen Cointegration test finds one cointegration equation. These results are very sensitive to the time period analyzed, and the assumed lag length (See Table 1). We conclude that it is likely that there is no linear combination of short term and long term interest rates that is stationary.

We fit a vector autoregression (VAR) to first differences of Treasury bond and Treasury bill rates. The Schwartz Information Criteria suggests VAR(2), in which case the residuals exhibit no significant serial correlation.

$$d(Tbill)_t = \gamma^1 d(Tbill)_{t-1} + \gamma^2 d(Tbill)_{t-2} + \gamma^3 d(Tbond)_{t-1} + \gamma^4 d(Tbond)_{t-2} + \varepsilon_t^{Tbill} \quad (15)$$

Table 1. Johansen cointegration test summary

Time period		2/1965–12/1999	
Lag interval		1	
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/No trend	Intercept/No trend	Intercept/Trend
L.R. Test	Rank = 1	Rank = 1	Rank = 0
Time period		2/1965–12/1999	
Lag interval		2	
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/No trend	Intercept/No trend	Intercept/Trend
L.R. Test	Rank = 0	Rank = 0	Rank = 0
Time period		2/1968–12/1999	
Lag interval		1	
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/No trend	Intercept/No trend	Intercept/Trend
L.R. Test	Rank = 1	Rank = 0	Rank = 0

$$d(Tbond)_t = \beta^1 d(Tbill)_{t-1} + \beta^2 d(Tbill)_{t-2} + \beta^3 d(Tbond)_{t-1} + \beta^4 d(Tbond)_{t-2} + \varepsilon_t^{Tbond} \tag{16}$$

The adjusted R^2 of the first model is 0.16, and the adjusted R^2 of the second model is 0.14. The correlation between the residuals of the two variables is 0.56.

Since the second moment does not exist for stable random variables with $\alpha < 2$, the dependence structure between the innovations of Treasury bill and Treasury bond rates cannot be modeled by using covariance measure in the generation of stable scenarios. We derive the dependence structure from truncated observations by leaving out the virtually impossible values. Mercury (1999) package is used to estimate the 'truncated' covariance matrix by exponential smoothing and to simulate the residuals.

Once the truncated covariance matrix for $d(Tbill)$ and $d(Tbond)$, $\Sigma = \{c_{ik}\}$, is estimated, we generate

$$Z = A^{1/2}X \sim S_{\alpha,0}(0,\sigma^*), \quad \text{where } c = \left(\frac{\sigma^{*2}}{\sigma^2}\right)[\cos(\pi\alpha/4)]^{2/\alpha}, X \sim N(0, 2\sigma^2),$$

and $A \sim S_{\alpha/2,0}(1,c)$, X and A being independent. The dependence structure between risk factors still remains: The stable random variable $Z = A^{1/2}X$ can be viewed informally as $N(0, 2\sigma^2 A)$-distributed, i.e. normal with random variance $2\sigma^2 A$.

The future correlated residuals for $d(Tbill)$ and $d(Tbond)$ are then simulated as follows:

1) We simulate N independent multivarite normal random variables with the truncated covariance matrix Σ between components of each vector-column:

$$G = \begin{bmatrix} X_{1,1} & X_{1,2} \\ \vdots & \vdots \\ X_{N,1} & X_{N,2} \end{bmatrix}, \quad \text{where very column is } N(0,\Sigma).$$

2) We simulate N independent identically distributed stable random variables, $A_{j,i} \sim S_{\alpha_j/2,0}(1, c_j)$

$$S = \begin{bmatrix} A_{1,1}^{1/2} & A_{1,2}^{1/2} \\ \vdots & \vdots \\ A_{N,1}^{1/2} & A_{N,2}^{1/2} \end{bmatrix}.$$

3) The matrix T which is the dot product of G and S will contain N simulations for $d(Tbill)$ and $d(Tbond)$ with the desired stable parameters:

$$T = G \circ S, \quad \text{where } T_{j,i} \sim S_{\alpha_j,0}(0, \sigma_j), \ \forall i = 1, 2, .. N, \text{ and } j = 1, 2.$$

T can informally be viewed as $N(0, \Sigma^*)$-distributed, with random covariance matrix Σ^*, where $\Sigma^* = \{\sigma_{ij}^*\}$, $\sigma_{jj}^* = c_{jj} \cdot A_j$, and $\sigma_{ij} = c_{ij} \cdot (A_i A_j)^{1/2}$. Note that A_j is the square of the realization from S for the j-th variable (corresponding to $d(Tbill)$ or $d(Tbond)$), and c_{ij} is an element of Σ.

