IDENTIFICATION OF MULTIPLE-INPUT
TRANSFER FUNCTION MODELS

Lon-Mu Liu*   Dominique M. Hanssens

Department of Biomathematics   Graduate School of Management
University of California   University of California
Los Angeles, California 90024   Los Angeles, California 90024

ABSTRACT

This paper proposes a procedure for transfer function identification (specification) based on least-squares estimates of transfer function weights using the original or filtered series. The corner method is then used to identify a parsimonious rational form of the transfer function. The procedure is illustrated in a simulated example; it is shown how this straightforward approach outperforms other identification methods such as Box and Jenkins' prewhitening and Haugh and Box' double prewhitening techniques.

1. INTRODUCTION

The transfer function model is one of the most widely used time-series models in several areas of application such as engineering, economics and management science. Yet in spite of its popularity, the identification or specification stage of transfer function analysis is not sufficiently developed, in particular for multiple-input models.

---

*Now at the Department of Quantitative Methods, University of Illinois at Chicago Circle, Box 4348, Chicago, Illinois 60680.

297

The best known identification approach is due to Box and Jenkins (1976), who propose a comprehensive procedure for the one-input situation by studying the sample cross-correlation function (SCCF) between the prewhitened input series and the corresponding filtered output series (filtered by the ARMA model of the input series). While their method is promising for the one-input situation, it is difficult to generalize it for multiple-input models. Priestley (1971) recommends prewhitening both input and output series and obtaining the transfer function weights of the prewhitened series using least-squares estimation. Haugh and Box (1977) also suggest studying the SCCF of the prewhitened input and output series from which they identify the transfer function model. We refer to the Priestley, and Haugh and Box approach as the double prewhitening method. Fask and Robinson (1977) generalize the double prewhitening method to multivariate dynamic models. Granger and Newbold (1977a,b) also apply this method for the identification of a two-way causal system. Although double prewhitening has been widely used, it is more difficult to apply than the Box and Jenkins method, even in the one-input situation. The major difficulties are that a model is frequently over-structured due to the prewhitening factors, and that the lag structure is often cumbersome to derive.

Tiao et al (1979) use sample cross-correlation and partial correlation matrices of a set of multiple series for the identification of multiple ARIMA models. Since the transfer function model can be viewed as a special case of the multiple ARIMA model, their method can be applied to transfer function identification. Unfortunately, their method imposes an extra structure between the factors on the residuals and the transfer function, causing the transfer function model to be over-structured.

As an alternative to time-domain analysis, transfer function identification has also been explored using a frequency-domain approach (spectral methods), e.g., Box and Jenkins (1976) and Priestley (1971). However, it is rather difficult to apply spectral methods in practice.

The present paper proposes a procedure which mainly applies linear least-squares estimation on the original or filtered series for transfer function identification, similar to Priestley's (1969) suggestion for the one-input situation. Another least-squares approach was developed by Caines, Sethi and Brotherton (1977), who used a Cholesky least-squares algorithm for transfer function analysis. However, their method is aimed at model estimation rather than identification and their results are difficult to extend to multiple-input models. Although the procedure in the present paper is simple and straightforward, it performs very well. In addition to its simplicity, the method is easily extended to models with intervention components, which are often difficult to handle with other methods.

2. STATISTICAL BACKGROUND

Without loss of generality, we study the following two-input transfer function model:

$$Y_t = c + \frac{\omega_1(B)}{\delta_1(B)} X_{1t} + \frac{\omega_2(B)}{\delta_2(B)} X_{2t} + \varepsilon_t , \qquad (2.1)$$
$$t = 1, 2, \ldots, N$$

where X_{1t} and X_{2t} are input series (or exogenous variables), Y_t is the output series (or endogenous variable), ε_t the noise series, and c a constant. All series are stationary and follow ARMA (autoregressive-moving average) processes. The ARMA process for the noise series is

$$\phi(B)\varepsilon_t = \theta(B)e_t \qquad (2.2)$$

with
$$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p ,$$
and
$$\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q ,$$

where B is a backward shift operator ($B\varepsilon_t = \varepsilon_{t-1}$) and e_t's are independently identically distributed. The rational polynomial $\omega_i(B)/\delta_i(B)$ is the transfer function between Y_t and X_{it} where

$$\omega_i(B) = (\omega_{i1} + \omega_{i2}B + \ldots + \omega_{is_i} B^{s_i-1}) B^{b_i}$$

and   (2.3)

$$\delta_i(B) = 1 - \delta_{i1}B - \cdots - \delta_{ir_i}B^{r_i} .$$

The roots of the polynomials $\delta_i(B)$, $\phi(B)$ and $\theta(B)$ lie outside the unit circle. The noise series ε_t must be independent of each input series, but the input series may be correlated with each other.

