Journal of Financial Economics 26 (1990) 97–121. North-Holland

The valuation of options on yields

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Received September 1989, final version received May 1990

Many contingent claims incorporate options on yield levels. I derive closed-form expressions for European yield-option prices using a general equilibrium model in which the underlying yield is the relevant state variable. The properties of these options differ markedly from those of conventional options on traded assets. For example, yield-call values can be less than their intrinsic value and can be decreasing functions of the underlying yield. These features have important hedging implications. I examine the empirical implications of the model using price data for the 13-week T-bill options traded on the Chicago Board Options Exchange.

1. Introduction

Unfavorable shifts in the term structure are one of the most basic risks facing financial-market participants. Volatile interest rates during the past decade have magnified this risk and have led to a dramatic increase in the number and types of contingent claims that incorporate options on the level of yields. These options differ from options on bonds in that the underlying state variable is a yield. For example, interest-rate caps, floors, locks, and floor-ceiling agreements are simple portfolios of options on yields. Floating-rate notes and bonds often include limits on the size of the rate adjustment, a feature that can be modeled as a yield option. A growing number of financial institutions issue certificates of deposit that guarantee a minimum renewal yield if the certificate is rolled over at maturity. This renewal guarantee is simply a put option on the future yield. Many other contingent claims, such

*I am grateful for the comments of Warren Bailey, Fischer Black, Steve Buser, Peter Carr, K.C. Chan, Bruce Grundy, Michael Hemler, Coleman Kendall, David Mayers, Huston McCulloch, Tony Sanders, Chester Spatt, René Stulz, and Finance workshop participants at Carnegie Mellon University, The Ohio State University, and the University of Minnesota. I also wish to thank Wai-Ming Fong for research assistance. The comments and suggestions of the editor, Clifford Smith, Jr., and Eduardo Schwartz, the referee, were particularly helpful. All errors are my responsibility.

as adjustable-rate mortgages, options on interest-rate swaps, options on the actively traded Eurodollar futures contract, and extendible corporate bonds, incorporate yield options. Conservative estimates of the total principal value of contingent claims covered by some form of a yield option are well in excess of $500 billion.

In this paper, I derive closed-form expressions for the values of European calls and puts on yields, using an extended version of the Cox, Ingersoll, and Ross (1985b) general-equilibrium term-structure framework. In this version, I use the T-maturity yield as the relevant state variable in pricing an option on the T-maturity yield. This approach allows the model to capture the level of the yield curve at the most relevant maturities for pricing yield options. I also derive closed-form expressions for the values of options on average yields and yield spreads by showing that these claims can be represented as portfolios of yield calls and puts.

The properties of yield-option prices are fundamentally different from those of conventional option prices. This is because yields – although simple nonlinear functions of bond prices – are not themselves the prices of traded assets. Consequently, they need not follow a martingale under the equivalent martingale measure described by Harrison and Kreps (1979) and Harrison and Pliska (1981). One implication is that the price of a call on a yield can exceed the yield's current numerical value. Similarly, the price of a yield call can be less than its intrinsic value. I show that the value of a yield call need not be a monotone increasing function of the underlying yield. An increase in the yield increases not only the expected payoff for the call, but also the discount rate for the payoff, which eventually dominates. This feature has important implications for hedging interest-rate risk with yield options. Furthermore, I show that yield-call prices can actually be decreasing functions of the time until expiration and the volatility of the underlying yield. Corresponding results hold for yield puts.

To examine the empirical implications of these results, I focus on the valuation of the recently introduced European options on the 13-week Treasury-bill yield traded on the Chicago Board Options Exchange (CBOE). The sample covers the first six months of trading, but excludes days on which the options are not actively traded. Over 450 call and put prices are included. Interestingly, most of the prices of in-the-money calls are below their intrinsic values. This is consistent with the pricing model of this paper, but incompatible with alternative pricing models based on the Black–Scholes (1973) or Black (1976) formulas. In addition, I show that deviations from the traditional put–call parity relation are positively related to the maturity of the options, as implied by the yield-option model. Like Rubinstein (1985), I test for model biases by inferring the volatility parameter from observed option prices and then examining whether the implied volatility estimates are systematically related to other variables. I find some evidence of a yield-related bias, but it accounts for only a small proportion of the variation in the implied volatilities. Finally, I examine the pricing errors and show that the model performs well on average. The average pricing error is on the order of 2% for call options and 6% for put options.

Section 2 presents the valuation model, derives closed-form expressions for yield-option prices, and discusses their analytical properties. Section 3 describes the data. Section 4 presents the empirical results. Section 5 summarizes the results and makes concluding remarks.

2. Yield option prices

In this section, I derive closed-form expressions for European yield option values using the general-equilibrium term-structure model of Cox, Ingersoll, and Ross (1985b). In this model, discount bond prices and yields are obtained by specializing the intertemporal general-equilibrium asset-pricing model of Cox, Ingersoll, and Ross (1985a) to a single-state-variable setting in which expected production returns and return variances are proportional to a fundamental state variable designated X. In addition, the representative investor is assumed to have time-additive state-independent logarithmic preferences and the state variable is assumed to follow a singular square root diffusion process of the type studied by Feller (1951).

Let Y_T denote the yield to maturity for discount bonds with a constant maturity of T. In this framework, the dynamics of the economy are time-homogeneous. Thus, without loss of generality, I designate the current time as time zero. Furthermore, for notational convenience, I denote the time t value of the T-maturity yield $[Y_T(t)]$ simply as Y_T when the value of t is clear from the context. Cox, Ingersoll, and Ross show that the yield on instantaneously maturing bonds Y_0 is proportional to X, and use this property to make a change of variables from X to Y_0. The resulting equilibrium dynamics for Y_0 can be written in the form

$$\mathrm{d}Y_0 = (\alpha - \kappa Y_0)\,\mathrm{d}t + \sigma\sqrt{Y_0}\,\mathrm{d}Z, \tag{1}$$

where α, κ, and σ^2 are positive constants. The parameter α corresponds to the term $\kappa\theta$ in Cox, Ingersoll, and Ross' (1985b) eq. 17. We use this simpler notation to conform more closely to Feller (1951), who shows that the behavior of the process as it approaches the singularity at zero is governed entirely by the relation between the parameters α and σ^2 and is independent of κ. The T-maturity yield can be expressed as

$$Y_T = A(T) + B(T)Y_0, \tag{2}$$

where

$$A(T)=\frac{2\alpha}{\sigma^2 T}\ln\left(\frac{(\gamma+\beta)(e^{\gamma T}-1)+2\gamma}{2\gamma\, e^{(\gamma+\beta)T/2}}\right),$$

$$B(T)=\frac{2(e^{\gamma T}-1)}{T((\gamma+\beta)(e^{\gamma T}-1)+2\gamma)},$$

$$\gamma=(\beta^2+2\sigma^2)^{1/2},$$

and β equals the sum of κ and the market price of interest-rate risk parameter λ. Using this notation, it is clear from Cox, Ingersoll, and Ross' (1985b) eq. 22 that interest-rate-dependent contingent-claim values depend on κ and the market price of interest-rate risk parameter λ only through their sum β. This follows because β, not κ, is the coefficient of Y_0 in the drift term for the risk-adjusted dynamics of Y_0. This feature allows contingent-claim values to be expressed in terms of just three parameters – α, β, and σ^2. An important advantage of this parameterization is that it eliminates the need to estimate the market price of risk as a separate parameter. General-equilibrium models of contingent-claim prices have often been criticized because of the perception that market prices of risk need to be estimated separately, and in addition to the structural parameters of the model. For example, see Ball and Torous (1983) and Heath, Jarrow, and Morton (1987). If T is held fixed, Y_T is a linear function of Y_0 and therefore of X. Thus for $T<\infty$, the mapping from X to Y_T is globally invertible and, without loss of generality, we are free to make a simple change of variables from the original-state variable X to Y_T (instead of Y_0) in deriving interest-rate-sensitive contingent-claim values.