3.2.2 Level 2: Inflation

Mulvey (1996) finds that inflation depends on previous inflation rates and the current yield curve. Since we avoid modeling the yield curve, we checked whether the inflation rate can be explained by changes of short term and long term interest rates. Changes in short rate do not explain inflation rates at 1 or 5 percent significance levels. However, changes in long rate have significant explanatory power at 5 percent level. The residuals exhibit ARMA(1,1) structure. There is a very significant peak in the partial autocorrelation function at lag 9. When ninth order autoregressive term is added, the serial correlation in the residuals becomes insignificant. However, if a different time horizon is considered, there is no longer a significant peak in the partial autocorrelation function at lag 9. Since there is no particular reason for its existence, we conclude that is an outlier.

We use the following time series model for price inflation:

$$Inf_t = c^{Inf} + \gamma^{Inf} d(Tbond)_t + resInf_t \tag{17}$$

$$resInf_t = \beta^{Inf}\, resInf_{t-1} + \alpha^{Tbond} \varepsilon_{t-1}^{Inf} + \varepsilon_t^{Inf}, \tag{18}$$

where Inf : log differences of seasonally adjusted monthly CPI values.

This model gives the highest Schwartz Information Criterion without leaving any significant serial correlation in the residuals. The Jarque-Bera statistic rejects that ε_t^{Inf} comes from normal distribution at 1% and 5% significance levels. The residuals have a kurtosis of 8.15 and a skewness of 0.49. The adjusted R^2 of the estimated model is .51.

3.2.3 Level 3: Stock dividend growth rate and stock dividend yield

Mulvey and Thorlacius (1998) suggest dividing the stock returns into two components: dividend and capital appreciation. They argue that by separating the base components as dividend growth and dividend yield, one can accurately depict cash income and the decomposed structure provides more accurate linkages to the key economic factors such as interest rates and inflation level. We adopt their approach.

Mulvey (1996) observes that growth of dividends net of inflation has been fairly stable over the last several decades. He suggests that dividend growth can be linked to inflation and past dividend growth.

The data reveals that the dividend growth rate can be explained by dividend growth rate of the previous two years and second order autoregressive terms:

$$Divg_t = \gamma^{Divg}\, Inf_t + resDivg_t \tag{19}$$

$$resDivg_t = \beta_1^{Divg}\, resDivg_{t-1} + \beta_2^{Divg}\, resDivg_{t-2} + \varepsilon_t^{Divg} \tag{20}$$

where $Divg$: log differences of dividend index of S&P 500.

The inclusion of changes in short and long rate directly in the model does not have any significant explanatory power. The residuals have no significant serial correlation left over. The kurtosis of the residuals is 6.63, and the skewness is −.002. The Jarque-Bera statistic rejects that ε_t^{Divg} comes from normal distribution at 1% and 5% significance levels. The adjusted R^2 of the estimated model is .23.

Dickey-Fuller test for unit root suggest dividend yield is first order integrated process. Hence, we model the change in the dividend yield rather than dividend yield itself. Mulvey suggests that dividend yield depends on the movement of short-term and long-term interest rates. However, our analysis shows that short term interest rate as proxied by 3 month Treasury rate has no significant explanatory power for explaining dividend yield movements. The current and the previous month's long rates have significant explanatory power (at 5% level) in explaining the change in dividend yield.

Using the Schwartz Information Criteria, the time series model we suggest for change in the dividend yield is as follows:

$$d(Divy)_t = \gamma_1^{Divy} d(Tbond)_t + \gamma_2^{Divy} d(Tbond)_{t-1} + \varepsilon_t^{Divy} \tag{21}$$

where $Divy$: logarithm of monthly dividend yield of S&P 500.

This model leaves no significant serial correlation in the residuals. The Jarque-Bera statistic rejects that ε_t^{Divy} comes from normal distribution at 1% and 5% significance levels. The residuals are kurtotic and skewed. The kurtosis is 12.02, and the skewness is 1.56. The adjusted R^2 of the estimated model is .13.