The transfer function component $\omega_i(B)/\delta_i(B)$ can also be expressed as

$$V_i(B) = v_{i0} + v_{i1}B + v_{i2}B^2 + \cdots ,$$

where $V_i(B)$, a linear form of the transfer function, has a finite number of terms if $\delta_i(B) = 1$ and an infinite number of terms if $\delta_i(B) \neq 1$. Since all roots of the $\delta_i(B)$ polynomial lie outside the unit circle, the transfer function $\omega_i(B)/\delta_i(B)$ can always be approximated by $V_i(B)$ with a finite number of terms, say K_i, in practice. We refer to v_{ij} as the j-th transfer function (or impulse response) weight for the input series X_{it}.

The task of transfer function identification is to find appropriate estimates of the $V_i(B)$ polynomials and to express them in rational forms $\omega_i(B)/\delta_i(B)$. Both elements are discussed below.

2.1 Estimating the Transfer Function Weights

For sufficiently large K_i's, (2.1) can be expressed as

$$Y_t = c + (v_{10} + v_{11}B + \cdots + v_{1K_1}B^{K_1}) X_{1t}$$

$$+ (v_{20} + v_{21}B + \cdots + v_{2K_2}B^{K_2}) X_{2t} + \varepsilon_t \qquad (2.4)$$

Using
$$K = \text{Max}(K_1, K_2),$$

$$n = N - K,$$

$$\underset{\sim}{\beta} = [c \ \ v_{10} \ \ v_{11} \ \cdots \ v_{1K_1} \ \ v_{20} \ \ v_{21} \ \cdots \ v_{2K_2}]'$$

and
$$\underset{\sim}{Y} = [Y_{K+1}, Y_{K+2}, \cdots, Y_{K+n}],$$

$$\underset{\sim}{X} = [1 \ \ \ \underset{\sim}{X}_1^0 \ \ \underset{\sim}{X}_1^1 \ \cdots \ \underset{\sim}{X}_1^{K_1} \ \ \underset{\sim}{X}_2^0 \ \ \underset{\sim}{X}_2^1 \ \cdots \ \underset{\sim}{X}_2^{K_2}],$$

where

$$\underset{\sim}{X}_i^j = B^j \underset{\sim}{X}_i^0 \quad \text{and} \quad \underset{\sim}{X}_i^0 = [X_{i(K+1)} \; X_{i(K+2)} \; \cdots \; X_{i(K+n)}]' \; ,$$

the ordinary least-squares (OLS) estimate of $\underset{\sim}{\beta}$ can be expressed as

$$\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Y} \; . \tag{2.5}$$

Two problems may arise in the above least-squares estimation. First, if one of the input series contains an autoregressive (AR) factor with roots close to one, i.e., if the series is close to nonstationarity, the $\underset{\sim}{X}'\underset{\sim}{X}$ matrix may be near-singular and result in ill-conditioning for matrix inversion. This can be observed as follows: if we standardize the matrix $\underset{\sim}{X}'\underset{\sim}{X}$ by its diagonal elements, the new matrix approximately contains the sample autocorrelations and cross-correlations of the X_{1t} and X_{2t} as its elements. Thus if one of the input series contains an autoregressive polynomial with roots close to the unit circle, a number of sample autocorrelations will be very close to one. This difficulty is less serious if an input series follows a moving (MA) process since the sample autocorrelations of an MA process will not be close to one even if the root of the MA polynomial is one (Box and Jenkins 1976). Second, the noise series ε_t may not be white noise, therefore the OLS estimates of β may not be efficient.

To avoid the difficulty of inverting an ill-conditioned $\underset{\sim}{X}'\underset{\sim}{X}$, we suggest to examine the ARMA models of the input series and then take appropriate action. As argued earlier, only the roots of the AR polynomials are important: if the absolute values of all these roots are rather large there is no problem in using the original data. However, if some of the roots are close to one, a common filter on the input and output series is used in order to make $\underset{\sim}{X}'\underset{\sim}{X}$ well-conditioned.