Intuitively, the choice of Y_T as the state variable is a natural one. For example, if we are interested in pricing options on the ten-year yield, we can choose the ten-year yield as the state variable. Similarly, if we wish to price options on the riskless rate Y_0, we can choose Y_0 as the relevant state variable. Thus, the linearity of yields in the state variable X in this framework actually provides a large family of possible state variables to work with, rather than just the instantaneous riskless rate. These changes of variable are all mathematically equivalent, and the contingent-claim values implied by this framework are independent of the change of variables employed. The advantage of using Y_T as the state variable is that it allows us to fit the level of the term structure at the most relevant maturity for pricing interest-rate options rather than at the short end of the maturity spectrum.

With this framework, we can now derive expressions for the values of contingent claims with payoffs that depend on Y_T. Let $F(Y_T)$ denote the payoff function for a contingent claim on Y_T at the maturity date τ. Cox, Ingersoll, and Ross (1985a) show that the value of the claim can be obtained by taking the expectation of the product of $F(Y_T)$ and the discount factor with respect to the risk-adjusted process for Y_T. In general, however, the Cox, Ingersoll, and Ross approach is difficult to apply directly because the stochastic discount factor is correlated with the payoff function, and directly evaluating the expectation of the product of the discount factor and the payoff function can be an intractable task.

The following separation theorem resolves this problem. Specifically, I show that the discounting and expectation-taking functions of valuing interest-rate-dependent contingent claims can be performed sequentially, rather than simultaneously, by making a further adjustment to the risk-adjusted process for Y_T. Thus, contingent claims can be valued by first taking the expectation of the payoff, and then discounting the expectation at the riskless rate. In this sense, this theorem provides a stochastic-interest-rate counterpart to the Cox and Ross (1976) risk-neutral valuation approach.

Separation Theorem. Let $F(Y_T)$ denote the payoff function for a (European-type) contingent claim on Y_T maturing at time τ. Let $D(\tau)$ denote the value of a τ-maturity discount bond. Then the value of the contingent claim in this framework is

$$D(\tau)\,\mathrm{E}[F(Y_T)], \tag{3}$$

where the expectation is taken with respect to Y_T which is distributed as

$$A(T) + \left[\sigma^2 \tau B(\tau) B(T)/4\right] \chi^2(\nu, \eta),$$

and where $\chi^2(\nu,\eta)$ is a noncentral chi-square variate[1] *with ν degrees of freedom and noncentrality parameter η, where*

$$\nu = 4\alpha/\sigma^2,$$

$$\eta = \frac{-4\gamma^2 \tau\, \mathrm{e}^{\gamma\tau} B(\tau) A(T)}{\sigma^2(\mathrm{e}^{\gamma\tau}-1)^2 B(T)} + \frac{4\gamma^2 \tau\, \mathrm{e}^{\gamma\tau} B(\tau)}{\sigma^2(\mathrm{e}^{\gamma\tau}-1)^2 B(T)} Y_T.$$

[1]The noncentral chi-square distribution is discussed in Johnson and Kotz (1970, ch. 28). A linear function of a noncentral chi-square variate can also be viewed as a special case of a quadratic form of a normal variate. See Johnson and Kotz (1970, ch. 29).

Proof. Let $H(Y_0, \tau)$ denote the value of an interest-rate-dependent contingent claim with payoff function $F(A(T)+B(T)Y_0)$ at time τ. From Cox, Ingersoll, and Ross (1985b), $H(Y_0, \tau)$ satisfies the partial differential equation

$$\frac{\sigma^2 Y_0}{2} H_{11} + (\alpha - \beta Y_0) H_1 - Y_0 H - H_2 = 0,$$

where the subscripts represent partial derivatives with respect to the indicated arguments. Let $H(Y_0, \tau) = D(\tau) G(Y_0, \tau)$, where $D(\tau)$ is the value of a τ-maturity discount bond. Differentiating H, substituting the partial derivatives into the partial differential equation, and observing that $D(\tau)$ also satisfies the above equation, leads to the following partial differential equation for $G(Y_0, \tau)$:

$$\frac{\sigma^2 Y_0}{2} G_{11} + \left[\alpha - \beta Y_0 - \sigma^2 \tau B(\tau) Y_0\right] G_1 - G_2 = 0,$$

subject to the same maturity condition. Thus, from Friedman's (1975) theorem 5.2,

$$H(Y_0, \tau) = D(\tau)\, \mathrm{E}\left[F(Y_T)\right],$$

where the expectation is taken with respect to the process

$$\mathrm{d}Y_0 = \left[\alpha - \left(\beta + \sigma^2 \tau B(\tau)\right) Y_0\right] \mathrm{d}t + \sigma \sqrt{Y_0}\, \mathrm{d}Z.$$

A simple change of scale and time of the type described by Capocelli and Ricciardi (1976) can now be used to show that the conditional distribution of $4Y_0/[\sigma^2 \tau B(\tau)]$ implied by this process corresponds to a noncentral chi-square with ν degrees of freedom and noncentrality parameter η. The result then follows by changing variables from Y_0 to Y_T. ■

2.1. *Yield calls*

The payoff function for a call on Y_T with maturity τ is $\max(0, Y_T - K)$, where K denotes the strike price (or exercise yield).[2] The payoff for the call is in monetary units such as dollars, since Y_T and K are expressed in numerical form. By expressing Y_T and K in numerical form, we avoid

[2]Since the support of Y_T is $(A(T), \infty)$, we assume that $K \geq A(T)$. Otherwise, the call option is always in the money at maturity [this type of call can be valued by simply setting $\phi = 0$ in (4)]. If $2\alpha < \sigma^2$, the value $A(T)$ is accessible and the support of Y_T is actually $[A(T), \infty)$. As in Cox, Ingersoll, and Ross (1985b), we require that the process satisfy the generalized reflecting barrier (zero flux) condition at $A(T)$ when $0 < 2\alpha < \sigma^2$. See Feller (1951).

confusing percentages with monetary units. To illustrate, if $Y_T = 0.09$ at time τ and $K = 0.07$, then the call payoff is 0.02. In many cases, the payoff function for actual yield calls may be scaled by multiplying the yield and strike price by 100 or even 1,000 as in the case of the 13-week Treasury-bill options traded on the CBOE. From the Separation Theorem, this scaling has no effect on the valuation of the options – the value of a call on 1,000 Y_T with strike price 1,000 K is just 1,000 times the value of a call on Y_T with strike price K.

Having specified the payoff function, we can obtain the value of a yield call $C(Y_T, \tau, K)$ by simply substituting the payoff function into (3) and taking the relevant expectation. The resulting expression for $C(Y_T, \tau, K)$ is

$$D(\tau)\left[Y_T Q(\phi, \nu+4, \eta)\xi - KQ(\phi, \nu, \eta) + \psi\right], \tag{4}$$

where

$$\xi = \left(\gamma\tau B(\tau)(e^{\gamma\tau}-1)^{-1} e^{\gamma\tau/2}\right)^2,$$

$$\phi = \frac{4(K - A(T))}{\sigma^2 \tau B(\tau) B(T)},$$

$$\psi = A(T)Q(\phi, \nu, \eta) + \alpha\tau B(\tau)B(T)Q(\phi, \nu+2, \eta)$$

$$- \xi A(T)Q(\phi, \nu+4, \eta),$$

and where $Q(\phi, \nu, \eta)$ is the complementary noncentral chi-square distribution function[3] with ν degrees of freedom and noncentrality parameter η.