3.3 *Simulation of future scenarios*

Future economic projections are simulated at monthly intervals. The scenarios have a tree structure. One set of scenarios is generated by assuming that the residuals of each variable is identical normally distributed. This is the classical assumption made in the literature. Another set is generated by assuming that the residuals are identical stable distributed. At each stage (month) we generate n possible alternative realizations. For each scenario, we first generate a normal or stable residual for Treasury bill, and calculate the corresponding Treasury bill rate for the proceeding month. Then, given this short rate, we generate Treasury bond rate, price inflation, dividend growth rate and dividend yield for that month according to the cascade structure and the time

Table 2. The estimated normal and stable parameters for the innovations

innovations of	normal dist.		stable dist.			
	μ	σ	α	β	μ	σ
Price Inflation (*Inf*)	6.15 e −06	.0021	1.7072	0.1073	6.15 e −06	0.0012
Dividend gr. (*Divg*)	9.89 e −4	.0195	1.7505	−0.0229	9.89 e −4	0.0114
Dividend yield (*d*(*Divy*))	−.002551	.0407	1.8076	0.2252	−.002551	0.0239
Treasury bill (*d*(*Tbill*))	.000336	.0579	1.5600	0	0	.0308
Treasury bond (*d*(*Tbond*))	.000818	.0339	1.9100	0	0	.0230

series models we have built. For instance, the inflation rate for next month is generated by using the Treasury bond rate, inflation rate and the surprise to expected inflation this month, and the normal or stable innovation of inflation rate next month. Note that we allow for innovation of each economic variable in each simulated month.

At the next stage, n new offspring scenarios are generated from the parent scenarios. This continues until the final time of interest. In this study, we generate 2 scenarios for each month, so 512 possible economic scenarios are considered over the next three quarters.

The estimated normal and stable parameters for the innovations of the time series models are given in Table 2. All of the innovations have indices of stability estimates less than 2. This indicates that these variables have fatter tails in comparison to the Gaussian distribution. They are also slightly skewed. This flexibility of stable distribution is more useful for significantly skewed financial variables, such as corporate bonds. The variable with the lowest index of stability and hence the fattest tails is the residuals of change in Treasury bill rate. Figure 2 depicts comparison of empirical probability density function, the stable fit and the normal fit for Treasury bill rate.

Rachev and Mittnik (2000) compare the empirical fit of several fat-tailed distributions to daily returns on S&P 500. The best fit in the tails of the distribution is achieved by log-stable and Student-t models. A major drawback of Student-t distribution is its lack of stability with respect to summation, i.e. a portfolio of Student-t distributed asset returns does not have Student-t distribution. It is not supported by a central limit theorem. Student-t distribution is a symmetric distribution and it cannot capture the possible skewness in financial data.

3.4 Valuation of assets

The monthly return of S&P 500 is derived using the dividend yield and the dividend index. Dividend index is calculated by multiplying price index with the dividend yield:

$DI_t = P_t * DY_t$, where DI_t is the dividend index for period t, P_t is the price index for period t, and DY_t is the dividend yield for period t. The dividend growth is just log differences of dividend indices.

The dividend yield and dividend growth rate are simulated as explained in the previous section. Hence, we can get back simulated future price index in period t under scenario s from the simulated dividend growth and dividend yield indices by

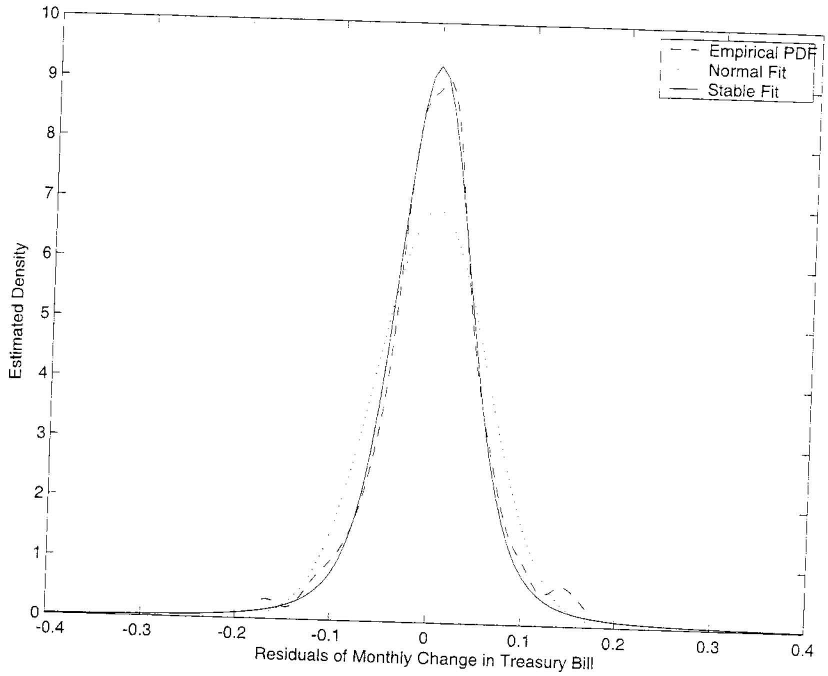

Fig. 2. Empirical, Stable and Normal Fit to the Residuals of Treasury Bill Rate

$$P_{st} = \frac{DI_{st}}{DY_{st}}.$$

Then, we can calculate the return for holding S&P 500 for a month under scenario s as