Common filters have been used previously, primarily because they do not alter the transfer function weights if the series are stationary (Priestley 1971, Granger and Newbold 1977b). For example, Sims (1972) employed an ad hoc filter $(1-.75B)^2$ in a well-

known study of causality between money and income. Although the
choice of a filter can be flexible, we do offer a selection cri-
terion, namely that it eliminates the AR factors with roots close
to one. If there are more than one such factors they can be com-
bined in one common filter. As an example, if three input series
X_1, X_2 and X_3 follow the ARMA processes

$$(1-.9B)(1-.4B^4) \ X_{1t} = a_{1t} \ ,$$

$$(1 - .3B)(1 - .8B^4) \ X_{2t} = (1 - .7B)a_{2t} \ ,$$

and $\qquad (1 - .5B) \ X_{3t} = (1 - .9B^4)a_{3t} \ ,$

a choice of $(1 - .9B)(1 - .8B^4)$ is recommended as a common filter.

It should be pointed out that common filtering is done for
numerical accuracy rather than statistical efficiency. Its nece-
ssity depends on the roots of the AR polynomials <u>and</u> the precision
of the computer. In some situations, filtering could be avoided
by using double precision computing. Alternatively, the multi-
collinearity could be handled by ridge regression analysis (e.g.,
Erickson 1981). Also, since a filter with roots close to unity
may create an MA polynomial near non-invertibility when ε_t is
white noise, it should be avoided. For instance, the filter
$(1-.7B) \ (1-.7B^4)$ may be safer in the above example.

The second problem can be resolved by estimating $\underset{\sim}{\beta}$ with
generalized least squares (GLS) rather than OLS. If the covari-
ance matrix $\underset{\sim}{\Sigma}$ of ε_t (t=K, K+1,...,N) is known the GLS estimator of
$\underset{\sim}{\beta}$ is:

$$\underset{\sim}{\tilde{\beta}} = (X' \underset{\sim}{\Sigma}^{-1} X')^{-1} X' \underset{\sim}{\Sigma}^{-1} Y \ , \qquad\qquad (2.6)$$

which is consistent and efficient. The direct computation of (2.6)
is complicated. It may be easier to obtain $\tilde{\beta}$ by using OLS esti-
mation on the transformed series $\underset{\sim}{\tilde{X}}$ and $\underset{\sim}{\tilde{Y}}$. Since $\underset{\sim}{\Sigma}^{-1}$ can be ex-
pressed as $\underset{\sim}{H}'\underset{\sim}{H}$, we obtain

$$\underset{\sim}{\tilde{\beta}} = (\underset{\sim}{\tilde{X}}'\underset{\sim}{\tilde{X}})^{-1}\underset{\sim}{\tilde{X}}'\underset{\sim}{\tilde{Y}} \ , \qquad\qquad (2.7)$$

where $\tilde{X} = H X$,   (2.8)

           $\tilde{Y} = H Y$.   (2.9)

Following Ljung and Box (1979) the transformations (2.8) and
(2.9) are approximately equivalent to filtering the input and
output series by the ARMA model of the noise series. In prac-
tice, the noise model is unknown but can be identified from
ARMA analysis on the OLS residuals. An iterative process of
filtering and OLS estimation could then be started until the
estimates converge. Such a procedure is similar to estimating
regression coefficients with the well-known Cochrane-Orcutt
(1949) method for simple AR(1) situations.

    Since the purpose of the least-squares procedures is <u>identi-
fication</u> of transfer functions, we find it not necessary to apply
the iterative process. In our experience, the OLS estimates
based on the filtered series are usually very satisfactory. For
more accurate estimates we recommend joint estimation of the
transfer function weights and noise parameters by using nonlinear
least-squares. This can be done easily with computer programs
such as BMDQ2T (Liu 1979).

2.2 Expressing the Transfer Function Weights in Rational Form

    In many instances a more parsimonious representation of
transfer function weights may be obtained by expressing the $V_i(B)$
in rational form. This task is equivalent to finding the values
r_i, s_i and b_i in (2.3). Box and Jenkins (1976) use the pattern of
the transfer function weights for this purpose, similar to their
ARMA identification method. A cut-off pattern implies that $r_i=0$
and is simple to model. A tail-out pattern implies that a denomi-
nator polynomial is present, but it appears difficult to apply Box
and Jenkins' guidelines in this case. The authors propose to use
a modification of the corner method for ARMA identification
(Beguin, Gourieroux and Monfort 1980) for this purpose.