From (4), the call price is actually a function of two yields: Y_τ (via the discount bond price) and Y_T. This is reasonable, since we would expect these two maturities to be the most relevant for pricing a τ-maturity option on the T-maturity yield. Eq. (4) has some features in common with both the Black–Scholes (1973) and Black (1976) option-pricing models. For example, the first two terms in (4) include the product of a cumulative distribution function with the underlying state variable Y_T and the strike price K, respectively.

[3]The complementary noncentral chi-square distribution function $Q(\phi, \nu, \eta)$ is simply $1 - \chi^2(\phi, \nu, \eta)$, where $\chi^2(\phi, \nu, \eta)$ is the noncentral chi-square distribution function. See Johnson and Kotz (1970, ch. 28, eq. 3.1). An extremely accurate algorithm for computing $\chi^2(\phi, \nu, \eta)$ is given by Sankaran (1963) and discussed by Johnson and Kotz. This algorithm approximates the noncentral chi-square distribution by a normal distribution. Thus, calculating yield-option values using this algorithm requires only slightly more computer time than calculating Black–Scholes option values. Also see Schroder (1989) and Longstaff (1990).

In general, however, the properties of yield-call values are fundamentally different from those of conventional call options. The primary reason is that Y_T, although a simple nonlinear function of a discount bond price, is itself not the price of a traded asset. Thus, Y_T need not follow a martingale under the equivalent martingale measure described by Harrison and Kreps (1979) and Harrison and Pliska (1981). To see this, note that the price of a call on Y_T with a strike price of zero $C(Y_T, \tau, 0)$ is not equal to the current value of Y_T. This implies that there is no self-financing portfolio with value equal to Y_T at every point in time. If there were, arbitrage profits could be generated by taking offsetting positions in this portfolio and in a call on Y_T with a strike price of zero.

To illustrate some of the differences between options on yields and options on traded assets, consider the rational restrictions on option prices derived by Merton (1973). For example, Merton shows that call prices are bounded above by the price of the underlying asset. In contrast, (4) implies that the price of a call on Y_T can exceed the current value of Y_T. The intuition for this result can be understood best in terms of the mean reversion of yields in this framework. If the current value of Y_T is below its unconditional value, future values of Y_T are likely to be higher. Of course, higher future yields are also associated with higher discount rates. For short-maturity calls, however, the prospect of higher yields in the future can outweigh the higher discount factor and lead to call values in excess of the current value of Y_T.

Merton also shows that calls on traded assets must be worth more than their intrinsic value in order to avoid arbitrage opportunities. The value of a yield call, in contrast, can be less than its intrinsic value $\max(0, Y_T - K)$. This is illustrated in fig. 1, which graphs the values of calls on the three-month yield as a function of the three-month yield. As shown, the call values are greater than their intrinsic value for small values of Y_T. As Y_T increases, however, the call values eventually drop below their intrinsic value.

The intuition for why the value of a yield call can be less than its intrinsic value is best understood by first examining the comparative statics for the option. For example, it is easily shown that the relationship between yield-call values and changes in Y_T is indeterminate. The reason is that changes in Y_T have two effects on the interest-rate call price. First, an increase in Y_T increases the expected payoff for the call. However, the higher Y_T, the higher the discount factor for the terminal payoff function for the call. For small values of Y_T, the first effect dominates and the call is an increasing function of Y_T. For some sufficiently large value of Y_T, however, the latter effect dominates and the call becomes a decreasing function of Y_T. Furthermore, as $Y_T \to \infty$, (4) implies that the call price converges to zero. This is because an increase in Y_T increases the expected payoff of the option linearly, whereas it has an exponential effect on the discount factor.[4] Of course, since the call price converges to zero as $Y_T \to \infty$, the continuity of the call price in Y_T

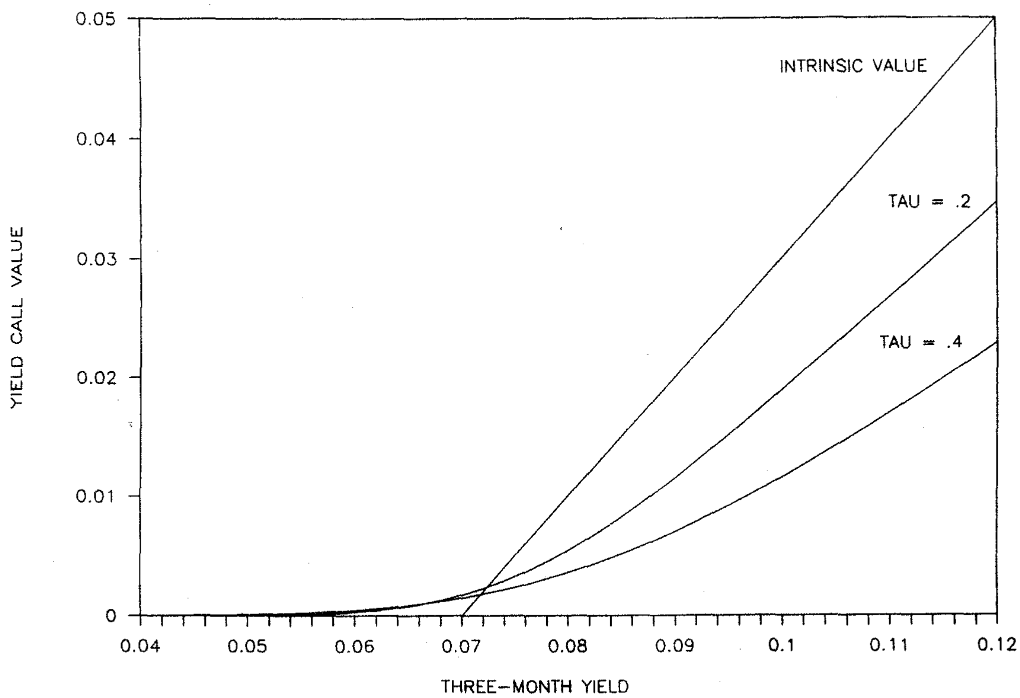

Fig. 1. Examples of the values of calls on the three-month Treasury-bill yield plotted as functions of the three-month Treasury-bill yield. The model parameters used are $\alpha = 0.04$, $\beta = 1.00$, and $\sigma^2 = 0.01$. The underlying strike price for the calls is 0.07. The call maturities (tau) are 0.2 and 0.4 years, respectively. The 45-degree line is the intrinsic value for in-the-money calls.

implies that the call will be below its intrinsic value for some sufficiently large Y_T. An important implication is that the early exercise of an American call option on Y_T can be optimal, even though the underlying state variable Y_T follows a process with a continuous sample path.

The nonmonotonicity of the yield call in Y_T has many important implications for the hedging properties of these options. Fig. 2 presents some additional examples of the values of calls on the three-month yield graphed as functions of the three-month yield, but with larger values of τ than in fig. 1. Yield calls can actually be perverse hedges against shifts in the yield curve for longer-maturity calls. This property is particularly important for the active interest-rate-cap market, since a typical cap agreement might involve calls on a short-term yield with option maturities ranging from 5 to 30 years. In

[4]This follows since $C(Y_T, \tau, 0)$ provides an upper bound for $C(Y_T, \tau, K)$ and because

$$\lim_{Y_T \to \infty} C(Y_T, \tau, 0) = \lim_{Y_T \to \infty} \xi D(\tau) Y_T = 0,$$

where the second equality follows from the Reimann–Lebesgue lemma.