$$r_{st} = \frac{P_{st} - P_{s(t-1)} + DI_{st}}{P_{s(t-1)}}.$$

4 Computational results

We first present the mean annualized return of S&P 500 in 100 repetitions of the scenario tree generated by using the Gaussian and stable distribution models (See Table 3). The table also depicts the percentiles of these return scenarios. It should be noted that the S&P 500 returns generated by stable scenarios have fatter tails than those of Gaussian scenarios. Hence, stable scenarios consider more extreme scenarios than Gaussian scenarios do. Rachev et al. (2000) report similar observations in their paper where they compute value at risk employing Gaussian and stable distributed daily returns. They

Table 3. Annualized return scenarios on S&P 500

	mean	1%	2.5%	25%	75%	97.5%	99%
Normal Scenarios	9.07	−122.34	−103.03	−31.97	48.54	129.66	152.90
Stable Scenarios	10.20	−149.17	−107.29	−27.45	44.96	128.68	171.16

Table 4. Optimal allocations under normal and stable scenarios (T = 3 quarters)

γ	Normal Scenarios Optimal Percentage Invested		Stable Scenarios Optimal Percentage Invested	
	S&P 500	Treasury Bill	S&P 500	Treasury Bill
0.80	100%	0%	100%	0%
1.00[a]	100	0	88	12
1.50	86	14	66	44
2.30	60	40	48	52
2.70	52	48	42	58
10.00	14	86	12	88

[a] Note that when $\gamma = 1$ the power utility function reduces to logarithmic utility function.

state that 5% percentile of normal and stable distribution are very close, but the 1% percentile of stable distribution is greater than that of the Gaussian.

The asset allocation problem has been solved for an investor that maximizes the power utility of final wealth. The optimal asset allocation depends on the risk aversion level of the agent. If his relative risk aversion coefficient is very low, such as 0.80, or very high, such as 10.00, then the Gaussian and stable scenarios result in similar asset allocations (See Table 4). The intuitive explanation for this is that, the investor who has very low risk aversion, does not mind the risk very much. Therefore, his decision does not change when the extreme events are modeled more realistically. Similarly, the investor who has very high risk aversion, is already scared away from the risky asset. The fatter tails do not affect his decision much either. On the other hand, an investor who would put 60% in S&P 500 if he were to use normal scenarios, will put only 48% in S&P 500 if he uses stable scenarios. The fact that stable scenarios model the extreme events more realistically, results in stable investor putting less in the risky asset than Gaussian investor does.

The time series models which generate the Gaussian and stable scenarios are the same except for the residuals being Gaussian or stable, respectively. In our computations, the mean return of Gaussian S&P 500 scenarios came out to be less than stable S&P 500 scenarios. The equity premium is 3.07% in the normal scenarios and 4.20% in the stable scenarios. Since the premium on equity is higher in stable scenarios, the equity is more attractive. However, the fact that the stable scenarios also have heavier tails outweighs this, and consequently the investor puts considerably less money in the stock index. If the equity premium were the same in both sets of scenarios, we contemplate that the allocation difference would be even more pronounced.

Table 5 depicts the change in the utility if the investor uses stable scenarios rather than Gaussian scenarios[2]. The improvement can be as large as 0.72% depending on the risk aversion level of the investor. Table 6 reports the improvement in the certainty equivalent final wealth (CEFW) if an investor uses stable scenarios rather than Gaussian scenarios. The computations show a 6 basis point improvement in the certainty equivalent wealth of the investor who

[2] Since Gaussian distribution is a special case of stable distribution, the stable model encompasses the Gaussian model. Therefore, the comparisons are made under the assumption that stable is the correct model.