    Let $v_{i,max}$ be the maximum value of $|v_{ij}|$, $j=0,1,2, \ldots, K_i$,
where the v_{ij}'s are the true transfer function weights for the

rational polynomial $\omega_i(B)/\delta_i(B)$ and let $\eta_{ij} = v_{ij}/v_{i,max}$. Then,
for each explanatory variable X_i we can construct a $(g \times g)$ matrix
$D(f,g)$ and its determinant $\Delta(f,g)$, where

$$D(f,g) = \begin{bmatrix} \eta_f & \eta_{f-1} & \cdots & \eta_{f-g+1} \\ \eta_{f+1} & \eta_f & \cdots & \eta_{f-g+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \eta_{f+g-1} & \eta_{f+g-2} & \cdots & \eta_f \end{bmatrix}$$

$f>0$, $g>1$, and $\eta_j = 0$ if $j<0$. The subscript i is omitted for
simplicity. Now, for any integer M large enough to be greater than
r_i and (s_i+b_i), we can build an $(M+1) \times M$ array C with $\Delta(f,g)$ as its
fg-th element, where $f=0,1,2, \ldots, M$, and $g=1,2, \ldots, M$. Then the
transfer function weights v_{ij} have a representation $\omega_i(B)/\delta_i(B)$
with orders r,s and b, if and only if the C array has the follow-
ing structure:

g \ f	1	2	...	r	r+1	...	M
0	0	0	...	0	0	...	0
1	0	0	...	0	0	...	0
.	.	.		.	.		.
.	.	.		.	.		.
.	.	.		.	.		
b-1	0	0	...	0	0	...	0
b	Δ(b,1)	Δ(b,2)	...	Δ(b,r)	Δ(b,r+1)	...	Δ(b,M)
.	.	.		.	.		.
.	.	.		.	.		.
s+b-1	Δ(s+b-1,1)	Δ(s+b-1,2)	...	x	x	...	x
s+b	Δ(s+b,1)	Δ(s+b,2)	...	x	0	...	0
.	.	.		.	.		.
.	.	.		.	.		.
M	Δ(M,1)	Δ(M,2)	...	x	0	...	0

where an "x" means that the term is different from zero. The
pattern implies the orders r, s and b if and only if the first b
rows and the lower right-hand corner starting at the (s+b+1)th row
and the (r+1)th column of the C-array are all zeroes. Details of
the corner method are discussed in Beguin, Gourieroux and Monfort
(1980).

In practice the v_{ij} are estimated by the $\hat{v}_{ij}$, which are
subject to random errors. Consequently, one will find some small
values in the C-array for the zeroes indicated above. However,
the upper and lower-right hand corner will show a sudden drop in
array values. As an illustration, the method is applied to the
gas furnace data in Box and Jenkins (1976). The estimated trans-
fer function weights are:

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\hat{v}_j$	-.02	10	-.06	-.53	-.63	-.38	-.52	-.32	-.06	.06	-.10	-.06	-.04	-.13

The C array is:

f \\ g	1	2	3	4	5	6	7
0	-.02	.00	-.00	.00	-.00	.00	-.00
1	.11	.01	.00	.00	-.00	-.00	-.00
2	-.07	.07	-.03	.01	-.00	.00	-.00
3	-.60	.31	-.17	.12	-.08	.05	-.03
4	-.72	-.09	.29	-.01	-.07	.02	.00
5	-1.00	.58	-.33	.15	-.07	.01	-.00
6	-.59	-.01	.09	.01	-.04	.01	.00
7	-.36	.09	-.02	.02	-.02	.01	-.01

and indicates clearly that b=3, s=3 and r=1, which is the final
model obtained in Box and Jenkins (1976).

In recent work, de Gooijer and Heuts (1981) examine the
application of the corner method to the identification of
ARMA(p,q) models and report that it fails to work well in practice.
However, we find that the C-arrays in their paper indicate the
simulated models quite clearly. The corner method indicates a few

candidates for the values of b,s and r, thereby providing an
efficient guideline to transfer function identification. The
principle of parsimony should be applied in cases where more than
one combination of parameters is feasible. Since the $\delta(B)$ poly-
nomials are usually simple in practice, the use of the corner
method to transfer function identification seems to be less diffi-
cult than for ARMA models.