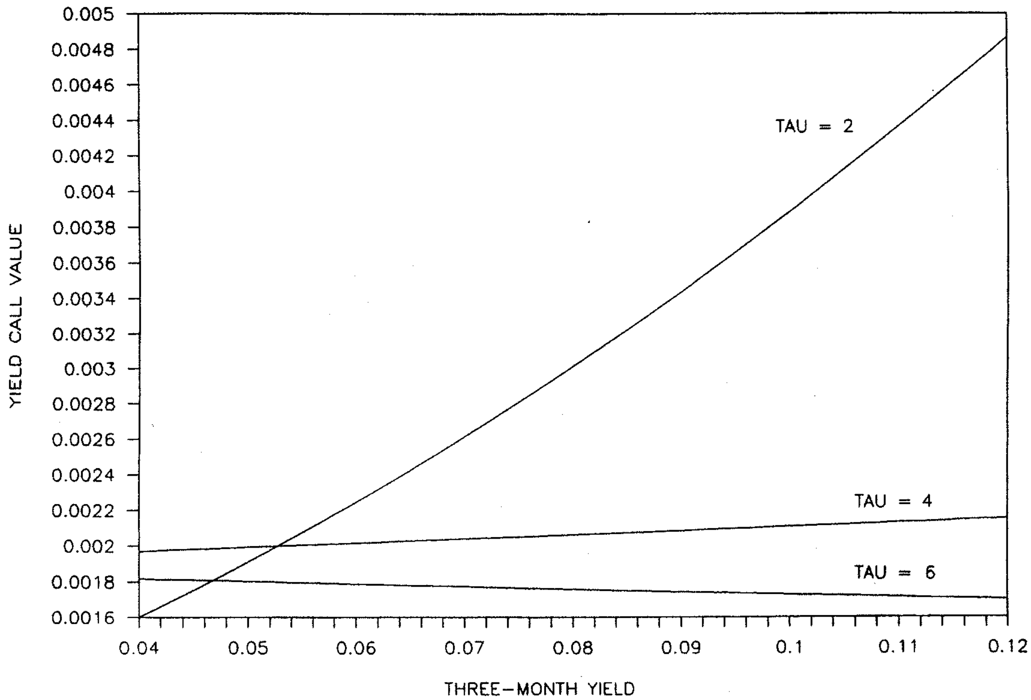

Fig. 2. Examples of the values of calls on the three-month Treasury-bill yield plotted as functions of the three-month Treasury-bill yield. The model parameters used are $\alpha = 0.06$, $\beta = 1.00$, and $\sigma^2 = 0.01$. The underlying strike price for the calls is 0.07. The call maturities (tau) are 2, 4, and 6 years, respectively.

addition, fig. 2 shows that there is some critical value of Y_T at which changes in Y_T leave the call value unaffected. This means that for some values of Y_T, an uncovered position in an in-the-money yield call can be perfectly hedged or immunized against interest-rate risk. Furthermore, it is possible to hedge a long-call position with another long call in some situations.

Fig. 2 also shows that calls on yields need not be increasing functions of the life of the option. In fact, for some values of Y_T, the value of a yield call can increase as τ increases, but can then decrease as τ increases further. The intuition for this comparative-statics result is similar to that for the relation between call values and yields. As $\tau \to \infty$, the discount factor approaches zero while the expected payoff for the option remains bounded because of the mean-reverting behavior of Y_T. Thus, the call must eventually become a decreasing value of τ since its price converges to zero as τ increases without bound.

Finally, we consider the relation between the call value and the underlying riskiness of changes in Y_T, which is governed by the parameter σ^2. In contrast to the Black–Scholes option prices, yield calls need not be increasing functions of the variance of changes in Y_T as measured by σ^2. The reason is again related to the mean reversion in Y_T. As shown by Cox, Ingersoll, and Ross, discount bond prices are increasing functions of σ^2. However, increases in σ^2 also tend to reduce the long-term mean of Y_T. For small values of Y_T the first effect dominates, whereas for larger Y_T the opposite is true. The remaining comparative-statics results for yield calls are indeterminate – call values can be either increasing or decreasing functions of α, β, and T.

2.2. *Yield puts*

The payoff function for a put option on Y_T is $\max(0, K - Y_T)$. The value of this put option $P(Y_T, \tau, K)$ is obtained by using the put–call parity relation for yield options,

$$P(Y_T, \tau, K) = C(Y, \tau, K) + KD(\tau) - C(Y_T, \tau, 0). \tag{5}$$

This put–call parity relation differs from that for options on traded assets. Again, this is because the value of a portfolio that pays Y_T at time τ is not the current value of Y_T, but the value of a call option on Y_T with a strike price of zero. As with yield calls, the value of the yield put is a function of two yields, Y_τ and Y_T, and captures the level of the yield curve at two points.

Many of the analytical properties of puts on yields are similar to those of yield calls. For example, since $Y_T > 0$ in this valuation setting, the upper bound for a put on Y_T is $KD(\tau)$, which converges to zero as $\tau \to \infty$. Thus, yield puts are not monotone increasing functions of the option's maturity – as τ increases without bound, the yield-put price must eventually decline. As for yield calls, yield-put values converge to zero as $Y_T \to \infty$. However, differentiating the yield-put price with respect to Y_T shows that the price is always a decreasing function of Y_T. The reason is that an increase in Y_T decreases both the present value of the payoff from the put and the amount of the payoff itself. In this respect, yield-put values are similar to Black–Scholes put prices, which are also monotone decreasing functions of the underlying price. Finally, an increase in the underlying volatility parameter σ^2 can either increase or decrease the value of the yield put. In addition, the signs of the partial derivatives of the yield put with respect to the parameters α, β, and T are indeterminate.

2.3. *Options on average yields*

Rather than providing an option on a specific yield, some types of yield options are based on the average of several yields. For example, the CBOE LTX options are European calls and puts on the arithmetic average of the yields on 7-, 10-, and 30-year Treasury notes and bonds. The linear relation between yields of all maturities in this valuation framework allows us to use the earlier results for calls and puts to derive closed-form expressions for options on average yields.

To show this, let Y_{T_1} and Y_{T_2} be the yields for maturities T_1 and T_2, respectively, where $T_1 < T_2$, and assume that we are interested in pricing an option on a weighted average of the two yields $wY_{T_1} + (1-w)Y_{T_2}$, where $0 \leq w \leq 1$. The payoff function for a call on this average is $\max(0, wY_{T_1} + (1-w)Y_{T_2} - K)$. From (2), we can express Y_{T_1} as a linear function of Y_{T_2},

$$Y_{T_1} = c_0 + c_1 Y_{T_2}, \tag{6}$$

where

$$c_0 = \frac{A(T_1)B(T_2) - A(T_2)B(T_1)}{B(T_2)} \leq 0,$$

$$c_1 = \frac{B(T_1)}{B(T_2)} \geq 1.$$

We could also express Y_{T_2} in terms of Y_{T_1}, which would lead to the option on the average yield's being represented as a portfolio of options on Y_{T_1}. The options on Y_{T_1} could potentially have negative exercise prices, however. This poses no real difficulties, since options with negative strike prices can be valued using (4) by setting $\phi = 0$. Substituting (6) into the payoff function for the call gives $\max[0, (wc_1 + [1-w])Y_{T_2} - (K - wc_0)]$ which can also be written

$$[1 + w(c_1 - 1)]\max\left(0, Y_{T_2} - \frac{K - wc_0}{1 + w(c_1 - 1)}\right). \tag{7}$$