Table 5. Comparison of utility achieved from normal and stable scenarios (T = 3 quarters)

γ	Normal Scenarios		Stable Scenarios		% Change
	% in S&P 500	Utility	% in S&P 500	Utility	in Utility
0.80	100%	5.0633	100%	5.0633	0.00
1.00	100	0.0600	88	0.0604	0.72
1.50	86	−1.9458	66	−1.9445	0.06
2.30	60	−0.7188	48	−0.7181	0.09
2.70	52	−0.5391	42	−0.5386	0.09
10.00	14	−0.0728	12	−0.0728	0.03

Table 6. Comparison of certainty equivalent wealth achieved from normal and stable scenarios (T = 3 quarters)

γ	Normal Scenarios		Stable Scenarios		% Change
	% in S&P 500	CEFW	% in S&P 500	CEFW	in CEFW
0.80 x	100%	1.0650	100%	1.0650	0.00
1.00	100	1.0618	88	1.0623	0.04
1.50	86	1.0565	66	1.0579	0.13
2.30	60	1.0536	48	1.0543	0.07
2.70	52	1.0526	42	1.0532	0.05
10.00	14	1.0480	12	1.0481	0.00

would put 60% in S&P 500. The difference could get larger or smaller depending on the risk aversion level of the decision maker.

The other 'utility' function we consider is an analog of mean-variance criterion. The computational results achieved are very similar to the constant relative risk aversion utility. The investor who has very low or very high risk aversion, does not gain much from using the stable model. However, the stable model makes a difference for the investors in the middle. Table 7 depicts that an investor who would put 60% in S&P 500 if he were to use normal scenarios, will put only 56% in S&P 500 if he uses stable scenarios. Table 8 reports the percentage improvement in the 'utility' function[3] if one uses stable model as opposed to Gaussian model. If there is any percentage improvement in the utility function, an investor can reduce the risk for a given level of mean return or increase the mean return for a given level of risk. This can be achieved by switching from Gaussian scenario generation to stable scenario generation.

5 Conclusion

Generating scenarios that realistically represent the future uncertainty is important for the validity of the results of asset allocation models. The assump-

[3] Since the risk corresponding to certainty equivalent return is zero, the certainty equivalent return is equal to the utility of return. Hence, the percentage improvement in the utility of return is equivalent to the percentage improvement in the certainty equivalent return.

Table 7. Optimal allocations under normal and stable scenarios (T = 3 quarters)

c	Normal Scenarios Optimal Percentage Invested		Stable Scenarios Optimal Percentage Invested	
	S&P 500	Treasury Bill	S&P 500	Treasury Bill
0.35	100%	0%	100%	0%
0.40	90	10	80	20
0.52	60	40	54	46
0.59	50	50	44	66
1.00	20	80	18	82

Table 8. Percentage change in utility achieved from normal and stable scenarios (T = 3 quarters)

c	Normal Scenarios		Stable Scenarios		% Change
	% in S&P 500	Utility	% in S&P 500	Utility	in Utility
0.35	100%	0.0583	100%	0.0583	0.00
0.40	90	0.0561	80	0.0562	0.28
0.52	60	0.0526	54	0.0527	0.10
0.59	50	0.0513	44	0.0514	0.12
1.00	20	0.0479	18	0.0480	0.08

tion underlying most of the scenario generation models used in the literature is the normal distribution. The validity of normal distribution has been questioned in the finance and macroeconomics literature. The leptokurtotic and asymmetric nature of economic variables can be better captured by using stable distribution as opposed to normal distribution.

We analyze the effects of the distributional assumptions on optimal asset allocation. A multistage dynamic asset allocation model with decision rules has been set up. The optimal asset allocations found under normal and stable scenarios are compared. The analysis suggests that the normal scenarios greatly underestimate risks. Stable scenario modeling leads to asset allocations that are up to 20% different than those of normal scenario modeling.

The effect of fat tailed returns on the asset allocation decision depends on the objective function of the investor. In this paper, we only analyzed power utility and mean-1.5 power of mean absolute deviation. Tokat et al. (2001) investigate the same effect when the investor uses Value-at-Risk and Conditional Value-at-Risk as risk measures, and they report that the impact of fat tailed returns on asset allocation are even greater for these risk measures since they concentrate on the tail of the return distribution.

We have used stable modeling for an equity index that is not as highly volatile as some other indices. We expect that the allocation and certainty equivalent returns differences will be greater for more volatile indices such as NASDAQ, Russell 2000 or Wilshire 5000. Ortobelli et al. (1999) observe such a behavior for several indices they compare.

Although the financial data exhibit time varying volatility and long range dependence as well as heavy tails, this study has only considered explicit modeling of heavy tails in the financial data. The conditional heteroskedastic models (ARMA-GARCH) utilizing stable distributions can be used to describe the time varying volatility along with the asymmetric and leptokurtic behavior. In addition to these, the long-range dependence can also be modeled if fractional-stable GARCH models are employed. These aspects of financial data will be considered in a later paper.

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