3. A PROCEDURE FOR THE IDENTIFICATION OF A TRANSFER FUNCTION MODEL

Based on the results in Section 2, we now present a five-step
procedure which introduces filtering and least-squares estimation
in the transfer function identification.

Step 1:

Build ARMA models for all input series after the series are
appropriately differenced to achieve stationarity. If no AR
factors are found or the roots of the AR factors are large (not
close to 1), proceed to Step 2. If there are processes with AR
roots close to 1, choose a common filter from the AR factors.
Apply this filter to all input series and the output series.

Step 2:

Perform least-squares estimation of the transfer function
weights for the series obtained from Step 1. The values K_i should
be chosen from subject-matter considerations and should be suffi-
ciently large to avoid truncation bias. It is also important to
check the SACF of the residuals since they provide information
about the reliability of the usual least-squares hypothesis test-
ing. It is recommended to omit the unnecessary terms in (2.4) if
it is clear that they can be deleted.

Step 3:

Build an ARMA model for the residuals computed from the
linear model selected in Step 2. If the residuals are white
noise, proceed to Step 5. If not, go to Step 4.

Step 4:

Using the Step 3 ARMA model as a filter, perform OLS estima-
tion of the transfer function weights based on the filtered series.
Alternatively, we can also estimate the full transfer function -
noise model jointly by nonlinear least squares. The significance
tests of the weights can be carried out in the usual regression
manner.

Step 5:

If no prefiltering was used in Step 1, the noise model is the
one obtained in Step 4. Otherwise compute the noise of the orig-
inal output series by using the transfer function weights from
Step 2 or 4 and identify an ARMA model for the noise. Then,
obtain a rational form $w_i(B)/\delta_i(B)$ for input series X_i by using
the corner method on $V_i(B)$, if necessary. Note that the corner
method should be used only if some of the impulse response weights
are significant.

4. EXAMPLE

To illustrate the identification method a simulated example
is presented in this section.

The simulation model is

$$Y_t = (2B^3 + 4B^4)X_{1t} + \frac{1.5B^2 + 3B^3}{1-B + .24B^2} X_{2t} + \varepsilon_t , \quad t=1,\ldots,100 \qquad (4.1)$$

The models for ε_t, X_{1t}, and X_{2t} are

$$(1 - 1.3B + .4B^2)\varepsilon_t = e_t , \qquad e_t \sim N(0,2) ,$$
$$(1 - 1.4B + .48B^2)X_{1t} = b_t , \qquad b_t \sim N(0,1) ,$$
and
$$(1 - .7B)X_{2t} = c_t , \qquad c_t \sim N(0,2) ,$$

where e_t is independent of b_t and c_t, and b_t and c_t are contem-
poraneously correlated with correlation 0.7.

Following Box and Jenkins' guidelines, the ARMA models for
Y_t, X_{1t} and X_{2t} are obtained as[1]

$$(1 - 2.17 B + 1.95 B^2 - 1.02 B^3 + .28 B^4)Y_t = \hat{a}_t , \qquad \hat{\sigma}_a^2 = 42.377 ,$$

$$(1 - 1.34 B + .37 B^2) X_{1t} = \hat{b}_t , \qquad \hat{\sigma}_b^2 = 1.102 ,$$

and $\qquad (1 - .69 B) X_{2t} = \hat{c}_t , \qquad \hat{\sigma}_c^2 = 2.118 .$

The AR polynomial for X_{1t} can be factored into $(1 - .95B)$ $(1 - .39B)$. Comparing the AR factors in the input series, we choose $(1 - .95B)$ as a common filter. Table I lists the OLS and