Observe that the payoff for the call on the average yield is identical to the payoff from holding $[1 + w(c_1 - 1)]$ interest-rate calls on Y_{T_2} with strike price $(K - wc_0)/[1 + w(c_1 - 1)]$ – a call on the average is equivalent to a portfolio of calls on Y_{T_2}. Thus, the value of the call on the average is simply

$$[1 + w(c_1 - 1)]C\left(Y_{T_2}, \tau, \frac{K - wc_0}{1 + w(c_1 - 1)}\right). \tag{8}$$

Note that if $T_1 \to T_2$, then $c_0 \to 0$, $c_1 \to 1$, and the value of a call converges to $C(Y_{T_2}, \tau, K)$. Similar arguments can be used to show that the value of a put on the average yield is

$$[1 + w(c_1 - 1)]P\left(Y_{T_2}, \tau, \frac{K - wc_0}{1 + w(c_1 - 1)}\right). \tag{9}$$

These results hold for options on averages of two yields. The analysis is easily generalized, however, to value options on the average of more than two yields. These options can be represented as portfolios of options on the longest-maturity yield.

2.4. *Options on yield spreads*

An increasingly popular type of yield option consists of calls and puts on the spread between the yields of two different maturities. For example, the SYCURVE options recently introduced by Goldman, Sachs & Co. are calls and puts on the spread between two yields, typically a short-term yield and an intermediate- or long-term yield.

The above approach for pricing options on average yields can be applied to value options on yield spreads. As an example, assume that we wish to value an option on the spread between Y_{T_2} and Y_{T_1}. If $K = 0$, the payoff function for this claim is simply $\max(0, Y_{T_2} - Y_{T_1})$. This option can also be viewed as an exchange option on Y_{T_2} [see Margrabe (1978)]. Using (6), the payoff function[5] for this claim can be expressed as $\max(0, -c_0 + (1 - c_1)Y_{T_2})$, or alternatively as

$$(c_1 - 1)\max\left(0, \frac{c_0}{1 - c_1} - Y_{T_2}\right). \tag{10}$$

However, this is just the payoff function for a portfolio of puts on Y_{T_2} with strike price $c_0/(1 - c_1)$. Thus, the value of this yield-spread option is

$$(c_1 - 1)P\left(Y_{T_2}, \tau, \frac{c_0}{1 - c_1}\right). \tag{11}$$

As $T_1 \to T_2$, the value of this option converges to zero. This is intuitively clear, since the right to exchange a variable for itself is worthless. Similar reasoning can be used to show that the value of an option on the spread

[5] If $K \neq 0$, then the payoff function is just $\max(0, -K - c_0 + (1 - c_1)Y_{T_2})$ and the following analysis can be applied to value the option (making the obvious modification in the strike price).

$Y_{T_1} - Y_{T_2}$ is given by

$$(c_1 - 1)C\left(Y_{T_2}, \tau, \frac{c_0}{1 - c_1}\right). \tag{12}$$

Finally, it can be shown that a contingent claim with a payoff function equal to the maximum of the two yields Y_{T_1} and Y_{T_2} can be replicated by a portfolio consisting of a call on Y_{T_1} and an exchange option on Y_{T_2}. Thus, the value of a claim that pays $\max(Y_{T_1}, Y_{T_2})$ at maturity τ is just

$$C(Y_{T_1}, \tau, 0) + (c_1 - 1)P\left(Y_{T_2}, \tau, \frac{c_0}{1 - c_1}\right). \tag{13}$$

Similarly, the value of a claim that pays $\min(Y_{T_1}, Y_{T_2})$ is

$$C(Y_{T_1}, \tau, 0) - (c_1 - 1)C\left(Y_{T_2}, \tau, \frac{c_0}{1 - c_1}\right). \tag{14}$$

3. The data

The valuation results derived in this paper have many potential applications. To examine the empirical implications of the results, I focus on the valuation of the recently introduced options on the 13-week U.S. Treasury-bill yield traded on the CBOE. These options are of particular interest because they have a European-style exercise feature and their market prices can be readily observed. Although this market may not be as large as some interest-rate option markets, trading in these options has been significant. The most important reason for studying these options, however, is the insight they provide into the pricing of yield options that are implicit in other contingent claims and do not have directly observable prices.

These yield options began trading on June 23, 1989. The underlying index for the options is 1,000 times the yield (annualized discount rate) on the most recently auctioned 13-week U.S. Treasury bill. For example, if the yield on the 13-week T-bill is 0.0745, the index is 74.50. The strike price intervals are in units of 2.5 index points. The options are cash-settled on the basis of the index-strike differential at the expiration date of the option. For example, if the strike price is 77.5 and the index level is 80.0, the payoff from a call option is 2.5. Option premiums are quoted in 1/16s for prices below 3, and in 1/8s for all other prices. A monthly and a quarterly cycle of expirations are listed and options expire on the Saturday following the third Friday (last trading day) of the expiration month.

Table 1

Summary statistics for 13-week Treasury bill-yield option prices and trading volume for the period June 23, 1989 to December 31, 1989.[a]

Expiration date	13-week T-bill yield[b]		Yield calls			Yield puts		
	Min.	Max.	N^c	Avg. τ^d	Avg. volume[e]	N^c	Avg. τ^d	Avg. volume[e]
22 Jul 89	0.0771	0.0809	31	15.0	789.5	19	15.6	810.8
19 Aug 89	0.0761	0.0809	71	27.9	557.9	64	24.3	553.5
16 Sep 89	0.0753	0.0809	125	48.7	506.7	68	48.9	626.7
21 Oct 89	0.0741	0.0808	7	37.1	251.4	11	66.7	454.7
18 Nov 89	0.0741	0.0805	11	21.5	450.6	2	76.5	221.5
16 Dec 89	0.0741	0.0789	33	22.8	434.2	10	27.0	241.0
20 Jan 90	0.0748	0.0784	8	49.3	347.4	—	—	—
All expiration dates	0.0741	0.0809	286	35.6	528.8	174	36.4	582.2

[a]Price and trading-volume data are from the *Wall Street Journal*. Yield-call data are included in the sample only for days on which total call-trading volume exceeds 200. Similarly for yield puts.

[b]These figures represent the minimum and maximum values of the 13-week Treasury-bill yield during the life of the options with the indicated expiration date (or from the initial listing date of the option to the end of the sample period for the options expiring 20 January 1990).

[c]N is the number of prices included in the sample for options with the indicated expiration date.

[d]The average number of days until maturity for the options included in the sample with the indicated expiration date.

[e]The average trading volume for all call or put options for those days on which a price for an option with the indicated expiration date is included in the sample.

Price data for the June 1989 to December 1989 period were collected from the *Wall Street Journal*. During this period, 40 calls and puts were traded, with expiration months ranging from July 1989 to January 1990. Table 1 presents summary statistics for the data. To avoid data problems arising from thin trading, I include call prices only if the total call volume for that day exceeds 200, and similarly for put prices. As shown, the sample consists of 286 call prices and 174 put prices. In 107 cases, simultaneous call and put prices are available. The average daily volume (for days included in the sample) is 528.8 for calls and 582.2 for puts.

4. Empirical results

In examining the empirical implications of the yield-option-pricing model, I focus first on the relationship between the prices and intrinsic values of yield calls for the 13-week Treasury-bill yield options traded on the CBOE. Next, I examine the put–call parity relationship using yield-option prices. I then use Rubinstein's (1985) approach to test whether the yield-option model developed here has any systematic pricing biases. Finally, I examine the magnitude of the model's pricing errors.