TABLE I

Estimates of Transfer Function Weights

	(1) OLS on original series		(2) OLS on prefiltered series		(3) GLS on prefiltered series		(4) OLS on prewhitened series	
	$\hat{v}_{ij}$	|t|*	$\hat{v}_{ij}$	|t|*	$\hat{v}_{ij}$	|t|	$\hat{v}_{ij}$	|t|*
c	-1.023	1.90	0.028	0.14	0.019	0.06	0.050	0.16
$\hat{v}_{10}$	0.148	0.24	0.304	1.13	-	-	-0.472	1.40
$\hat{v}_{11}$	0.233	0.23	-0.126	0.42	-	-	0.002	0.01
$\hat{v}_{12}$	-0.362	0.36	-0.051	0.17	-	-	0.208	0.62
$\hat{v}_{13}$	1.871	1.34	1.910	6.54	1.903	7.30	1.686	4.99
$\hat{v}_{14}$	3.389	3.13	3.770	12.54	3.350	16.25	2.269	5.04
$\hat{v}_{15}$	0.028	0.03	0.259	0.36	-	-	-2.238	6.58
$\hat{v}_{16}$	0.150	0.14	0.091	0.30	-	-	1.493	4.25
$\hat{v}_{17}$	-0.807	0.77	-0.270	0.92	-	-	-0.965	2.31
$\hat{v}_{18}$	0.494	0.35	0.513	1.99	-	-	0.410	1.13
$\hat{v}_{20}$	0.154	0.33	-0.320	1.57	-0.111	0.85	0.158	0.61
$\hat{v}_{21}$	0.190	0.35	-0.081	0.40	-0.036	0.22	0.119	0.47
$\hat{v}_{22}$	1.560	2.87	1.197	5.09	1.262	7.21	1.094	4.34
$\hat{v}_{23}$	4.384	3.05	4.116	20.50	4.313	19.92	2.603	10.14
$\hat{v}_{24}$	4.187	7.49	4.001	20.02	4.066	18.52	-1.073	4.25
$\hat{v}_{25}$	2.967	5.30	2.547	12.65	2.333	14.01	0.162	0.64
$\hat{v}_{26}$	1.532	2.85	1.371	7.00	1.619	3.30	-0.328	1.29
$\hat{v}_{27}$	1.716	3.39	1.134	5.91	1.165	5.11	0.203	0.32
$\hat{v}_{28}$	1.373	3.90	0.374	2.36	0.780	4.09	-0.590	2.77
$\hat{v}_{29}$	-	-	-	-	0.305	1.65	-0.211	1.19
$\hat{v}_{210}$	-	-	-	-	0.336	1.85	0.307	1.69
$\hat{v}_{211}$	-	-	-	-	0.432	2.54	0.011	0.06
$\hat{v}_{212}$	-	-	-	-	0.129	0.31	-0.134	0.74
$\hat{v}_{213}$	-	-	-	-	0.004	0.03	0.171	0.94
ρ	-	-	-	-	0.334	2.91	-	-
$\hat{\sigma}_e^2$	15.231		3.164		2.609		5.570	

*The t-values in columns 1, 2 and 4 are provided for information only. They do not necessarily reflect the correct significance levels of the estimates.

[1] The backcasting method is used in the estimation of ARIMA models.

GLS estimates of the transfer function weights and their t-values for the original and the prefiltered series.

The computations in TABLE I (except for the first column) are performed in single precision on IBM 3032 using BMDQ2T (Liu 1979). When single precision is used in computing the OLS estimates of transfer function weights for the original series, the computation is rejected because the $\underset{\sim}{X}'\underset{\sim}{X}$ matrix is too close to singularity. The estimates can be obtained (as shown in column 1) by using double precision. The results in column 1 seem to exhibit the simulated pattern well even though X_{1t} is highly autocorrelated. However, in other simulation tests we found that erroneous results are obtained even if double precision is used. When prefiltering is performed prior to OLS estimation (column 2), the transfer function weights can be estimated in single precision. These weights also exhibit the simulated pattern well. Since the residuals of the model in column 2 follow an AR(1) process, the transfer function weights are also estimated by using nonlinear least squares, incorporating this noise process (column 3). Longer lags are used since they are shown to be necessary in the previous two analyses. It is easy to identify the orders $r_1=0$, $s_1=2$ and $b_1=3$ from these estimates. As far as the model for X_{2t} is concerned, we may use the corner method described in Section 2.2. The $\underset{\sim}{C}$-array for the transfer function weights of X_{2t} is shown in TABLE II and indicates that $r_2=2$, $s_2=2$ and $b_2=2$. An examination of the residuals of the model in column 3 revealed an AR(2) process, consis-