4.1. Lower boundary results

Merton (1973) shows that the lower boundary for the price of an in-the-money call on a traded asset is equal to the underlying asset price minus the present value of the strike price. This lower boundary is greater than or equal to the intrinsic value of the call: Merton's results imply that European call prices cannot be less than the option's intrinsic value. As discussed earlier, however, the yield-option-pricing model implies that call values can be less than their intrinsic value $\max(0, Y_T - K)$. Furthermore, since $C(Y_T, \tau, K) \to 0$ as $Y_T \to \infty$, the price of a sufficiently deep-in-the-money yield call must be below its intrinsic value.

Fig. 3 plots the prices of the 13-week Treasury-bill calls as a function of their 'moneyness' as measured by the difference $Y_T - K$. *Most of the in-the-money call prices are below their intrinsic value*. Specifically, 144 of the 191 in-the-money call values – 75.4% – are below their intrinsic value. The mean difference between in-the-money call prices and the intrinsic value is -1.41. This difference is highly statistically significant, with a t-statistic for the mean of -10.46. Note the strong similarity between fig. 3 and the values of short-term calls on the three-month riskless yield shown in fig. 1.

These simple but striking results are consistent with the yield-option-pricing model developed in this paper. On the other hand, these results are incompatible with the frequently used approach of assuming that Y_T is the underlying asset price and applying the Black–Scholes formula – usually with

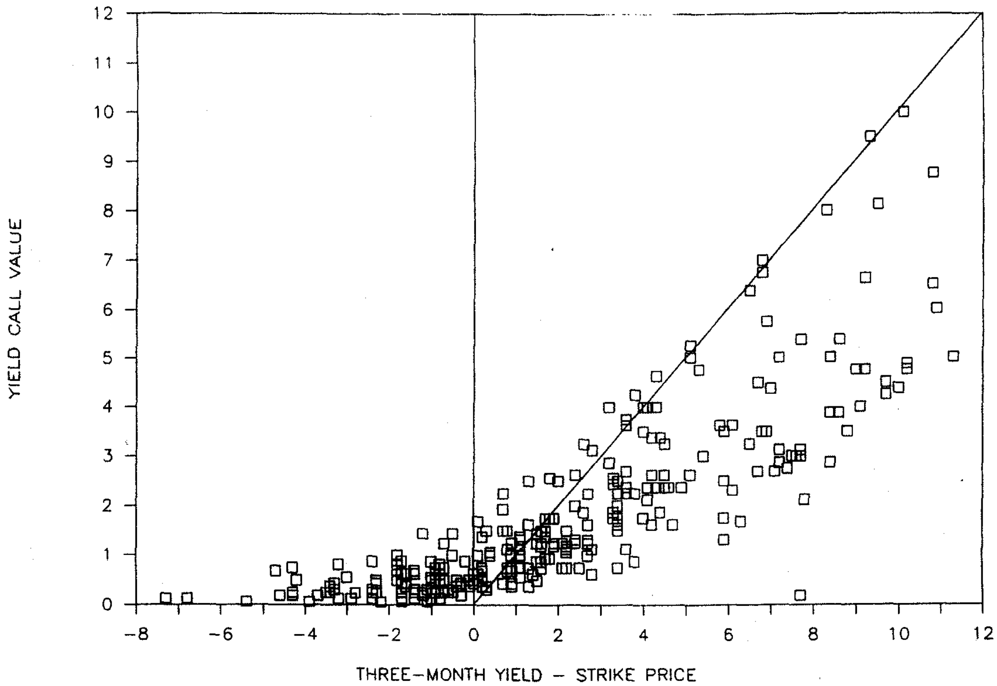

Fig. 3. CBOE 91-day Treasury-bill yield-call values for the June 1989 to December 1989 period plotted as a function of the moneyness of the options (the 91-day Treasury-bill yield index minus the strike price of the option). The 45-degree line is the intrinsic value for the in-the-money calls. 286 observations.

some adjustment to the variance of the yield process – to value options on yields.[6] This follows because there is no possible implied variance for the Black–Scholes model consistent with call prices that are below the call's intrinsic value. A similar criticism is applicable to the related approach of assuming that the forward rate $F_{\tau T}$ for the period from time τ to $\tau + T$ is a forward price, and then using the Black (1976) model for options on futures – over 69% of the in-the-money calls violate the lower bound $D(\tau)(F_{\tau T} - K)$ implied by the Black option-pricing model [see Black's (1976) eq. 16]. As discussed by Sundaresan (1990), forward rates are fundamentally different from forward prices, because a forward rate is earned over time, whereas a forward price applies to a specific point in time. Applying the Black model to forward rates to value options on yields ignores this fundamental difference.

[6]For example, see Hull (1989, pp. 260–265) for a discussion of several ways in which the Black–Scholes (1973) option-pricing model and the Black (1976) futures-option-pricing model have been modified and applied to value yield options. Also see Goldman, Sachs & Co. (1989).

4.2. Put–call parity

Let X denote the deviation from the traditional put–call parity relation,

$$X = P(Y_T, \tau, K) - C(Y_T, \tau, K) - KD(\tau) + Y_T. \tag{15}$$

If Y_T were the price of a traded asset, then standard arbitrage considerations would imply $X = 0$. Merton (1973) shows that this result is distribution-free. The yield-option-pricing model derived in this paper, however, implies that X need not equal zero in general. Specifically, from (5),

$$X = Y_T - C(Y_T, \tau, 0). \tag{16}$$

Although the precise value of X implied by the yield-option model depends on the model's parameters, several general statements about the properties of X follow from the comparative-statics results derived earlier. For example, differentiating $C(Y_T, \tau, 0)$ with respect to τ implies that X can be negative, but only for small values of τ. As τ increases, however, $C(Y_T, \tau, 0)$ converges to zero independent of the value of Y_T. Thus, while X may be initially negative, the model implies that X must eventually be a positive and increasing function of τ.

Fig. 4 plots the values of the deviations from the put–call parity relation as a function of the days to expiration for the 13-week T-bill yield options. As illustrated, the relation of the deviations to the life of the options is consistent with the implications of the model. For example, only the shortest-maturity options lead to negative values of X. The deviations are positive and strongly positively related to maturity for the remaining options. The mean deviation from the put–call parity relation is 3.3387. The t-statistic for the mean is 14.02. The correlation between the deviations and τ is 0.8597. This evidence of systematic deviations from the standard put–call parity relation is again consistent with the implications of the yield-option-pricing model but inconsistent with pricing models in which the underlying state variable is the price of a traded asset or a forward price.