TABLE II

C-Array for the Transfer Function Weights of X_{2t}

f \ g	1	2	3	4	5	6	7
0	-0.03	0.00	-0.00	0.00	-0.00	0.00	-0.00
1	-0.01	0.01	0.00	0.00	0.00	0.00	0.00
2	0.29	0.09	0.01	-0.00	-0.00	-0.00	-0.00
3	1.00	0.72	0.50	0.35	0.24	0.17	0.12
4	0.94	0.23	-0.00	-0.05	-0.03	0.01	0.02
5	0.66	0.08	0.02	0.01	0.00	0.00	0.00
6	0.38	-0.04	0.00	0.00	0.00	-0.00	0.00
7	0.27	0.01	0.00	0.00	0.00	0.00	0.00

tent with the originally simulated model. The final estimates and
standard errors (in parentheses) of the parameters are:

$$\hat{\omega}_{11} = 1.698 \ (.213) \qquad\qquad \hat{\delta}_{21} = .965 \ (.043)$$

$$\hat{\omega}_{12} = 3.987 \ (.191) \qquad\qquad \hat{\delta}_{22} = -.228 \ (.038)$$

$$\hat{\omega}_{21} = 1.404 \ (.194) \qquad\qquad \Phi_1 = 1.271 \ (.094)$$

$$\omega_{22} = 3.176 \ (.194) \qquad\qquad \Phi_2 = -.429 \ (.093)$$

$$\hat{\sigma}_e^2 = 2.466$$

For the purpose of comparison, we also performed two other
identification methods on the simulated data. First, following
Priestley (1971), and Haugh and Box (1977), we may regress $\hat{a}_t$ on
$\hat{b}_t$ and $\hat{c}_t$ using the model in (2.4). The least-squares estimates
of the transfer function weights and their t-values are listed in
the last column of TABLE I. From these weights we obtain the
model for the prewhitened series as

$$\hat{a}_t = (\omega_{11} + \omega_{12}B + \omega_{13}B^2 + \omega_{14}B^3 + \omega_{15}B^4) \ B^3 \ \hat{b}_t$$

$$+ \ (\frac{\omega_{21} + \omega_{22}B + \omega_{23}B^2 + \omega_{24}B^3}{1 - \delta_{21} B}) \ B^2 \ \hat{c}_t + e_t \ .$$

Substituting $\hat{a}_t$, $\hat{b}_t$, and $\hat{c}_t$ with the corresponding prewhitening
models, we obtain a tentative model for the system. Note that
this tentative model is much more complicated than the simulated
model. It requires a great deal of experience and effort to
simplify it to the actual model.

Secondly, if we apply Box and Jenkins'(1976) single pre-
whitening method to each pair of input and out series, we obtain
the SCCF between the prewhitened input series and the corres-
ponding filtered output series (i.e., $r_{\hat{b}\hat{a}'}(\ell)$ and $r_{\hat{c}\hat{a}''}(\ell)$) as
shown in TABLE III. The patterns of the SCCF in this situation
are somewhat misleading due to the high correlation between the
two input series.

TABLE III

SCCF's for the Simulated Example

Lag	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$r_{\hat{\delta}\hat{a}}(\ell)$	.05	-.01	-.09	-.07	-.10	-.06	-.13	.09	.06	.12	.42*	.36*	-.08	-.05	.00
$r_{\hat{c}\hat{a}''}(\ell)$	.13	.20	.16	.13	.08	.07	.07	.06	.06	.12	.41*	.51*	.33*	.20*	.16

The standard errors of the sample cross-correlatons are about .10, "" is used to denote significance at $p < .05$. The definition of $r_{xy}(\ell)$ is the same as in Box and Jenkins (1976).

In conclusion the simulated example illustrates the performance and simplicity of the proposed identification procedure while the other two methods provide not as good results.

6. DISCUSSION

This paper proposes filtering and least-squares estimation to obtain transfer function weights and the corner method to identify the rational form of the model. In practice, it is not necessary to prefilter the data unless one or more input series have autoregressive roots close to one in absolute value. In that case, the choice of a common filter is flexible because it does not alter the transfer function of a stationary system.

Unlike the proposed procedure, the double prewhitening method produces results which are heavily dependent on the choice of a prewhitening model. A simple prewhitening model may not whiten the series completely, thereby casting doubt on the use of the nice theoretical results in this approach. An elaborate model, on the other hand, will typically cause the transfer function to be over-structured.