4.3. Bias tests

The preceding results provide general support for the empirical implications of the yield-option-pricing model of this paper. It is also important, however, to examine the specific pricing implications of the model. I use an approach similar to that of Rubinstein (1985), who tests several stock-option-pricing models by examining whether implied volatility estimates are systematically related to variables such as the strike/stock-price ratio or the time until expiration. In these tests, I also infer the parameters of the pricing

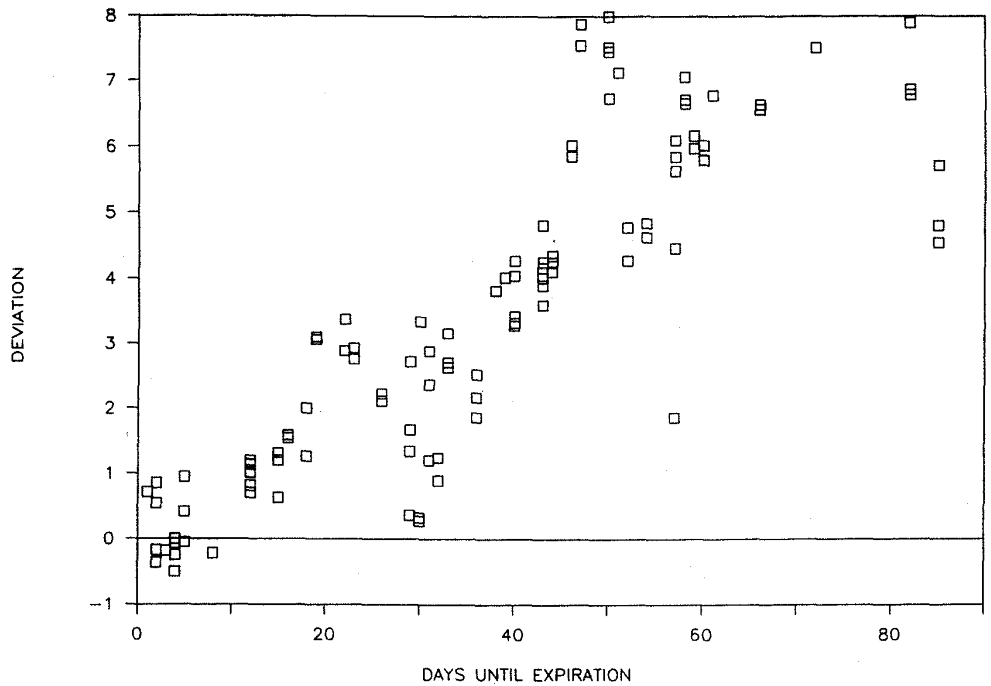

Fig. 4. Deviations from the put–call parity relation for options on traded assets computing using CBOE 91-day Treasury-bill yield options for the June 1989 to December 1989 period plotted as a function of the number of days until expiration for the options. 107 observations.

model from the data and then test whether the model is misspecified by examining whether the implied values display any systematic relation to other variables. An important advantage of this approach is that it provides specific information about the model's biases.

Recall that the model has three parameters – α, β, and σ^2. In principle, implied estimates of these parameters can be obtained in a straightforward manner from the prices of three interest-rate-sensitive contingent claims. For example, estimates of α, β, and σ^2 could be obtained from the price of a τ-maturity option on Y_T and the prices of the discount bonds with maturities τ and $\tau + T$. Rather than inferring the values of all three parameters from the data, however, I use the following equivalent procedure in the empirical tests. First, I select realistic values for the parameters α and β. Conditional on these values, I then infer a value of σ^2 for each option in the sample.[7] Next, I test whether these implied estimates of σ^2 display systematic biases. I then repeat the procedure for another choice of the parameters α and β.

[7]In rare instances, there may not be a unique solution for σ^2 because of the complicated relation between σ^2 and the yield-option price. In these cases, the extraneous solutions can usually be discarded on economic grounds because they are implausibly large.

This test approach has a number of advantages. For example, it allows me to determine the sensitivity of the test results to different choices of parameters. In addition, since only one parameter is implied from the data, any pricing error is concentrated in the σ^2 estimates, leading to tests that are potentially more powerful. Finally, this approach makes it possible to test for biases using simple and easily interpreted univariate regressions, rather than more complex multivariate regressions.

I test for four possible types of pricing bias, by regressing the implied values of σ^2 obtained from the 13-week T-bill call and put prices on the 13-week T-bill yield Y_T, the option's life τ, the degree to which the option is in or out of the money $Y_T - K$, and the absolute value of $Y_T - K$. Including these last two variables provides a test for linear and nonlinear strike price biases. And, by using the estimated value of σ^2 as the dependent variable in the regression, I avoid the possibility of sampling or measurement error in σ^2 causing an errors in variables problem. The regressions are estimated by ordinary least squares using a standard Cochrane–Orcutt procedure to guard against serially correlated errors. Because of the possibility of conditional heteroskedasticity in the residuals, all t-statistics reported are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix. To minimize the possibility of outliers, I restrict the tests to options with ten or more days until expiration. This reduces the sample size slightly, to 234 calls and 151 puts. The values used for α and β in the tests are consistent with a long-run average value of Y_0 in the range from 0.04 to 0.06 and a first-order serial correlation for monthly observations of Y_0 in the range from 0.90 to 0.95.

Table 2 summarizes the implied values of σ^2 and presents the regression results for the yield calls. The average value of the σ^2 estimates ranges from 0.00232 to 0.01001. These values are consistent with an unconditional variance for the riskless rate ranging from about 80 to 120 basis points, which agrees well with recent interest-rate behavior. The standard deviations for the σ^2 estimates are similar to the means.

The results show some evidence of systematic bias in the option-pricing model. In particular, the bias related to the level of the 13-week T-bill yield is statistically significant in all of the regressions. This uniformity suggests that the bias is probably due to an actual misspecification of the model rather than to the choice of α and β. Part of the apparent misspecification, however, could also be due to the bid–ask spread implicit in the observed transaction prices. If the bid–ask spread is related to the level or volatility of yields, some yield bias might be introduced into the data [see Phillips and Smith (1980)]. The other sources of bias are not robust to the choice of the parameters. $Y_T - K$ and its absolute value are significant in only two of the six regressions.

The bias accounts for only a very small part of the total variation in the implied values of σ^2. For example, the adjusted R^2 coefficients for the

Table 2

Results from regressions of implied volatility estimates σ^2 from yield-call values[a] (using the indicated values of α and β) on the 13-week Treasury-bill yield[b] Y_T, the time until expiration for the options in years τ, the difference between Y_T and the strike price K, and the absolute value of this difference.[c] Calls with fewer than ten days to expiration are excluded. White (1980) heteroskedasticity-consistent t-statistics are reported in parentheses. 234 observations.

$$\sigma^2 = \gamma_0 + \gamma_1 Y_T + \gamma_2 \tau + \gamma_3 (Y_T - K) + \gamma_4 |Y_T - K| + \varepsilon$$

α	β	Avg. σ^2	S.D. σ^2	γ_0	γ_1	γ_2	γ_3	γ_4	Adj. R^2
0.04	1.00	0.00542	0.00463	−0.02818	0.69575	−0.00506	−0.16161	0.34341	0.035
				(−2.59)	(2.84)	(−0.70)	(−1.36)	(1.71)	
0.05	1.00	0.00365	0.00380	−0.02473	0.55773	−0.00860	−0.26684	0.19508	0.045
				(−2.61)	(2.89)	(−1.49)	(−3.32)	(1.16)	
0.06	1.00	0.00232	0.00290	−0.02628	0.42528	−0.00712	−0.31902	0.18388	0.114
				(−2.81)	(3.06)	(−1.75)	(−7.23)	(1.54)	
0.04	1.25	0.01001	0.00633	−0.02974	0.89223	0.00677	0.15573	0.63236	0.117
				(−2.22)	(2.45)	(0.70)	(0.76)	(2.65)	
0.05	1.25	0.00776	0.00559	−0.02953	0.82999	0.00143	−0.05911	0.48098	0.059
				(−2.38)	(2.61)	(0.16)	(−0.33)	(2.08)	
0.06	1.25	0.00572	0.00495	−0.03169	0.79120	−0.00546	−0.19254	0.36551	0.040
				(−2.76)	(3.01)	(−0.70)	(−1.49)	(1.71)	

[a]An implied value of the volatility parameter σ^2 is obtained numerically, conditional on the given values of the model parameters α and β, for each option price in the sample. The mean value of σ^2 and the standard deviation of the estimates S.D. σ^2 are computed using all 234 estimates of the volatility parameter.

[b]The underlying yield for the calls is actually 1,000 times the 13-week T-bill yield. In computing the implied values of σ^2, I rescale the call prices, underlying yields, and strike prices for the options by dividing by 1,000.

[c]The regressions are estimated by ordinary least squares using a standard Cochrane–Orcutt correction for serially correlated residuals.

Table 3

Results from regressions of implied volatility estimates σ^2 from yield-put values[a] (using the indicated values of α and β) on the 13-week Treasury-bill yield[b] Y_T, the time until expiration for the options in years τ, the difference between Y_T and the strike price K, and the absolute value of this difference.[c] Puts with fewer than ten days to expiration are excluded. White (1980) heteroskedasticity-consistent t-statistics are reported in parentheses. 151 observations.

$$\sigma^2 = \gamma_0 + \gamma_1 Y_T + \gamma_2 \tau + \gamma_3 (Y_T - K) + \gamma_4 |Y_T - K| + \varepsilon$$

α	β	Avg. σ^2	S.D. σ^2	γ_0	γ_1	γ_2	γ_3	γ_4	Adj. R^2
0.04	1.00	0.00252	0.00297	0.02397	−0.35474	−0.01018	−0.00765	0.22094	0.076
				(3.61)	(−3.25)	(−2.47)	(−0.04)	(0.92)	
0.05	1.00	0.00443	0.00423	0.04358	−0.66635	−0.00527	−0.27248	0.42941	0.083
				(4.41)	(−4.07)	(−0.86)	(−0.92)	(1.23)	
0.06	1.00	0.00702	0.00551	0.05399	−0.88208	0.00735	−0.73314	0.71725	0.157
				(4.65)	(−4.28)	(0.83)	(−2.22)	(1.91)	
0.04	1.25	0.00183	0.00124	0.00394	−0.06287	−0.00683	0.09710	0.08548	0.279
				(1.76)	(−1.51)	(−4.45)	(4.54)	(2.91)	
0.05	1.25	0.00268	0.00210	0.01371	−0.20139	−0.00962	0.03736	0.19851	0.138
				(3.05)	(−2.75)	(−3.75)	(0.34)	(1.54)	
0.06	1.25	0.00385	0.00338	0.03036	−0.46116	−0.00971	−0.00336	0.22367	0.074
				(4.08)	(−3.73)	(−2.09)	(−0.01)	(0.84)	

[a] An implied value of the volatility parameter σ^2 is obtained numerically, conditional on the given values of the model parameters α and β, for each option price in the sample. The mean value of σ^2 and the standard deviation of the estimates S.D. σ^2 are computed using all 151 estimates of the volatility parameter.

[b] The underlying yield for the puts is actually 1,000 times the 13-week Treasury-bill yield. In computing the implied values of σ^2, I rescale the put prices, underlying yields, and strike prices for the options by dividing by 1,000.

[c] The regressions are estimated by ordinary least squares using a standard Cochrane–Orcutt correction for serially correlated residuals.

Table 4

Summary statistics for the pricing errors of the yield-option-pricing model.[a]

	Yield calls	Yield puts
Mean percentage pricing error[b]	2.228%	−5.963%
Std. dev. of mean percentage error	3.575%	4.632%
t-statistic	0.62	−1.29
Root mean squared error	4.212%	7.551%
Correlation of model and actual prices	0.905	0.932

[a]The model prices are computed using the parameters $\alpha = 0.05$, $\beta = 1.00$, and $\sigma^2 = 0.004$. The model prices are then compared with the actual prices of the options. Options with fewer than ten days until expiration are excluded. This results in a sample of 234 call prices and 151 put prices.

[b]The percentage errors are computed in relation to the model price.

regressions range from 0.035 to 0.117 with a mean of 0.068. These adjusted R^2 coefficients, in conjunction with the standard deviations of the σ^2 estimates and the low elasticity[8] of the call price with respect to σ^2, suggest that the pricing effects of the model's bias are small.

The results for yield puts are summarized in table 3. In general, the results are similar to those for yield calls. For example, the yield bias is significant in five of the six regressions, and inferences about the significance of $Y_T - K$ and its absolute value are again parameter-dependent. The maturity of the yield put, however, is significant in four regressions. Although the adjusted R^2 coefficients are roughly twice as large as for the yield-call regressions, they are still fairly small in absolute terms, averaging 0.134. Comparing table 2 with table 3 shows that the implied values of σ^2 from yield calls and yield puts are roughly comparable on average, but can differ for specific parameter values. When $\alpha = 0.05$ and $\beta = 1.00$, the average estimate of σ^2 is approximately 0.004 for both calls and puts.

4.4. *Pricing errors*

To provide some information about the magnitude of the pricing errors of the model, I use the parameter values $\alpha = 0.05$, $\beta = 1.00$, and $\sigma^2 = 0.004$ to compute model values and compare these with actual prices. These in-sample results are summarized in table 4. As shown, the model tends to overprice

[8]For example, numerical estimates of the elasticity of in-the-money yield calls with respect to σ^2 – using the parameters $\alpha = 0.05$, $\beta = 1.00$, $\sigma^2 = 0.004$, $T = 0.25$, and $\tau = 0.1$ – range from approximately 0.20 for calls that are 50 basis points in the money to 0.05 for calls that are 100 basis points in the money.

calls by about 2.22% on average. The t-statistic for this difference is only 0.62, however. In contrast, the model underprices put options by approximately 5.96%. The t-statistic for this error is -1.29. The larger error for puts is consistent with the implications of the bias tests. The correlation of actual option prices with the option prices implied by the model is 0.905 for calls and 0.932 for puts. Although the option-pricing model does have some systematic pricing error, it performs fairly well on average.

5. Conclusion

The pricing of options on the level of yields is important because yield options are incorporated into a wide variety of commonly encountered contingent claims. I derive closed-form expressions for European calls and puts on yields using an extended version of the Cox, Ingersoll, and Ross (1985b) term-structure model. This approach has the important advantage of expressing interest-rate-dependent contingent-claim values in terms of yields. This allows the pricing model to capture the level of the yield curve at the most relevant maturities. I show that options on average yields and options on yield spreads can be represented as portfolios of yield options.

The analytical properties of these options are very different from those of options on traded assets. This is because the underlying yield need not follow a martingale under the risk-neutral pricing process. As examples of the differences, I show that yield calls can be worth more than the underlying yield or worth less than the call's intrinsic value. The comparative-statics analysis indicates that yield calls can actually be perverse hedges against yield changes and are not monotone increasing functions of their maturity or of the volatility of interest rates.

To examine the empirical implications of the model, I focus on the pricing of the recently introduced options on the 13-week Treasury-bill yield traded on the CBOE. I find that over 75% of the in-the-money call prices in the sample are below the intrinsic value of the options, which is consistent with the pricing model derived here. In addition, the deviations from put–call parity are positively related to the life of the options, as implied by the model. Using a technique similar to Rubinstein's (1985), I test for bias in the implied values of the model's volatility parameter. I find some evidence of a yield bias, but its explanatory power is small in economic terms. I then examine the model's pricing errors and find that the average mispricing is about 2% for calls and 6% for puts. Neither of these differences is statistically significant.

This analysis provides a number of important results and insights into the behavior of yield-option prices. This topic merits further study and research, however. The option-pricing literature focuses primarily on options on traded assets, but yield options have the potential to become even more significant for financial markets than stock options.

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