The procedure is simpler when applied to original as opposed to prefiltered series. Therefore, it may be advisable in some cases to analyze the original data by using double precision computations or even ridge regression techniques. However, since model identification is an exploratory process, it is desirable to work with stationary as opposed to near-nonstationary input series.

Efficient estimates of the transfer function weights may be obtained by applying OLS estimation on the series, filtered by the ARMA model of the output residuals, or by performing joint estimation of transfer function weights and ARMA coefficients by using nonlinear least-squares. In most cases the OLS and GLS estimates of the weights are rather close even if the OLS residuals are not white noise, but their t-values may be quite different.

The corner method provides a mechanical way of finding the orders of $w_i(B)$ and $\delta_i(B)$ when $\delta_i(B)$ is not 1. Other methods, such as the S-array technique by Gray, Kelley and McIntire (1978), may also be applied to identify a rational form. However, the corner method appears to be very informative and easy to use.

ACKNOWLEDGEMENTS

BIBLIOGRAPHY

Beguin, J.-M, C. Gourieroux and A. Monfort (1980). Identification of a Mixed Autoregressive-Moving Average Process: the Corner Method. Time Series (Anderson, O.D., ed.). Amsterdam: North-Holland.

Box, G.E.P. and G. M. Jenkins (1976). Time-Series Analysis: Forecasting and Control. Revised Edition. San Francisco: Holden Day.

Caines, P.E., S.P. Sethi and T.W. Brotherton (1977). Impulse Response Identification and Causality Detection for the Lydia Pinkham Data. Annals of Economic and Social Measurement, 6, 147-63.

Cochrane, D. and G.H. Orcutt (1949). Application of Least Squares Regression to Relationships Containing Autocorrelated Error Terms. Journal of the American Statistical Association, 44, 32-61.

de Gooijer, J.G. and R.M.J. Heuts (1981). The Corner Method: An Investigation of an Order Discrimination Procedure for General ARMA Processes. Journal of the Operational Research Society, forthcoming.

Erickson, G. M. (1981). Using Ridge Regression to Directly Estimate Lagged Effects in Marketing. Journal of the American Statistical Association, forthcoming.

Fask, A. and P. B. Robinson (1977). Identification of Multivariate Dynamic Econometric Models. American Statistical Association, 1977 Proceedings of the Business and Economic Statistics Section, 284-289.

Granger, C.W.J. and P. Newbold (1977a). Identification of Two-way Causal Systems. Frontiers in Quantitative Economics, Vol. IIIA (Intriligator, M.D., ed.). Amsterdam: North-Holland.

Granger, C.W.J. and P. Newbold (1977b). The Time Series Approach to Econometric Model Building. New Methods in Business Cycle Research: Proceedings from a Conference (sponsored by the Federal Reserve Bank of Minneapolis).

Gray, H.L., G. Kelley and D. McIntire (1978). A New Approach to ARMA Modeling. Communications in Statistics, B7, 1-77.

Haugh, L.D. and G.E.P. Box (1977). Identification of Dynamic Regression (Distributed Lag) Models Connecting Two Time Series. Journal of the American Statistical Association, 72, 121-130.

Liu, L-M. (1979). User's Manual for BMDQ2T(TSPACK): Time Series Analysis (Box-Jenkins). Technical Report 57, Department of Biomathematics, University of California, Los Angeles.

Ljung, G.M. and G.E.P. Box (1979). The Likelihood Function of Stationary Autoregressive-Moving Average Models. Biometrika 66, 265-270.

Pack, D.J. (1977). A Computer Program for the Analysis of Time Series Models Using Box-Jenkins Philosophy. Automatic Forecasting Systems, Hatboro, PA.

Priestley, M.B. (1969). Estimation of Transfer Functions in Closed Loop Stochastic Processes. Automatica 5, 623-632.

Priestley, M.B. (1971). Fitting Relationships Between Time Series. Bulletin of the International Statistical Institute 34, 295-324.

Sims, C.A. (1972). Money, Income, and Causality. American Economic Review 62, 540-552.

Tiao, G.C., G.E.P. Box, M.R. Grupe, G.B. Hudak, W.R. Bell, and
    I. Chang (1979). The Wisconsin Multiple Time Series Program,
    A Preliminary Guide. Department of Statistics, University of
    Wisconsin, Madison.